ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

99-218

E2-99-218

E.E.Donets[1], A.I.Pashnev[2], J.J.Rosales[3], M.M.Tsulaia[4]

$N = 4$ SUPERSYMMETRIC MULTIDIMENSIONAL QUANTUM MECHANICS, PARTIAL SUSY BREAKING AND SUPERCONFORMAL QUANTUM MECHANICS

Submitted to «Physical Review D»

[1]E-mail: edonets@sunhe.jinr.ru
[2]E-mail: pashnev@thsun1.jinr.ru
[3]E-mail: rosales@thsun1.jinr.ru
[4]E-mail: tsulaia@thsun1.jinr.ru

1999

Донец Е.Е. и др. E2-99-218
Многомерная $N = 4$ суперсимметричная квантовая механика,
частичное нарушение суперсимметрии
и суперконформная квантовая механика

Многомерная $N = 4$ суперсимметричная квантовая механика рассмотрена с использованием суперполевого подхода. В результате получены компонентные формы соответствующих классических и квантовых Лагранжианов и Гамильтонианов. В рассматриваемой теории классическая и квантовая $N = 4$ алгебры содержат центральные заряды и это открывает различные возможности для частичного нарушения суперсимметрии. Показано, что квантовомеханические модели с одной четвертью, половиной и тремя четвертями ненарушенных (нарушенных) суперсимметрий могут существовать в рамках многомерной $N = 4$ суперсимметричной квантовой механики, тогда как соответствующая одномерная теория, построенная ранее, допускает только полное или половинное нарушение суперсимметрии. Полученный общий формализм проиллюстрирован на точно решаемом примере, который есть прямое многомерное обобщение $N = 4$ одномерной суперконформной квантовомеханической модели. Вкратце обсуждены некоторые открытые вопросы и возможные приложения к известным точно решаемым системам, а также к проблемам квантовой космологии.

Работа выполнена в Лаборатории теоретической физики им. Н.Н.Боголюбова и в Лаборатории высоких энергий ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1999

Donets E.E. et al. E2-99-218
$N = 4$ Supersymmetric Multidimensional Quantum Mechanics,
Partial SUSY Breaking and Superconformal Quantum Mechanics

The multidimensional $N = 4$ supersymmetric quantum mechanics (SUSY QM) is constructed using the superfield approach. As a result the component form of the classical and quantum Lagrangian and Hamiltonian is obtained. In the considered SUSY QM both classical and quantum $N = 4$ algebras include central charges and it opens various possibilities for the partial supersymmetry breaking. It is shown, that the quantum mechanical models with one quarter, one half and three quarters of the unbroken (broken) supersymmetries can exist in the framework of the multidimensional $N = 4$ SUSY QM, while the one-dimensional $N = 4$ SUSY QM, constructed earlier, admits only the one half or total supersymmetry breakdown. We illustrate the constructed general formalism, as well as all possible cases of the partial SUSY breaking on the example, which is the direct multidimensional generalization of the one-dimensional $N = 4$ superconformal quantum mechanical model. Some open questions and possible applications of the constructed multidimensional $N = 4$ SUSY QM to the known exactly integrable systems and to the problems of quantum cosmology are briefly discussed.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics and at the Laboratory of High Energies, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1999

I. INTRODUCTION

The supersymmetric quantum mechanics (SUSY QM), being introduced first in Refs. [1] - [2] for the $N = 2$ case turns out to be a convenient tool for investigating problems of the supersymmetric field theories, since it provides the simple and at the same time quite adequate understanding of various phenomena, arising in the relativistic theories.

The important question of all modern theories of fundamental interactions, including superstrings and M – theory is the problem of the spontaneous breakdown of supersymmetry. Supersymmetry, as a fundamental symmetry of the nature, if exists, has to be spontaneously broken at low energies, since particles with all equal quantum numbers, except the spin, are not observed experimentally. The several (rather different) mechanisms of the spontaneous breakdown of supersymmetry have been proposed in particle physics in order to resolve this problem. One of them is to add to the supersymmetric Lagrangian the so called D –, or F – terms, which are invariant under supersymmetry transformations as well but break the supersymmetry spontaneously due to the nonzero vacuum expectation values or, alternatively, to introduce into the theory some soft breaking mass terms "by hand"; the latter procedure does not spoil the nonrenormalization theorem of the supersymmetric field theories and was successfully applied for constructing the Minimal Supersymmetric extension of the Standard Model (see Ref. [3] and Refs. therein). The next mechanism of SUSY breaking is the dynamical (nonperturbative) breakdown of the supersymmetry, caused by instantons (see, for example, [4] and Refs. therein). In this case the energy of tunneling between topologically distinct vacua produces the energy shift from the zero level, hence leading to the spontaneous breakdown of supersymmetry. And, finally, the mechanism of the partial spontaneous breaking of the $N = 2$ supersymmetry in the field theory was proposed recently in Ref. [5]. This mechanism is based on the including into the Lagrangian two types of the Fayet – Iliopoulos terms – electric and magnetic ones and it leads to the corresponding modification of the $N = 2$ SUSY algebra of Ref. [6].

The problem of the spontaneous breakdown of supersymmetry could be investigated in the framework of the supersymmetric quantum mechanics as well. The conjecture, that supersymmetry can be spontaneously broken by instantons [1] – [2] was investigated in details by several authors for the case of $N = 2$ SUSY QM [7] – [9]. However the most physically interesting case is provided by the $N = 4$ supersymmetric quantum mechanics, since it can be applied to the description of the systems, resulted from the "realistic" $N = 1$ supersymmetric field theories (including supergravity) in four ($D = 4$) dimensions after the dimensional reduction to one dimension.

One-dimensional $N = 4$ SUSY QM was constructed first in Refs. [10] – [11]. The partial breaking of supersymmetry, caused by the presence of the central charges in the corresponding superalgebra was also discussed in Ref. [11]. It was the first example of the partial breaking of supersymmetry in framework of SUSY QM and the corresponding mechanism is in full analogy to that in Ref. [5] in the field theory. The main point is that the presence of the central charges in the superalgebra allows the partial supersymmetry breakdown, whereas according to Witten's theorem [1], no partial supersymmetry breakdown is possible if the SUSY algebra includes no central charges. The main goal of our paper is further generalization of the construction, proposed in Ref. [11] for the multidimensional case and investigation of partial breaking of supersymmetry under the consideration.

The question of the particular importance is to consider the supersymmetric algebra with the central charges for the several reasons. First, it provides a good tool to study the dyon solutions of quantum field theory since in such theories the mass and electric and magnetic charges turns out to be the central charges [12]. Second, the presence of the central charges produces the reach structure of supersymmetry breaking. Namely it is possible to break a part of all supersymmetries, leaving all others exact [13]. In fact, the invariance of a state with respect to the supersymmetry transformation means the saturation of the Bogomol'ny bound and this situation takes place in $N = 2$ and $N = 4$ supersymmetric Yang – Mills theory [14] – [15] as well as in theories of an extended supergravity [16].

The investigation of supersymmetric properties of branes in M – theory also revealed that the partial breakdown of supersymmetry takes place. Namely the ordinary branes break half of the supersymmetries, while "intersecting" and rotating branes can leave only 1/4, 1/8, 1/16 or 1/32 of the supersymmetries unbroken [17]. The known examples of the breakdown of the supersymmetries in supersymmetric quantum mechanics are the cases, where either all the supersymmetries are broken(exact) or only half of all the supersymmetries are broken(exact) [11]. In this paper we demonstrate the possibility of the three – quarters or one – quarter of the supersymmetry breakdown in the framework of multidimensional $N = 4$ SUSY QM. The later case (3/4 of the supersymmetries are exact) has not been observed before neither in supersymmetric field theories nor in SUSY QM and seems to be quite interesting by itself even without specifying the physical origin of the phenomena.

