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SUPERSYMMETRIC **KP** HIERARCHY IN $N = 1$ SUPERSPACE AND ITS $N = 2$ REDUCTIONS

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Суперсимметричная иерархия КП в $N = 1$ суперпространстве
и ее $N = 2$ редукции

Описан широкий класс $N = 2$ редукций суперсимметричной иерархии КП в $N = 1$ суперпространстве. Этот класс включает новое $N = 2$ суперсимметричное обобщение цепочной иерархии Тоды. Для этой иерархии построены представления Лакса для ее бозонных и фермионных потоков, локальные и нелокальные гамильтонианы, конечные и бесконечные дискретные симметрии, первые две гамильтоновы структуры и рекурсионный оператор. Обсуждена ее вторичная редукция в новую $N = 2$ суперсимметричную модифицированную иерархию КдФ.

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Lechtenfeld O., Sorin A.S. E2-99-216
Supersymmetric KP Hierarchy in $N = 1$ Superspace
and Its $N = 2$ Reductions

A wide class of $N = 2$ reductions of the supersymmetric KP hierarchy in $N = 1$ superspace is described. This class includes a new $N = 2$ supersymmetric generalization of the Toda chain hierarchy. The Lax pair representations of the bosonic and fermionic flows, local and nonlocal Hamiltonians, finite and infinite discrete symmetries, first two Hamiltonian structures and the recursion operator of this hierarchy are constructed. Its secondary reduction to new $N = 2$ supersymmetric modified KdV hierarchy is discussed.

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1 Introduction

Since the first $N = 1$ supersymmetric generalization of the bosonic KP hierarchy —the Manin-Radul $N = 1$ supersymmetric KP hierarchy [1]— and its reduction —the Manin-Radul $N = 1$ supersymmetric KdV hierarchy— appeared this subject has attracted permanent attention both for purely academic reasons and because of various applications. Let us mention, for example, the problem of finding a supersymmetric hierarchy relevant for the longstanding yet unsolved problem of constructing supermatrix models which differ non-trivially from bosonic ones. During past years several generalizations of the Manin-Radul supersymmetric KP hierarchy were proposed. They possess a new type of fermionic flows [2, 3], an enlarged number of bosonic and fermionic flows [4], or additional supersymmetries (see, e.g. the recent paper [5] and references therein). Recently, a large class of new reductions of the Manin-Radul $N = 1$ KP hierarchy was discussed in the important work [6] where bosonic and fermionic flows respecting the original algebraic structure were constructed. Even more recently [5], an $N = 4$ supersymmetric KP hierarchy was proposed and a wide class of its reductions was described in the Lax-pair framework. It is remarkable that the supersymmetric KP hierarchy in $N = 2$ superspace actually displays an $N = 4$ supersymmetry. This doubling of supersymmetry also occurs in $N = 1$ superspace where the supersymmetric KP hierarchy is actually $N = 2$ supersymmetric [4]. It is an interesting question to find different reductions of the latter hierarchy which preserve its $N = 2$ algebraic structure. This is the main goal of the present paper.

Using a dressing formalism, we describe a wide class of $N = 2$ reductions of the supersymmetric KP hierarchy in $N = 1$ superspace. One such reduction is considered in considerable detail: we derive its bosonic and fermionic flows, Hamiltonians, Hamiltonian structures, recursion operator, finite and infinite discrete symmetries and its reduction to a new $N = 2$ supersymmetric modified KdV hierarchy. As a byproduct we obtain a new version of the $N = 2$ supersymmetric Toda chain equation which is related with the infinite discrete symmetries of the reduced $N = 2$ KP hierarchy.

We point out that our construction of a new class of $N = 2$ supersymmetric integrable hierarchies in $N = 1$ superspace builds upon some recent results on supersymmetric hierarchies [4, 7, 6, 8, 5]. Let us describe the content of this paper. In section 2 we review the supersymmetric KP hierarchy in $N = 1$ superspace within the framework of the dressing approach and demonstrate that it is $N=2$ supersymmetric. In section 3 we discuss a consistent reduction of the $N=2$ supersymmetric KP hierarchy preserving its algebraic structure. We find its finite and infinite discrete symmetries and use them to obtain the new Lax operators. Finally, we construct its local and nonlocal Hamiltonians, first two Hamiltonian structures and recursion operator. Section 4 then describes its secondary reduction to a new version of the $N=2$ supersymmetric modified KdV hierarchy. Section 5 presents generalizations of the reduced $N=2$ KP hierarchy to the matrix case and closes with some open questions. An appendix contains the new version of the $N=2$ supersymmetric Toda chain equation, includes its zero-curvature representation and explains the origin of the Lax operators of section 3.