The paper is organized as follows. In Sec. II we present a formal construction of $N = 4$ multidimensional supersymmetric quantum mechanics: classical and quantum Hamiltonian and Lagrangian, SUSY transformations, supercharges algebra and so on. In Sec. III partial supersymmetry breaking is investigated and all possible cases of the partial SUSY breakdown are listed. In Sec. IV we give an exactly solvable example, which illustrates main properties of the introduced formal constructions. This example is interesting by itself since we consider the multidimensional generalization of the $N = 4$ superconformal quantum mechanics [11], [18], which is naturally related to the extremal RN black holes in "near horizon" limit and adS/CFT correspondance [19]. In Sec. V we conclude with some open questions and further perspectives.

II. D – DIMENSIONAL $N = 4$ SUSY QUANTUM MECHANICS

In this section we describe the general formalism of D – dimensional ($D \geq 1$) $N = 4$ supersymmetric quantum mechanics, starting with the superfield approach and concluding with the component form of the desired Lagrangian and Hamiltonian.

Consider $N = 4$ SUSY transformations

$$\begin{aligned} \delta t &= \frac{i}{2}(\epsilon^a \bar{\theta}_a + \bar{\epsilon}^a \theta_a), \\ \delta\bar{\theta}_a &= \bar{\epsilon}_a, \\ \delta\theta^a &= \epsilon^a, \end{aligned} \tag{2.1}$$

in the superspace, spanned by the even coordinate t and mutually complex conjugated odd coordinates θ^a and $\bar{\theta}_a$. The parameters of $N = 4$ SUSY transformations ϵ^a and $\bar{\epsilon}_a$ are complex conjugate to each other as well.[1] The generators of the above supersymmetry transformations

$$Q_a = \frac{\partial}{\partial\theta^a} + \frac{i}{2}\bar{\theta}_a\frac{\partial}{\partial t}, \quad \bar{Q}^a = \frac{\partial}{\partial\bar{\theta}_a} + \frac{i}{2}\theta^a\frac{\partial}{\partial t}, \tag{2.2}$$

along with the time translation operator $H = -i\frac{\partial}{\partial t}$ obey the following (anti)commutation relations:

$$\begin{aligned} \{Q_a, \bar{Q}^b\} &= -\delta_a^b H, \\ [H, Q_a] &= [H, \bar{Q}^a] = 0. \end{aligned} \tag{2.3}$$

The automorphism group for a given algebra is $SO(4) = SU(2) \times SU(2)$ and the generators of the $N = 4$ SUSY transformations are in the spinor representation of one of the $SU(2)$ groups.

The next step is to construct irreducible representations of the algebra (2.3). The usual way of doing this is to use the supercovariant derivatives

$$D_a = \frac{\partial}{\partial\theta^a} - \frac{i}{2}\bar{\theta}_a\frac{\partial}{\partial t}, \quad \bar{D}^a = \frac{\partial}{\partial\bar{\theta}_a} - \frac{i}{2}\theta^a\frac{\partial}{\partial t}, \tag{2.4}$$

and impose some constraints on the general superfield. Hereafter we deal with the superfield Φ^i ($i = 1, \ldots, D$), subjected to the following constraints:

$$\begin{aligned} [D_a, \bar{D}^a]\Phi^i &= -4m^i, \\ D^a D_a \Phi^i &= -2n^i, \\ \bar{D}_a \bar{D}^a \Phi^i &= -2\bar{n}^i, \end{aligned} \tag{2.5}$$

where m^i are real constants, while n^i and $\bar{n}^i$ are mutually complex conjugated constants. The explicit form of the superfield Φ^i is the following:

$$\begin{aligned} \Phi^i &= \phi^i + \theta^a \bar{\psi}_a^i - \bar{\theta}_a \psi^{ia} + \theta^a B_a^{bi} \bar{\theta}_b + \\ &+ m^i(\theta\bar{\theta}) + \frac{1}{2}n^i(\theta\theta) + \frac{1}{2}\bar{n}^i(\bar{\theta}\bar{\theta}) + \\ &+ \frac{i}{4}(\theta\theta)\bar{\theta}_a \dot{\psi}^{ai} - \frac{i}{4}(\bar{\theta}\bar{\theta})\theta^a \dot{\bar{\psi}}_a^i + \frac{1}{16}(\theta\theta)(\bar{\theta}\bar{\theta})\,\ddot{\phi}^i, \end{aligned} \tag{2.6}$$

($\dot{} \equiv \partial_t$). Note, that in the case when all the constants m^i, n^i, $\bar{n}^i$ are equal to zero, the superfield (2.6) represents D "trivial" copies of the superfield Φ, given in Ref. [20], which describes the irreducible representation of one-dimensional $N = 4$ SUSY QM. The latter superfield contains one bosonic field ϕ, four fermionic fields ψ^a and $\bar{\psi}_a$ and three auxiliary bosonic fields $B_a^b = (\sigma_I)_a^b B^I$, where $(\sigma_I)_a^b$ ($I = 1, 2, 3$) are ordinary Pauli matrices.

Another irreducible representation of the algebra (2.3) can be constructed after making the appropriate generalization of the constraints given in Ref. [10]:

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[1] Our conventions for spinors are as follows: $\theta_a = \theta^b \varepsilon_{ba}$, $\theta^a = \varepsilon^{ab}\theta_b$, $\bar{\theta}_a = \bar{\theta}^b \varepsilon_{ba}$, $\bar{\theta}^a = \varepsilon^{ab}\bar{\theta}_b$, $\bar{\theta}_a = (\theta^a)^*$, $\bar{\theta}^a = -(\theta_a)^*$, $(\theta\theta) \equiv \theta^a\theta_a = -2\theta^1\theta^2$, $(\bar{\theta}\bar{\theta}) \equiv \bar{\theta}_a\bar{\theta}^a = (\theta\theta)^*$, $\varepsilon^{12} = 1$, $\varepsilon_{12} = 1$.

$$(\varepsilon^{ac}D_c\bar{D}^b + \varepsilon^{bc}D_c\bar{D}^a)\Phi = 0. \tag{2.7}$$

The technique of constructing $N = 4$ SUSY invariant Lagrangians is absolutely the same for both cases and therefore we shall not consider the second one separately.

The components of the superfield (2.6) transform under the $N = 4$ transformations as follows:

$$\begin{aligned}
\delta\phi^i &= \epsilon^a\bar{\psi}^i_a - \bar{\epsilon}_a\psi^{ai},\\
\delta\psi^{ai} &= \epsilon^b B^{ai}_b + \frac{i}{2}\epsilon^a\dot{\phi}^i + \epsilon^a m^i - \bar{\epsilon}^a\bar{n}^i,\\
\delta B^{ai}_b &= -\frac{i}{2}\epsilon_b\dot{\bar{\psi}}^{ai} - \frac{i}{2}\epsilon^a\dot{\bar{\psi}}^i_b - \frac{i}{2}\bar{\epsilon}^a\dot{\psi}^i_b - \frac{i}{2}\bar{\epsilon}_b\dot{\psi}^{ai}.
\end{aligned} \tag{2.8}$$

Now one can write down the most general form of the Lagrangian, invariant under the above-mentioned $N = 4$ SUSY transformations:

$$L = -8\Big(\int d^2\theta d^2\bar{\theta}(A(\Phi^i)) + \frac{1}{16}\lambda^a_{bi}B^{bi}_a\Big), \tag{2.9}$$

where $A(\Phi^i)$ is an arbitrary function of the superfield Φ^i, called the superpotential. The second term is the Fayet – Iliopoulos term and $\lambda^a_{bi} = (\sigma_I)^b_a\Lambda^I_i$ are just constants. The expression for the Lagrangian (2.9) is the most general one in the sense, that any other $N = 4$ SUSY invariant terms added will lead with necessity to the higher derivatives in the component form.

After the integration with respect to the Grassmanian coordinates θ^a and $\bar{\theta}_a$ one obtains the component form of the Lagrangian (2.9):

$$L = K - V, \tag{2.10}$$

where

$$K = \frac{1}{2}\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}\dot{\phi}^i\dot{\phi}^j + i\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}(\bar{\psi}^i_a\dot{\psi}^{aj} + \psi^{ai}\dot{\bar{\psi}}^j_a), \tag{2.11}$$

and

$$\begin{aligned}
V = {}& 2\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}(m^i m^j + n^i\bar{n}^j) + \frac{\partial^2 A}{\partial\phi^i\partial\phi^j}B^{ai}_b B^{bj}_a +\\
&+ \frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}(2\bar{\psi}^i_a\psi^{aj}m^k + \psi^{ai}\psi^j_a n^k + \bar{\psi}^i_a\bar{\psi}^{aj}\bar{n}^k) + \frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^p}(\bar{\psi}^i_a\psi^{bj} + \bar{\psi}^{ib}\psi^j_a)B^{ap}_b +\\
&+ \frac{1}{2}\frac{\partial^4 A}{\partial\phi^i\partial\phi^j\partial\phi^k\partial\phi^l}(\bar{\psi}^i_a\bar{\psi}^{ja})(\psi^{bk}\psi^l_b) - \frac{1}{2}\lambda^a_{bi}B^{bi}_a.
\end{aligned} \tag{2.12}$$

Expressing the auxiliary field B^{bi}_a in terms of the physical fields

$$B^{ai}_b = \Big(\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}\Big)^{-1}\Big(\frac{1}{4}\lambda^a_{bj} - \frac{1}{2}\frac{\partial^3 A}{\partial\phi^j\partial\phi^k\partial\phi^p}(\bar{\psi}^k_a\psi^{bp} + \bar{\psi}^{bk}\psi^p_a)\Big), \tag{2.13}$$

using it's equation of motion and inserting it back into the Lagrangian (2.10), one obtains the final form of the potential term:

$$\begin{aligned}
V = {}& \frac{1}{16}\lambda^a_{bi}\lambda^b_{aj}\Big(\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}\Big)^{-1} + 2\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}(m^i m^j + n^i\bar{n}^j) +\\
&+ \frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}(2\bar{\psi}^i_a\psi^{aj}m^k + \psi^{ai}\psi^j_a n^k + \bar{\psi}^i_a\bar{\psi}^{aj}\bar{n}^k) -\\
&- \frac{1}{2}\lambda^a_{bp}\Big(\frac{\partial^2 A}{\partial\phi^p\partial\phi^k}\Big)^{-1}\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}\bar{\psi}^i_a\psi^{bj} +\\
&+ \frac{1}{2}\frac{\partial^4 A}{\partial\phi^i\partial\phi^j\partial\phi^k\partial\phi^l}(\bar{\psi}^i_a\bar{\psi}^{ja})(\psi^{bk}\psi^l_b) - \Big(\frac{\partial^2 A}{\partial\phi^p\partial\phi^q}\Big)^{-1}\Big(\frac{\partial^3 A}{\partial\phi^i\partial\phi^k\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^j\partial\phi^l} +\\
&+ \frac{1}{2}\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^k\partial\phi^l}\Big)\bar{\psi}^j_a\bar{\psi}^k_b\psi^{bl}\psi^{ai},
\end{aligned} \tag{2.14}$$

where the identity

$$\begin{aligned}
&\Big(\frac{\partial^2 A}{\partial\phi^p\partial\phi^q}\Big)^{-1}\frac{\partial^3 A}{\partial\phi^i\partial\phi^k\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^j\partial\phi^l}\bar{\psi}^i_a\bar{\psi}^{al}\psi^{jb}\psi^k_b =\\
&= \Big(\frac{\partial^2 A}{\partial\phi^p\partial\phi^q}\Big)^{-1}\frac{\partial^3 A}{\partial\phi^i\partial\phi^k\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^j\partial\phi^l}(\bar{\psi}^i_a\psi^{aj}\bar{\psi}^l_b\psi^{bk} + \bar{\psi}^i_a\psi^{ak}\bar{\psi}^l_b\psi^{bj})
\end{aligned} \tag{2.15}$$

was used.

The formulae given above, can be rewritten in a different and more natural form, using the geometrical notations. Let us introduce the metric of some "target" manifold in the following way:

$$g_{ij} = \frac{\partial^2 A}{\partial\phi^i\partial\phi^j}, \tag{2.16}$$

along with the corresponding Christoffel connection and the Riemann curvature

$$\Gamma^i_{jk} = \frac{1}{2}\frac{\partial^3 A}{\partial\phi^p\partial\phi^j\partial\phi^k}\Big(\frac{\partial^2 A}{\partial\phi^p\partial\phi^i}\Big)^{-1}, \tag{2.17}$$

$$R_{ij,kl} = \frac{1}{4}\Big(\frac{\partial^2 A}{\partial\phi^p\partial\phi^q}\Big)^{-1}\Big(\frac{\partial^3 A}{\partial\phi^i\partial\phi^l\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^j\partial\phi^k} - \frac{\partial^3 A}{\partial\phi^i\partial\phi^k\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^j\partial\phi^l}\Big). \tag{2.18}$$

Now the Lagrangian (2.10) is rewritten in terms of these geometrical quantities as follows:

$$K = \frac{1}{2}g_{ij}\dot{\phi}^i\dot{\phi}^j + ig_{ij}(\bar{\psi}^i_a\dot{\psi}^{aj} + \psi^{ai}\dot{\bar{\psi}}^j_a), \tag{2.19}$$

and

$$\begin{aligned}
V = {}& \frac{1}{16}\lambda^a_{bi}\lambda^b_{aj}g^{ij} + 2g_{ij}(m^i m^j + n^i\bar{n}^j) +\\
&+ 4\bar{\psi}^i_a\psi^a_j D_i m^j + 2\psi^{ai}\psi_{aj}D_i n^j + 2\bar{\psi}^i_a\bar{\psi}^a_j D_i\bar{n}^j + \bar{\psi}^i_a\psi^{bj}D_i\lambda^a_{bj} +\\
&+ (D_i\Gamma_{jkl} + R_{ik,lj})\bar{\psi}^i_a\bar{\psi}^{aj}\psi^{bk}\psi^l_b + R_{jl,ki}\bar{\psi}^i_a\psi^{aj}\bar{\psi}^k_b\psi^{bl},
\end{aligned} \tag{2.20}$$

where D_i is a standard covariant derivative, defined with the help of the introduced Christoffel connection (2.17). Using Noether theorem technique one can find the classical expressions for the conserving supercharges, corresponding to the SUSY transformations (2.8), leaving the invariance of the Lagrangian:

$$\bar{Q}_a = \bar{\psi}^i_a p_i - 2i\bar{\psi}^i_a m^j \frac{\partial^2 A}{\partial\phi^i\partial\phi^j} + 2i\psi^i_a n^j \frac{\partial^2 A}{\partial\phi^i\partial\phi^j} + \\ + \frac{i}{2}\bar{\psi}^i_c\bar{\psi}^{ci}\psi^k_a \frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k} - \frac{1}{2}i\lambda^c_{ai}\bar{\psi}^i_c, \tag{2.21}$$

$$Q^b = \psi^{ib} p_i + 2i\psi^{bi} m^j \frac{\partial^2 A}{\partial\phi^i\partial\phi^j} + 2i\bar{\psi}^{bi}\bar{n}^j \frac{\partial^2 A}{\partial\phi^i\partial\phi^j} + \\ + \frac{i}{2}\bar{\psi}^{bi}\psi^{ci}\psi^k_c \frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k} + \frac{1}{2}i\lambda^b_{di}\psi^{di}. \tag{2.22}$$

These formulae for the conserved supercharges complete the classical description of the desired $N = 4$ SUSY multidimensional mechanics and now to quantize it we should analyze it's constraints.