2 N=2 supersymmetric KP hierarchy

In this section we discuss the hierarchy which is usually called the Manin-Radul $N=1$ supersymmetric KP hierarchy and focus on the remarkable fact that it actually possesses an $N=2$ supersymmetry. This section is essentially based on the results obtained in [4].

Our starting object is the $N=1$ supersymmetric dressing operator W

$$W \equiv 1+\sum_{n=1}^{\infty}\,(w_n^{(b)}+w_n^{(f)}D)\,\partial^{-n}, \tag{1}$$

where the all functions $w_n \equiv w_n(Z)$ involved into the W are the $N=1$ superfields depending on coordinates $Z\equiv(z,\theta)$. Further, D is the fermionic covariant derivative which, together with the supersymmetry generator Q, form the algebra[1]

$$\{D,D\}=+2\partial, \quad \{Q,Q\}=-2\partial \tag{2}$$

with the standard superspace realization:

$$D\equiv\frac{\partial}{\partial\theta}+\theta\partial, \quad Q\equiv\frac{\partial}{\partial\theta}-\theta\partial. \tag{3}$$

Our aim now is to construct a maximal set of consistent Sato equations for the dressing operator W which represent the flows of the extended supersymmetric KP hierarchy in $N=1$ superspace.

[1]We explicitly present only non-zero brackets in this paper.

We begin by choosing a basis $\{D,\ Q,\ [\theta,D],\ \partial\}$ of first-order differential operators which are point-wise (z) linearly independent. We then consider arbitrary powers of these and dress them by the dressing operator W, obtaining the operators

$$L_l\equiv WD^lW^{-1}, \quad M_l\equiv WQ^lW^{-1}, \quad N_l\equiv W\frac{1}{2}[\theta,\ D]\partial^lW^{-1} \tag{4}$$

with the obvious properties:

$$L_l\equiv(L_1)^l, \quad M_l\equiv(M_1)^l, \quad L_{2l}\equiv(-1)^lM_{2l}=W\partial^lW^{-1}. \tag{5}$$

Using the operators (4) we construct consistent Sato equations for W,

$$\begin{aligned} &\tfrac{\partial}{\partial t_l}W=-(L_{2l})_-W, && U_lW=-(N_l)_-W,\\ &D_lW=-(L_{2l-1})_-W, && Q_lW=-(M_{2l-1})_-W, \end{aligned} \tag{6}$$

where the subscript $-$ ($+$) denotes the purely pseudo-differential (differential) part of an operator. The bosonic (fermionic) evolution derivatives $\{\frac{\partial}{\partial t_l},\ U_l\}$ ($\{D_l,\ Q_l\}$) generating bosonic (fermionic) flows of the hierarchy under consideration have the following length dimensions:

$$[\tfrac{\partial}{\partial t_l}]=[U_l]=-l, \quad [D_l]=[Q_l]=-l+\frac{1}{2}. \tag{7}$$

We would like to recall that the subset of flows $\{\frac{\partial}{\partial t_l},\ D_l\}$ by itself forms a hierarchy which is usually called the Manin-Radul $N=1$ supersymmetric KP hierarchy [1]. The extra flows $\{U_l,\ Q_l\}$, when added to the $N=1$ KP hierarchy, produce an extended hierarchy possessing a richer algebraic structure. This extended supersymmetric hierarchy was called the maximal SKP hierarchy in [4].