Following the standard procedure of quantization of the system with bosonic and fermionic degrees of freedom [21], we introduce the canonical Poisson brackets:

$$\{\phi^i, p_j\} = \delta^i_j, \quad \{\psi^{ai}, p_{(\psi),bj}\} = -\delta^a_b\delta^i_j, \quad \{\bar{\psi}^i_a, p^b_{(\bar{\psi}),j}\} = -\delta^a_b\delta^i_j, \tag{2.23}$$

where $p_i, p_{(\psi),ai}$, and $p^a_{(\bar{\psi}),i}$ are the momenta, conjugated to ϕ^i, ψ^{ai} and $\bar{\psi}^i_a$. From the explicit form of the momenta:

$$p_i = g_{ij}\dot{\phi}^i, \tag{2.24}$$

$$p_{(\psi),ai} = -ig_{ij}\bar{\psi}^j_a, \quad p^a_{(\bar{\psi}),i} = -ig_{ij}\psi^{aj}, \tag{2.25}$$

with the metric g_{ij} given by (2.16), one can conclude , that the system possesses the second – class fermionic constraints:

$$\chi_{(\psi),ai} = p_{(\psi),ai} + ig_{ij}\bar{\psi}^j_a, \quad \text{and} \quad \chi^a_{(\bar{\psi}),i} = p^a_{(\bar{\psi}),i} + ig_{ij}\psi^{aj}, \tag{2.26}$$

since

$$\{\chi^a_{(\bar{\psi}),i}, \chi_{(\psi),bj}\} = -2ig_{ij}\delta^a_b. \tag{2.27}$$

Therefore, the quantization has to be done using the Dirac brackets, defined for any two functions V_a and V_b as

$$\{V_a, V_b\}_{Dirac} = \{V_a, V_b\} - \{V_a, \chi_c\}\frac{1}{\{\chi_c, \chi_d\}}\{\chi_d, V_b\}. \tag{2.28}$$

As a result we obtain the following Dirac brackets for the canonical variables:

$$\{\phi^i, p_j\}_{Dirac} = \delta^i_j, \quad \{\psi^{ai}, \bar{\psi}^j_b\}_{Dirac} = -\frac{i}{2}\delta^a_b\left(\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}\right)^{-1} = -\frac{i}{2}\delta^a_b g^{ij},$$

$$\{\psi^{ai}, p_j\}_{Dirac} = -\frac{1}{2}\psi^{ap}\frac{\partial^3 A}{\partial\phi^p\partial\phi^m\partial\phi^j}\left(\frac{\partial^2 A}{\partial\phi^m\partial\phi^i}\right)^{-1} = -\psi^{ak}\Gamma^i_{jk}, \\ \{\bar{\psi}^i_a, p_j\}_{Dirac} = -\frac{1}{2}\bar{\psi}^p_a\frac{\partial^3 A}{\partial\phi^p\partial\phi^m\partial\phi^j}\left(\frac{\partial^2 A}{\partial\phi^m\partial\phi^i}\right)^{-1} = -\bar{\psi}^k_a\Gamma^i_{jk}, \tag{2.29}$$

and, finally,

$$\{p_i, p_j\}_{Dirac} = -\frac{i}{2}\left(\frac{\partial^2 A}{\partial\phi^p\partial\phi^q}\right)^{-1}\left(\frac{\partial^3 A}{\partial\phi^i\partial\phi^k\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^j\partial\phi^l} - \frac{\partial^3 A}{\partial\phi^i\partial\phi^l\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^j\partial\phi^k}\right)\bar{\psi}^k_a\psi^{al} \\ = 2iR_{ij,kl}\bar{\psi}^k_a\psi^{al}. \tag{2.30}$$

The classical Hamiltonian, obtained after the usual Legendre transformation from the Lagrangian (2.10) has the form:

$$H_{class.} = \frac{1}{2}\left(\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}\right)^{-1} p_i p_j + V. \tag{2.31}$$

The supercharges and the Hamiltonian form the following $N = 4$ SUSY algebra with respect to the introduced Dirac brackets:

$$\{\bar{Q}_a, Q^b\}_{Dirac} = -i\delta^b_a H_{class.} - i\lambda^b_{ai}m^i, \\ \{\bar{Q}_a, \bar{Q}_b\}_{Dirac} = -i\lambda_{abi}n^i, \quad \{Q^a, Q^b\}_{Dirac} = i\lambda^{ab}_i\bar{n}^i. \tag{2.32}$$

Note the appearance of the central charges in the algebra. This fact is extremely important especially for the investigation of partial supersymmetry breaking, given in the next Section.

Replacing the Dirac brackets by (anti)commutators using the rule

$$i\{,\}_{Dirac} = \{,\}, \tag{2.33}$$

one obtains the quantum algebra:

$$\{\bar{Q}_a, Q^b\} = \delta^b_a H_{quant.} + \lambda^b_{ai}m^i, \\ \{\bar{Q}_a, \bar{Q}_b\} = \lambda_{abi}n^i, \quad \{Q^a, Q^b\} = -\lambda^{ab}_i\bar{n}^i, \tag{2.34}$$

provided the definite choice of operator ordering in the supercharges (2.21) – (2.22) and in the Hamiltonian (2.31):

$$\bar{Q}_a = \bar{\psi}^i_a R_i - 2i\bar{\psi}^i_a m^j\frac{\partial^2 A}{\partial\phi^i\partial\phi^j} + 2i\psi^i_a n^j\frac{\partial^2 A}{\partial\phi^i\partial\phi^j} - \frac{1}{2}i\lambda^c_{ai}\bar{\psi}^i_c, \tag{2.35}$$

$$Q^b = L_i\psi^{bi} + 2i\psi^{bi}m^j\frac{\partial^2 A}{\partial\phi^i\partial\phi^j} + 2i\bar{\psi}^{bi}\bar{n}^j\frac{\partial^2 A}{\partial\phi^i\partial\phi^j} + \frac{1}{2}i\lambda^b_{di}\psi^{di}, \tag{2.36}$$

$$H_{quant.} = \frac{1}{2}L_i(\frac{\partial^2 A}{\partial\phi^i\partial\phi^j})^{-1}R_j + \frac{1}{16}\lambda^a_{bi}\lambda^b_{aj}(\frac{\partial^2 A}{\partial\phi^i\partial\phi^j})^{-1} + 2\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}(m^i m^j + n^i\bar{n}^j) +$$
$$+ \frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}([\bar{\psi}^i_a\psi^{aj}]m^k + \psi^{ai}\psi^j_a n^k + \bar{\psi}^i_a\bar{\psi}^{aj}\bar{n}^k) -$$
$$- \frac{1}{4}\lambda^a_{bp}(\frac{\partial^2 A}{\partial\phi^p\partial\phi^k})^{-1}\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}[\bar{\psi}^i_a, \psi^{bj}] + \frac{1}{2}\frac{\partial^4 A}{\partial\phi^i\partial\phi^j\partial\phi^k\partial\phi^l}(\bar{\psi}^i_a\bar{\psi}^{ja})(\psi^{bk}\psi^l_b) -$$
$$- (\frac{\partial^2 A}{\partial\phi^p\partial\phi^q})^{-1}(\frac{\partial^3 A}{\partial\phi^i\partial\phi^k\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^j\partial\phi^l} +$$
$$+ \frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^k\partial\phi^l})\bar{\psi}^j_a\bar{\psi}^k_b\psi^{bl}\psi^{ai}, \tag{2.37}$$

where

$$L_i = p_i + i\bar{\psi}^j_a\psi^{ak}\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k} - \frac{i}{2}(\frac{\partial^2 A}{\partial\phi^j\partial\phi^k})^{-1}\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k},$$
$$R_i = p_i - i\bar{\psi}^j_a\psi^{ak}\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k} + \frac{i}{2}(\frac{\partial^2 A}{\partial\phi^j\partial\phi^k})^{-1}\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}. \tag{2.38}$$

The momenta operators are Hermitean with respect to the integration measure $d^D\phi\sqrt{|det(\frac{\partial^2 A}{\partial\phi^i\partial\phi^j})^{-1}|}$ if they have the following form:

$$p_i = -i\frac{\partial}{\partial\phi^i} - \frac{i}{4}\frac{\partial}{\partial\phi^i}ln(|detg_{ik}|) - 2i\omega_{i\alpha\beta}\bar{\psi}^\alpha_a\psi^{a\beta}, \tag{2.39}$$

with the new fermionic variables $\bar{\psi}^\alpha_a$ and $\psi^{a\beta}$, connected with the old ones via the tetrad e^α_i $(e^\alpha_i e^\beta_j \eta_{\alpha\beta} = g_{ij})$:

$$\bar{\psi}^\alpha_a = e^\alpha_i\bar{\psi}^i_a \quad \text{and} \quad \psi^\alpha_a = e^\alpha_i\psi^i_a, \tag{2.40}$$

and $\omega_{i\alpha\beta}$ in (2.39) is the corresponding spin connection. Therefore, the quantum supercharges (2.35) – (2.36) are mutually Hermitean conjugated and the resulting quantum Hamiltonian $H_{quant.}$ is a Hermitean self – adjoint operator as well.

As a result, the equations (2.34) – (2.40) completely describe the general formalism of $N = 4$ SUSY D – dimensional quantum mechanics and this provides the basis for the analysis of it's main properties.

III. PARTIAL SUSY BREAKING

Let us investigate in details the question of the partial supersymmetry breakdown in the framework of the constructed $N = 4$ SUSY QM in an arbitrary D number of dimensions. As it was mentioned in the Introduction, the problem of partially broken supersymmetry is very important for applications in supergravity, superstring theories and in M – theory as well, and supersymmetric quantum mechanics turns out to be an adequate mighty tool for investigating of the corresponding problems in supersymmetric field theories.

We shall see that in contrast to the one-dimensional $N = 4$ SUSY QM, the multidimensional one provides also the possibilities when either only one quarter of all supersymmetries is exact (for $D \geq 2$), or one quarter of all supersymmetries is broken (for $D \geq 3$).

In order to study partial SUSY breaking it is convenient to introduce a new set of real valued supercharges:

$$S^a = Q_a + \bar{Q}^a, \tag{3.1}$$

$$T^a = i(Q_a - \bar{Q}^a). \tag{3.2}$$

In the equations above the $SU(2)$ covariance is obviously damaged. This is the price we pay for passing to the real–valued supercharges. However, for the further discussion the loss of the covariance does not cause any problems. The label "a" has to be considered now just as the number of the supercharges, denoted as S and T.

The new supercharges form the following $N = 4$ superalgebra with the central charges:

$$\{S^a, S^b\} = H(\delta^a_b + \delta^b_a) + (\lambda^a_{bi} + \lambda^b_{ai})m^i + (\lambda_{abi}n^i - \lambda^{ab}_i\bar{n}^i), \tag{3.3}$$
$$\{T^a, T^b\} = H(\delta^a_b + \delta^b_a) + (\lambda^a_{bi} + \lambda^b_{ai})m^i - (\lambda_{abi}n^i - \lambda^{ab}_i\bar{n}^i), \tag{3.4}$$
$$\{S^a, T^b\} = i(\lambda^a_{bi} - \lambda^b_{ai})m^i + i(\lambda_{abi}n^i + \lambda^{ab}_i\bar{n}^i). \tag{3.5}$$

The algebra (3.3) - (3.5) is still nondiagonal. However, some particular choices of the constant parameters m^i, n^i and λ^b_{ai} bring the algebra to the standard form, i.e., to the form when the right hand side of (3.5) vanishes and the right hand sides of (3.3) and (3.4) are diagonal with respect to the indexes "a" and "b".

Now we consider several cases separately.

A. Four supersymmetries exact / Four supersymmetries broken

If we put equal to zero all central charges, appearing in the algebra, then no partial breakdown of supersymmetry is possible. In this case all supersymmetries are exact, if the energy of the ground state is zero; otherwise all of them are broken. This statement is obviously independent of the number of dimensions D.

B. Two supersymmetries exact

The case of the partial supersymmetry breakdown, when the half of the supersymmetries are exact, have been considered earlier [11] in the framework of one-dimensional $N = 4$ SUSY QM, but we shall describe it for completeness as well. Consider one-dimensional ($D = 1$) $N = 4$ SUSY QM and put all constants entered to the right hand sides of (3.3) - (3.5) equal to zero, except

$$m^1 \quad \text{and} \quad \Lambda^3_1. \tag{3.6}$$

Then the algebra (3.3) - (3.5) takes the form

$$
\begin{aligned}
\{S^1, S^1\} &= 2H + 2m^1\Lambda^3_1, \\
\{S^2, S^2\} &= 2H - 2m^1\Lambda^3_1, \\
\{T^1, T^1\} &= 2H + 2m^1\Lambda^3_1, \\
\{T^2, T^2\} &= 2H - 2m^1\Lambda^3_1.
\end{aligned} \tag{3.7}
$$

It means that if the energy of the ground state is equal to $m^1\Lambda^3_1$ and the last-mentioned product is positive, then S^2 and T^2 supersymmetries are exact, while the other two are broken. If $m^1\Lambda^3_1$ is negative, then S^1 and T^1 supersymmetries are exact, provided the energy of the ground state is equal to $-\, m^1\Lambda^3_1$.

C. One supersymmetry exact

The case of the three – quarters breakdown of supersymmetry is possible if the dimension of $N = 4$ SUSY QM is at least two ($D \geq 2$). Indeed, let us keep for $D = 2$ the following set of parameters nonvanished:

$$
\Lambda^3_1, \Lambda^1_2, m^1 \quad \text{and} \quad Re(n^2) \equiv N^2. \tag{3.8}
$$

Then one obtains

$$
\begin{aligned}
\{S^1, S^1\} &= 2H + 2m^1\Lambda^3_1 - 2\Lambda^1_2 N^2, \\
\{S^2, S^2\} &= 2H - 2m^1\Lambda^3_1 + 2\Lambda^1_2 N^2, \\
\{T^1, T^1\} &= 2H + 2m^1\Lambda^3_1 + 2\Lambda^1_2 N^2, \\
\{T^2, T^2\} &= 2H - 2m^1\Lambda^3_1 - 2\Lambda^1_2 N^2.
\end{aligned} \tag{3.9}
$$

Further choice

$$
m^1\Lambda^3_1 = \Lambda^1_2 N^2, \tag{3.10}
$$

leads to the case, when only T^2 supersymmetry is exact, while all others are broken if the energy of the ground state is equal to $2m^1\Lambda^3_1$, and $m^1\Lambda^3_1 > 0$. If $m^1\Lambda^3_1$ is negative, then T^1 is exact, provided the energy of the ground state is equal to $- - m^1\Lambda^3_1$.