In order to calculate the flow algebra of the extended hierarchy, one can use a supersymmetric generalization [9, 4] of the Radul map [10] which is a homomorphism between the flow algebra we are looking for and the algebra of the operators L_l, M_l and N_l (4). The resulting nonzero brackets are

$$\Big\{D_k\,,\,D_l\Big\}=-2\,\frac{\partial}{\partial t_{k+l-1}}, \quad \Big\{Q_k\,,\,Q_l\Big\}=+2\,\frac{\partial}{\partial t_{k+l-1}}, \tag{8}$$

$$\Big[U_k\,,\,D_l\Big]=Q_{k+l}, \quad \Big[U_k\,,\,Q_l\Big]=D_{k+l}. \tag{9}$$

The algebra (8–9) may be realized in the superspace $\{t_k, \theta_k, \rho_k, h_k\}$,

$$D_k = \frac{\partial}{\partial\theta_k} - \sum_{l=1}^{\infty}\theta_l\frac{\partial}{\partial t_{k+l-1}}, \quad Q_k = \frac{\partial}{\partial\rho_k} + \sum_{l=1}^{\infty}\rho_l\frac{\partial}{\partial t_{k+l-1}},$$
$$U_k = \frac{\partial}{\partial h_k} - \sum_{l=1}^{\infty}(\theta_l\frac{\partial}{\partial\rho_{k+l}} + \rho_l\frac{\partial}{\partial\theta_{k+l}}), \tag{10}$$

where t_k, h_k (θ_k, ρ_k) are bosonic (fermionic) abelian evolution times with length dimensions

$$[t_k] = [h_k] = k, \quad [\theta_k] = [\rho_k] = k - \frac{1}{2}. \tag{11}$$

A simple inspection of the superalgebra (8–9) shows that the flows $\frac{\partial}{\partial t_1}$, U_0, D_1 and Q_1 form a finite-dimensional subalgebra which is isomorphic to the well-known $N = 2$ supersymmetry algebra including its $u(1)$ automorphism. For this reason the maximal SKP hierarchy may also be called the $N = 2$ supersymmetric KP hierarchy.

It is instructive to introduce a new basis,

$$\{D, Q, D_l, Q_l\} \Longrightarrow \{\mathcal{D}, \overline{\mathcal{D}}, \mathcal{D}_l, \overline{\mathcal{D}}_l\}, \tag{12}$$

$$\mathcal{D} \equiv \frac{1}{\sqrt{2}}(Q + D), \qquad \overline{\mathcal{D}} \equiv \frac{1}{\sqrt{2}}(Q - D),$$
$$\mathcal{D}_k \equiv \frac{1}{\sqrt{2}}((-1)^{l-1}Q_k + D_k), \qquad \overline{\mathcal{D}}_k \equiv \frac{1}{\sqrt{2}}((-1)^{l-1}Q_k - D_k) \tag{13}$$

in which the algebras (2) and (8–9) read

$$\{\mathcal{D}, \overline{\mathcal{D}}\} = -2\partial, \qquad \left\{\mathcal{D}_k\,, \overline{\mathcal{D}}_l\right\} = +2\,\frac{\partial}{\partial t_{k+l-1}},$$
$$\left[U_k\,, \mathcal{D}_l\right] = \mathcal{D}_{k+l}, \quad \left[U_k\,, \overline{\mathcal{D}}_l\right] = -\overline{\mathcal{D}}_{k+l}. \tag{14}$$

In this basis the flows (6) take the form

$$\frac{\partial}{\partial t_l}W = -(W\partial^l W^{-1})_- W, \qquad U_l W = -(W\theta\frac{\partial}{\partial\theta}\partial^l W^{-1})_- W,$$
$$\mathcal{D}_l W = -(W\mathcal{D}\partial^{l-1}W^{-1})_- W, \quad \overline{\mathcal{D}}_l W = -(W\overline{\mathcal{D}}\partial^{l-1}W^{-1})_- W, \tag{15}$$

and one can easily recognize that the subflows $\{\frac{\partial}{\partial t_l}, \mathcal{D}_l\}$ form the Mulase-Rabin $N = 1$ supersymmetric KP hierarchy [2, 3]. As we have already seen earlier, the maximal SKP hierarchy includes the Manin-Radul $N = 1$ supersymmetric hierarchy as well. Therefore, we come to the conclusion that it actually comprises both the Manin-Radul and Mulase-Rabin $N = 1$ supersymmetric KP hierarchies.