D. Three supersymmetries exact

The situation of the one – quarter breakdown of supersymmetry can exist, if we add to the consideration one more dimension i.e., consider three dimensional $D = 3$ $N = 4$ supersymmetric quantum mechanics.

Keeping the following set of the parameters nonvanished

$$
\Lambda^3_1, \Lambda^1_2, \Lambda^2_3, m^1, N^2 \quad \text{and} \quad Im(n^3) \equiv M^3, \tag{3.11}
$$

the following algebra is obtained :

$$
\begin{aligned}
\{S^1, S^1\} &= 2H + 2m^1\Lambda^3_1 - 2\Lambda^1_2 N^2 - 2\Lambda^2_3 M^3, \\
\{S^2, S^2\} &= 2H - 2m^1\Lambda^3_1 + 2\Lambda^1_2 N^2 - 2\Lambda^2_3 M^3, \\
\{T^1, T^1\} &= 2H + 2m^1\Lambda^3_1 + 2\Lambda^1_2 N^2 + 2\Lambda^2_3 M^3, \\
\{T^2, T^2\} &= 2H - 2m^1\Lambda^3_1 - 2\Lambda^1_2 N^2 + 2\Lambda^2_3 M^3.
\end{aligned} \tag{3.12}
$$

If

$$
m^1\Lambda^3_1 = \Lambda^1_2 N^2, \tag{3.13}
$$

$$
\Lambda^1_2 N^2 = -\Lambda^2_3 M^3, \tag{3.14}
$$

and

$$
m^1\Lambda^3_1 < 0, \tag{3.15}
$$

then T^2 supersymmetry is broken, while all others are exact under the condition, that the energy of the ground state is equal to $m^1\Lambda^3_1$. If the last-mentioned product is positive, then T^2 supersymmetry is exact, while all others are broken, provided that the energy of the ground state is $3m^1\Lambda^3_1$ and we arrive to the three-dimensional generalization of the case C.

Obviously, when considering three-dimensional $N = 4$ SUSY QM one can either keep the parameters (3.8) under the condition (3.10), or the parameters (3.6), or put all of them equal to zero, and therefore obtain all particular cases of spontaneous breakdown of supersymmetry, discussed earlier. It is also obvious, that all this cases can be obtained from higher dimensional ($D \geq 3$) $N = 4$ supersymmetric quantum mechanics.

To summarize this section one should note, that according to the given general analysis of partial SUSY breaking in $N = 4$ multidimensional SUSY QM, there exist possibilities to construct the models with $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ supersymmetries unbroken, as well as models with totally broken or totally unbroken supersymmetries. However, the answer on the question which one of these possibilities can be realized for the considered system, depends crucially on the form of the chosen superpotential and on the imposed boundary conditions of the quantum mechanical problem.

IV. EXPLICIT EXAMPLE

For the better illustration of the ideas of the previous Section it is useful to consider a particular choice of the superpotential $A(\Phi^i)$. As it was mentioned before for considering all possible cases of the partial supersymmetry breakdown the minimal amount of the superfields needed is three. Therefore let us take three superfields of the type (2.6) and choose the constants $m^i, n^i, \bar{n}^i$ and λ^a_{bi} in accordance with the expressions (3.6), (3.8) and (3.11).

The simple and at the same time interesting example is the case, when the superpotential is the direct sum of terms, each being a function of only one superfield. This gives the possibility of the considerable simplification of the classical and quantum Hamiltonians and the supercharges as well [11]. We choose the explicit form of the superpotential as

$$
A(\Phi^i) = \Phi^i ln \Phi^i, \quad i = 1, 2, 3 \tag{4.1}
$$

and consider the physical bosonic components of the superfields Φ^i as the functions of the new variables x^i, namely:

$$\phi^i = (x^i)^2. \tag{4.2}$$

Making the following redefinition of the fermionic variables

$$\xi^{ai} = \psi^{ai}\sqrt{2\frac{\partial^2 A}{(\partial\phi^i)^2}}, \quad \bar{\xi}^i_a = \bar{\psi}^i_a\sqrt{2\frac{\partial^2 A}{(\partial\phi^i)^2}}, \tag{4.3}$$

where no summation over the repeated indices is assumed, one obtains the canonical commutation relations between fermions

$$\{\xi^{ai}, \bar{\xi}^j_b\} = \delta^a_b\delta^{ij}. \tag{4.4}$$

Inserting the expressions (4.1), (4.2) and (4.3) into (2.37), one obtains three-dimensional superconformal $N = 4$ quantum mechanics [18] with

$$H_{quant.} = H^1 + H^2 + H^3, \tag{4.5}$$

i.e., as it could be concluded from the fact that the superpotential is diagonal with respect to the superfields considered, the total Hamiltonian is also the direct sum of three Hamiltonians, each of them containing the bosonic and fermionic operators of only one type. The explicit form of the Hamiltonians H^i, ($i = 1, 2, 3.$) is

$$\begin{aligned} H^1 = & -\frac{1}{8}\frac{d^2}{(dx^1)^2} + \frac{1}{4}\Lambda^3_1(\sigma_3)^b_a\bar{\xi}^1_b\xi^{a1} + \frac{1}{8}(\Lambda^3_1)^2(x^1)^2 \\ & + \frac{1}{(x^1)^2}(2(m^1)^2 + \frac{3}{32} - m^1(\bar{\xi}^1_a\xi^{a1} - 1) - \frac{1}{4}\bar{\xi}^1_a\xi^{a1} + \frac{1}{8}(\bar{\xi}^1_a\xi^{a1})(\bar{\xi}^1_b\xi^{b1})), \end{aligned} \tag{4.6}$$

$$\begin{aligned} H^2 = & -\frac{1}{8}\frac{d^2}{(dx^2)^2} + \frac{1}{4}\Lambda^1_2(\sigma_1)^b_a\bar{\xi}^2_b\xi^{a2} + \frac{1}{8}(\Lambda^1_2)^2(x^2)^2 \\ & + \frac{1}{(x^2)^2}(2(N^2)^2 + \frac{3}{32} - \frac{1}{2}N^2(\xi^2_a\xi^{a2} + \bar{\xi}^2_a\bar{\xi}^{a2}) - \frac{1}{4}\bar{\xi}^2_a\xi^{a2} + \frac{1}{8}(\bar{\xi}^2_a\xi^{a2})(\bar{\xi}^2_b\xi^{b2})), \end{aligned} \tag{4.7}$$

$$\begin{aligned} H^3 = & -\frac{1}{8}\frac{d^2}{(dx^3)^2} + \frac{1}{4}\Lambda^2_3(\sigma_2)^b_a\bar{\xi}^3_b\xi^{a3} + \frac{1}{8}(\Lambda^2_3)^2(x^3)^2 \\ & + \frac{1}{(x^3)^2}(2(M^3)^2 + \frac{3}{32} - \frac{1}{2}M^3(\xi^3_a\xi^{a3} + \bar{\xi}^3_a\bar{\xi}^{a3}) - \frac{1}{4}\bar{\xi}^3_a\xi^{a3} + \frac{1}{8}(\bar{\xi}^3_a\xi^{a3})(\bar{\xi}^3_b\xi^{b3})). \end{aligned} \tag{4.8}$$

The next step is to find the energy spectrum of the quantum Hamiltonian (4.5).

Since the bosonic and fermionic variables of each type are completely separated, the eigenfunctions of the Hamiltonian (4.5) is the direct product of the eigenfunctions of the Hamiltonians (4.6) - (4.8) and the total energy is just a sum of the energies, corresponding to the Hamiltonians H^i.