It is easy to see that the flows (15) are form–invariant with respect to the $U(1)$ automorphism transformation of the $N = 2$ supersymmetry algebra (14),

$$\begin{aligned}
(\frac{\partial}{\partial t_l}, \quad U_l) &\Longrightarrow & (\frac{\partial}{\partial t_l}, \quad U_l), \\
(\mathcal{D}, \quad \mathcal{D}_l) &\Longrightarrow & \exp{(+i\phi)}\ (\mathcal{D}, \quad \mathcal{D}_l), \\
(\overline{\mathcal{D}}, \quad \overline{\mathcal{D}}_l) &\Longrightarrow & \exp{(-i\phi)}\ (\overline{\mathcal{D}}, \quad \overline{\mathcal{D}}_l),
\end{aligned} \tag{16}$$

where ϕ is an arbitrary parameter. Nevertheless, it is a very non-trivial task to find a realization of these transformations for the superfunctions w_n^f and w_n^b involved in the dressing operator W (1). We will return to discuss this point for the case of the reduced $N = 2$ KP hierarchy (see paragraph after eqs. (76)). Let us finally emphasize that the $U(1)$ covariance of the flows was hidden in the basis (3), (10), while it becomes manifest in the new basis (13).

3 Reduction of the N=2 KP hierarchy

3.1 Bosonic and fermionic flows

In this subsection we consider a reduction of the $N = 2$ supersymmetric KP hierarchy which preserves its flow algebra (8–9).

Let us introduce the following constraint on the operator M_1 (4):

$$M_1 = \mathcal{M} \equiv Q + vD^{-1}u \tag{17}$$

(its nature is explained in the appendix). The operator $\mathcal{M}$ possesses the following important property[2]:

$$(\mathcal{M}^l)_- = \sum_{k=0}^{l-1}(\mathcal{M}^{l-k-1}v)D_+^{-1}((\mathcal{M}^k)^T u), \quad l = 0,1,2\ldots \quad , \tag{18}$$

which can be proved by induction similarly to analogous formula in the bosonic case [11]. Equation (18) coincides with formula of ref. [6] where other reductions of the Manin-Radul supersymmetric hierarchy [1] were discussed.

Substituting the expression (4) for M_1 in terms of the dressing operator W (1) into the constraint (17), the latter becomes

$$WQW^{-1} = Q + vD^{-1}u \tag{19}$$

and gives an equation for W which can be solved iteratively. The unique solution $\mathcal{W}$ (1) is determined by

$$\begin{aligned} w_1^{(f)} &\equiv -Q^{-1}(uv), \quad w_1^{(b)} \equiv -Q^{-1}(vDu + uvQ^{-1}(uv)), \\ w_2^{(f)} &\equiv -Q^{-1}(vu\,' + uvDQ^{-1}(uv)) \\ &\quad +(Q^{-1}(uv))Q^{-1}(vDu + uvQ^{-1}(uv)), \quad \ldots . \end{aligned} \tag{20}$$

Replacing W by $\mathcal{W}$ in eqs. (4) one can obtain the reduced operators $\mathcal{L}_l$ and $\mathcal{N}_l$ as well. As an example, we present a few terms of the D^{-1} expansion of $\mathcal{L}_1$,

$$\begin{aligned} \mathcal{L}_1 \equiv \mathcal{W}D\mathcal{W}^{-1} = D &+ 2w_1^{(f)} - (Dw_1^{(f)})D^{-1} - ((Dw_1^{(b)}) - 2w_2^{(f)} \\ &+ w_1^{(f)}Dw_1^{(f)} + 2w_1^{(f)}w_1^{(b)})D^{-2} \\ &- ((D(w_2^{(f)} - w_1^{(f)}w_1^{(b)})) - (Dw_1^{(f)})^2)D^{-3} + \ldots, \end{aligned} \tag{21}$$

where the functions $w_n^{(b)}$ and $w_n^{(f)}$ are defined in eqs. (20).

The most complicated task now is to construct a consistent set of Sato equations for the reduced $\mathcal{W}$, generalizing the unreduced equations (6) and preserving their algebraic structure (8–9). Recently, a similar task

[2]Let us recall the operator conjugation rules: $D^T = -D$, $(OP)^T = (-1)^{d_O d_P} P^T O^T$, where O (P) is an arbitrary operator with the Grassmann parity d_O (d_P), and $d_O = 0$ ($d_O = 1$) for bosonic (fermionic) operators O. All other rules can be derived using these. Hereafter, we use the notation (Of) for an operator O acting only on a function f inside the brackets.

was carried out in [6] for some reductions of the Manin-Radul $N = 1$ supersymmetric KP hierarchy [1] as well as in [5] for the reduced $N = 4$ KP hierarchy, and we essentially use the ideas developed there. We succeeded in this construction only for the reduced $\frac{\partial}{\partial t_l}$, D_l, and Q_l flows. Nevertheless, as will be clear in what follows, the remaining U_l flows can be restored using the zero flow U_0 (28) and the recurrence relations (71) (see the paragraph after eqs. (74)).

The resulting Sato equations have the following form:

$$\begin{aligned} \frac{\partial}{\partial t_l}\mathcal{W} &= -(\mathcal{L}_{2l})_-\mathcal{W}, \quad D_l\mathcal{W} = -(\mathcal{L}_{2l-1})_-\mathcal{W}, \\ Q_l\mathcal{W} &= -((\mathcal{M}_{2l-1})_- - \widetilde{\mathcal{M}}_{2l-1})\mathcal{W}, \end{aligned} \tag{22}$$

where a new operator $\widetilde{\mathcal{M}}_{2l-1}$ has been introduced,

$$\widetilde{\mathcal{M}}_{2l-1} \equiv 2\sum_{k=0}^{l-2}(\mathcal{M}^{2(l-k)-3}v)D^{-1}((\mathcal{M}^{2k+1})^T u), \tag{23}$$

which is necessary for the consistency of the equations. The flows can easily be rewritten in Lax-pair form,

$$\begin{aligned} \frac{\partial}{\partial t_l}\mathcal{M} &= -[(\mathcal{L}_{2l})_-, \mathcal{M}] = [(\mathcal{L}_{2l})_+, \mathcal{M}], \\ D_l\mathcal{M} &= -\{(\mathcal{L}_{2l-1})_-, \mathcal{M}\} = \{(\mathcal{L}_{2l-1})_+, \mathcal{M}\}, \\ Q_l\mathcal{M} &= -\{(\mathcal{M}_{2l-1})_- - \widetilde{\mathcal{M}}_{2l-1}, \mathcal{M}\} \\ &= \{(\mathcal{M}_{2l-1})_+ + \widetilde{\mathcal{M}}_{2l-1}, \mathcal{M}\} - 2\mathcal{M}^{2l}, \end{aligned} \tag{24}$$

and, with the help of equation (18), lead to the following flow equations for the superfields v and u:

$$\begin{aligned} \frac{\partial}{\partial t_l}v &= ((\mathcal{L}_{2l})_+v), \qquad \frac{\partial}{\partial t_l}u = -((\mathcal{L}_{2l})_+^T u), \\ D_l v &= ((\mathcal{L}_{2l-1})_+v), \quad D_l u = -((\mathcal{L}_{2l-1})_+^T u), \\ Q_l v &= (((\mathcal{M}_{2l-1})_+ + \widetilde{\mathcal{M}}_{2l-1} - 2\mathcal{M}_{2l-1})v), \\ Q_l u &= -(((\mathcal{M}_{2l-1})_+^T + (\widetilde{\mathcal{M}}_{2l-1} - 2\mathcal{M}_{2l-1})^T)u). \end{aligned} \tag{25}$$

Using eqs. (25) for the bosonic and fermionic flows, we present the first few of them, for illustration[3],

$$\frac{\partial}{\partial t_0}\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} +v \\ -u \end{pmatrix}, \quad \frac{\partial}{\partial t_1}\begin{pmatrix} v \\ u \end{pmatrix} = \partial\begin{pmatrix} v \\ u \end{pmatrix},$$

[3]We have rescaled some evolution derivatives to simplify the presentation of some formulae.

$$\frac{\partial}{\partial t_2} v = +v\,'' - 2uv(DQv) + (DQv^2u) + v^2(DQu) - 2v(uv)^2,$$
$$\frac{\partial}{\partial t_2} u = -u\,'' - 2uv(DQu) + (DQu^2v) + u^2(DQv) + 2u(uv)^2, \quad (26)$$