Let us find the energy spectrum of the Hamiltonian H^1. Consider the general state in the "reduced" Fock space, spanned by the fermionic creation and annihilation operators $\bar{\xi}^1_a$ and ξ^{a1}, obeying the anticommutation relations (4.4) with $i = 1$:

$$|\rho\rangle = X_1(x^1)|0\rangle + Y^a_1(x^1)\bar{\xi}^1_a|0\rangle + Z_1(x^1)\bar{\xi}^1_a\bar{\xi}^{a1}|0\rangle. \tag{4.9}$$

The operator H^1 acting on the state vector (4.9) gives the following four Shrödinger equations on the unknown functions $X_1(x^1)$, $Y^a_1(x^1)$ and $Z_1(x^1)$:

$$(-\frac{1}{2}\frac{d^2}{(dx^1)^2} + \frac{1}{2}(\Lambda^3_1)^2(x^1)^2 + \frac{1}{(x^1)^2}(8(m^1)^2 + 4m^1 + \frac{3}{8}))X_1(x^1) = 4E^1_I X_1(x^1), \tag{4.10}$$

$$(-\frac{1}{2}\frac{d^2}{(dx^1)^2} + \Lambda^3_1 + \frac{1}{2}(\Lambda^3_1)^2(x^1)^2 + \frac{1}{(x^1)^2}(8(m^1)^2 - \frac{1}{8}))Y^1_1(x^1) = 4E^1_{II}Y^1_1(x^1), \tag{4.11}$$

$$(-\frac{1}{2}\frac{d^2}{(dx^1)^2} - \Lambda^3_1 + \frac{1}{2}(\Lambda^3_1)^2(x^1)^2 + \frac{1}{(x^1)^2}(8(m^1)^2 - \frac{1}{8}))X_1(x^1) = 4E^1_{III}Y^2_1(x^1). \tag{4.12}$$

$$(-\frac{1}{2}\frac{d^2}{(dx^1)^2} + \frac{1}{2}(\Lambda^3_1)^2(x^1)^2 + \frac{1}{(x^1)^2}(8(m^1)^2 - 4m^1 + \frac{3}{8}))Z_1(x^1) = 4E^1_{IV}Z_1(x^1). \tag{4.13}$$

The wave functions and the energy spectrum of the Hamiltonian of the type

$$\mathcal{H} = -\frac{1}{2}\frac{d}{dx^2} + \frac{1}{2}x^2 + g\frac{1}{x^2}, \tag{4.14}$$

have been investigated in details for the nonsupersymmetric theory in Refs. [22] - [23] and in the framework of $N = 2$ supersymmetric quantum mechanics in Refs. [23] - [26] as well. The most detailed and complete study has been done by Das and Pernice [23], where the eigenfunctions and the energy spectrum of the Hamiltonian of the type (4.14) where found after appropriate regularization of the potential and superpotential, depending on either one considers nonsupersymmetric or $N = 2$ supersymmetric problem. However, as it can be seen from (4.1) and (4.2) the superpotential in our $N = 4$ case for the Hamiltonian with $\frac{1}{x^2}$ term in the potential energy is regular in contrast to the case of $N = 2$ supersymmetric quantum mechanics and therefore we use the results of paper [23], which are obtained after the regularization of the potential, but not of the superpotential.

For the problem considered one obtains (we take the value of the parameter Λ^3_1 without loss of generality to be equal $+1$).

For $m^1 < -\frac{1}{4}$,

$$\begin{aligned} 4E^1_I &= 2k^1_I - 4m^1, \\ 4E^1_{II} &= 2k^1_{II} - 4m^1 + 2, \\ 4E^1_{III} &= 2k^1_{III} - 4m^1, \\ 4E^1_{IV} &= 2k^1_{IV} - 4m^1 + 2, \end{aligned} \tag{4.15}$$

where $k_M^A = 0, 1, 2, ...,$ $(A = 1, 2, 3, 4)$ and $(M = I, II, III, IV)$. Each energy level E_M^A corresponds to the couple (even and odd) of wave functions and therefore is doubly degenerate. The minimal energy corresponds to the minima of E_I^1 and E_{III}^1 for $k_I^1 = k_{III}^1 = 0$ and equals to $-m^1$. Let us denote the corresponding states by $\pi_I^{1\pm}$ and $\pi_{III}^{1\pm}$.

For $-\frac{1}{4} < m^1 < 0$ one has:

$$
\begin{aligned}
4E_I^1 &= 2k_I^1 + 4m^1 + 2, \\
4E_{II}^1 &= 2k_{II}^1 - 4m^1 + 2, \\
4E_{III}^1 &= 2k_{III}^1 - 4m^1, \\
4E_{IV}^1 &= 2k_{IV}^1 - 4m^1 + 2.
\end{aligned} \tag{4.16}
$$

The minimal energy corresponds to the minimum of E_{III}^1 for $k_{III}^1 = 0$ and equals to $-m^1$. We denote the corresponding ground states by $\pi_{III}^{1\pm}$.

For $0 < m^1 < \frac{1}{4}$ one has:

$$
\begin{aligned}
4E_I^1 &= 2k_I^1 + 4m^1 + 2, \\
4E_{II}^1 &= 2k_{II}^1 + 4m^1 + 2, \\
4E_{III}^1 &= 2k_{III}^1 + 4m^1, \\
4E_{IV}^1 &= 2k_{IV}^1 - 4m^1 + 2.
\end{aligned} \tag{4.17}
$$

The minimal energy is m^1 for $k_{III}^1 = 0$, the corresponding ground state is again $\pi_{III}^{1\pm}$. And finally, for $m^1 > \frac{1}{4}$:

$$
\begin{aligned}
4E_I^1 &= 2k_I^1 + 4m^1 + 2, \\
4E_{II}^1 &= 2k_{II}^1 + 4m^1 + 2, \\
4E_{III}^1 &= 2k_{III}^1 + 4m^1, \\
4E_{IV}^1 &= 2k_{IV}^1 + 4m^1,
\end{aligned} \tag{4.18}
$$

The minimal energy is m^1 for $k_{III}^1 = k_{IV}^1 = 0$, the corresponding ground states are $\pi_{III}^{1\pm}$ and $\pi_{IV}^{1\pm}$.

The points $\pm\frac{1}{4}$, and 0 are the branching points and when m^1 gets these values, then the corresponding energies and wave functions of the system in the regions of the parameter, divided by these points just coincide.

If we also choose $\Lambda_2^1 = \Lambda_3^2 = 1$, the energy spectra of the Hamiltonians H^2 and H^3 are absolutely the same as in (4.15) – (4.18). The only difference is, that the parameter m^1 should be replaced by N^2 or M^3 respectively. However, the eigenfunctions, corresponding to E_I^2 and E_{IV}^2 are the linear combinations of the states of zero and two fermionic sectors, since the fermionic number operator $\bar{\xi}_a^2 \xi^{a2}$ does not commute with the Hamiltonian H^2. The energies E_{II}^2 and E_{III}^2 are also the linear combinations of the both states of one fermionic sector, because the matrix $(\sigma_1)_a^b$ is not diagonal. The analogous situation takes place for the Hamiltonian H^3.

Now we are in a position to describe partial supersymmetry breaking following the lines of the previous Section.

First, let us consider the one dimensional case, with m^1 equal to zero. As it was mentioned above, zero value of m^1 is the branching point and therefore the energy spectra (4.16) and

(4.17), as well as the wavefunctions in these regions completely coincide. Therefore, one has the couple of supersymmetric ground states $\pi_{III}^{1\pm}$ and all supersymmetries are exact.

As it was mentioned in the previous Section, in order to describe the halfbreaking of supersymmetry it is enough to consider only the spectrum of the Hamiltonian H^1. Inserting the corresponding eigenvalues of the operator H^1 for each range of the parameter m^1 into equations (3.7), one obtains that half of supersymmetries are always broken.

Considering the spectra of the Hamiltonians H^1 and H^2 one can obtain the three – quarter breakdown of supersymmetry. Indeed, from the equations (3.9), (3.10) and (4.15) – (4.18) one can conclude, that either T^1 or T^2 supersymmetries are exact, depending on the range of parameter m^1. The corresponding ground state wavefunctions obviously are for $m^1 < -\frac{1}{4}$:

$$
\pi_I^{1\pm} \times \pi_I^{2\pm}, \quad \pi_I^{1\pm} \times \pi_{III}^{2\pm}, \quad \pi_{III}^{1\pm} \times \pi_I^{2\pm}, \quad \pi_{III}^{1\pm} \times \pi_{III}^{2\pm}, \tag{4.19}
$$

for $-\frac{1}{4} < m^1 < 0$:

$$
\pi_{III}^{1\pm} \times \pi_{III}^{2\pm}, \tag{4.20}
$$

for $0 < m^1 < \frac{1}{4}$

$$
\pi_{III}^{1\pm} \times \pi_{III}^{2\pm}, \tag{4.21}
$$

for $m^1 > \frac{1}{4}$:

$$
\pi_{III}^{1\pm} \times \pi_{III}^{2\pm}, \quad \pi_{III}^{1\pm} \times \pi_{IV}^{2\pm}, \quad \pi_{IV}^{1\pm} \times \pi_{III}^{2\pm}, \quad \pi_{IV}^{1\pm} \times \pi_{IV}^{2\pm}. \tag{4.22}
$$

In order to study the possibility of the one-quarter breakdown of supersymmetry one has to consider the three-dimensional case, i.e., the spectra and the wavefunctions of the Hamiltonians H^1, H^2 and H^3. Using the equations (3.12), (3.13) – (3.15) and (4.15) – (4.18) one can conclude, that for the considered model the one-quarter supersymmetry breakdown is impossible, since the energy of the ground state equals to $3m^1$, rather than to m^1, as it is required for the annihilation of the ground state by the operators S^1, S^2 and T^1. This obviously does not mean that one-quarter supersymmetry breakdown is impossible in principle, it means instead that this effect is impossible for the simple model we considered.

Indeed, let us consider the same three-dimensional problem, but restricting ourselves now with the nonnegative values of coordinates, i.e., $x^1 \geq 0$, $x^2 \geq 0$ and $x^3 \geq 0$.

The spectrum of $\mathcal{H}$ (4.14) when x belongs to the nonnegative half-axis is slightly different [22][2] and it opens the possibility to construct the ground state, invariant under three unbroken supersymmetries. According to Ref. [22] we have

$$
E_k^{(\pm\alpha)} = 2k \pm \alpha + 1, \tag{4.23}
$$

where α is given by

---

[2]In fact, as it was recently shown by A. Das and S. Pernice [23], the energy spectrum, obtained in [22] is correct, if one considers the problem only on the half-axis.

$$\alpha = +\frac{1}{2}\sqrt{1+8g}, \tag{4.24}$$

and k is the nonnegative integer. If $\alpha \geq 1$ then the energies $E_k^{(-\alpha)}$ must be excluded from the spectrum since the corresponding wavefunctions are no longer normalizable. Otherwise one has to consider both sets of solutions. Applying these results to the problem under consideration, and putting again $\Lambda_1^3 = \Lambda_2^1 = \Lambda_3^2 = 1$ one obtains for H^1:

$$\begin{aligned} \alpha_I^1 &= |4m^1 + 1|, \\ \alpha_{II}^1 &= |4m^1|, \\ \alpha_{III}^1 &= |4m^1|, \\ \alpha_{IV}^1 &= |4m^1 - 1|. \end{aligned} \tag{4.25}$$

And therefore the energy spectra have the form

$$\begin{aligned} 4E_I^{1,(\pm)} &= 2k_I^1 \pm |4m^1 + 1| + 1, \\ 4E_{II}^{1,(\pm)} &= 2k_{II}^1 \pm |4m^1|, \\ 4E_{III}^{1,(\pm)} &= 2k_{III}^1 \pm |4m^1| + 2, \\ 4E_{IV}^{1,(\pm)} &= 2k_{IV}^1 \pm |4m^1 - 1| + 1, \end{aligned} \tag{4.26}$$

and the same expressions for the spectra of H^2 and H^3, with m^1 substituted by N^2 and M^3 respectively. The both signs before the second terms has to be taken for E_I, if $-\frac{1}{4} \leq m^1 < 0$; for E_{II} and for E_{III}, if $0 \leq m^1 < \frac{1}{4}$; for E_{IV}, if $\frac{1}{4} \leq m^1 < \frac{1}{2}$. Let us further restrict the value of the parameter to belong to the open interval $-\frac{1}{4} < m^1 < 0$. Then due to the equations (3.13) - (3.15) and (4.26) the minimal energy of the system with $k_I^1 = k_I^2 = k_{III}^3 = 0$ is:

$$E_{min.} = E_I^{1,-} + E_I^{2,-} + E_{III}^{3,-} = -m^1, \tag{4.27}$$

and according to (3.12) we have the supersymmetric ground state with three supersymmetries being unbroken.

In this Section we have considered quite schematically the one-, two- and three-dimensional $N = 4$ supersymmetric versions of the quantum oscillator with an additional $\frac{1}{x^2}$ term in the potential energy. However we believe, that even this simple analysis gives a good illustration of all possible cases of the partial supersymmetry breakdown in multidimensional $N = 4$ SUSY QM. One should also stress the crucial meaning of the boundary conditions in the question of the partial supersymmetry breakdown, as it was shown for the case of the one-quarter supersymmetry breakdown in the considered example.

V. DISCUSSION

In this paper we have described the general formalism of multidimensional $N = 4$ supersymmetric quantum mechanics and studied the various possibilities of partial supersymmetry breaking, illustrating them by the exactly solvable example.

However, the several questions, which seem to be of a particular importance are left still opened. Indeed, it would be interesting to investigate other possibilities of changes of the

bosonic end fermionic variables, namely for the cases, when in contrast with (4.1) and (4.2), the superpotential $A(\Phi^i)$ is not a direct sum of the terms, each containing only one superfield Φ^i and when the bosonic components of these superfields depend on several variables x^i. The detailed study of this problem can lead to the possible $N = 4$ supersymmetrization and quantization of various pure bosonic integrable systems, such as n -particle Calogero and Calogero – Moser models, which are related to the RN black hole quantum mechanics and to $D = 2$ SYM theory [27]. This approach also can answer the question about the general class of the potentials, which lead to the superconformal $N = 4$ theories in higher dimensions.

Another topic which has been left uncovered in this paper is the possible application of the constructed multidimensional $N = 4$ SUSY QM to the problems of quantum cosmology. The possible embedding of pure bosonic effective Lagrangians, derived from the homogeneous cosmological models to the $N = 4$ SUSY QM can shed a new light on the old problems of boundary conditions and spontaneous SUSY breaking in quantum cosmology, which were investigated recently in the framework of $N = 2$ supersymmetric sigma – model approach [28] - [29].

All these questions are under intensive study now and will be reported elsewhere.

VI. ACKNOWLEDGMENTS

We would like to thank E. A. Ivanov for the helpful and stimulating discussions and A. V. Gladyshev and C. Sochichiu for some comments. Work of A. P. was supported in part by INTAS Grant 96-0538 and by the Russian Foundation of Basic Research, Grant 99-02-18417. Work of M. T. was supported in part by INTAS Grant 96-0308. J. J. R. would like to thank CONACyT for support under the program: Estancias Posdoctorales en el Extranjero, and Bogoliubov Laboratory of JINR for hospitality.

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Received by Publishing Department
on July 30, 1999.