$$\begin{aligned}
D_1 v &= -Dv + 2vQ^{-1}(uv), \quad D_1 u = -Du - 2uQ^{-1}(uv), \\
Q_1 v &= -Qv - 2vD^{-1}(uv), \quad Q_1 u = -Qu + 2uD^{-1}(uv), \\
D_2 v &= -Dv\,' + 2v\,'Q^{-1}(uv) + (Dv)Q^{-1}D(uv) + vQ^{-1}[uv\,' + (Dv)(Du)], \\
D_2 u &= +Du\,' + 2u\,'Q^{-1}(uv) + (Du)Q^{-1}D(uv) + uQ^{-1}[vu\,' + (Du)(Dv)], \\
Q_2 v &= -Qv\,' - 2v\,'D^{-1}(uv) + (Qv)D^{-1}Q(uv) - vD^{-1}[uv\,' - (Qv)(Qu)], \\
Q_2 u &= +Qu\,' - 2u\,'D^{-1}(uv) + (Qu)D^{-1}Q(uv) - uD^{-1}[vu\,' - (Qu)(Qv)],
\end{aligned} \quad (27)$$

$$U_0 \begin{pmatrix} v \\ u \end{pmatrix} = \theta D \begin{pmatrix} v \\ u \end{pmatrix}, \quad (28)$$

where the zero flow U_0 was constructed by hand so that it satisfies the algebra (8–9).

We would like to close this subsection with a few remarks.

First, observing eqs. (25) we learn that the reduced $N = 2$ KP flows (except for $\frac{\partial}{\partial t_l}$) are nonlocal in general (for an example, see eqs. (27)). This property of flows is just the result of the reduction.

Second, the flows $\{\frac{\partial}{\partial t_1}, U_0, D_1, Q_1\}$ forming the $N = 2$ supersymmetry algebra are non-locally and non-linearly realized in terms of the initial superfields v and u. However, there exists another superfield basis $\{\widehat{v}, \widehat{u}\}$, defined as

$$\widehat{v} \equiv v \exp\{+[\theta, D^{-1}](uv)\}, \quad \widehat{u} \equiv u \exp\{-[\theta, D^{-1}](uv)\}, \quad (29)$$

which localizes and linearizes the $N = 2$ supersymmetry realization into

$$\frac{\partial}{\partial t_1} = \partial, \quad D_1 = -D, \quad Q_1 = Q, \quad U_0 = \theta D. \quad (30)$$

However, in this basis even the flows $\frac{\partial}{\partial t_l}$ for $l \geq 2$ are nonlocal.

Let us finally stress that, in distinction to all known Lax operators used before in the supersymmetric literature, the Lax operators proposed in this subsection do not respect supersymmetry because they contain both the fermionic covariant derivative D and the supersymmetry generator Q. Nevertheless, the resulting hierarchy is $N = 2$ supersymmetric.

3.2 Discrete symmetries, Darboux-Bäcklund transformations and solutions

In this subsection we discuss finite and infinite discrete symmetries of the reduced hierarchy, and use them to construct its solutions and new Lax operators.

Direct verification shows that the flows (26–28) admit the two involutions:

$$\begin{aligned}
&(v, u)^* = i(u, v), \quad (z, \theta)^* = (z, \theta), \\
&(t_p, U_p, D_p, Q_p)^* = (-1)^{p-1}(t_p, -U_p, D_p, Q_p),
\end{aligned} \quad (31)$$

$$\begin{aligned}
&(v, u)^\dagger = (u, v), \quad (z, \theta)^\dagger = (-z, \theta), \\
&(t_p, U_p, D_p, Q_p)^\dagger = (-t_p, U_p, Q_p, D_p),
\end{aligned} \quad (32)$$

which are consistent with their algebra (2), (8–9). A third involution can easily be derived by multiplying these two.

It is a simple exercise to check that all the flows (24) (or (25)) also possess the involution (31), using the following involution property of the dressing operator $\mathcal{W}$:

$$\mathcal{W}^* = (\mathcal{W}^{-1})^T \quad (33)$$

which results from eq. (19) and its consequences

$$\begin{aligned}
&(\mathcal{L}_l)^* = (-1)^{\frac{l(l+1)}{2}}(\mathcal{L}_l)^T, \quad (\mathcal{M}_l)^* = (-1)^{\frac{l(l+1)}{2}}(\mathcal{M}_l)^T, \\
&(\widetilde{\mathcal{M}}_{2l-1})^* = (-1)^l(\widetilde{\mathcal{M}}_{2l-1})^T,
\end{aligned} \quad (34)$$

for the operators entering eqs. (24). As regards to the involution (32), we do not have a direct proof for it due to the very complicated transformation property of the dressing operator. However, a simple proof can be given using the recurrence relations (71), to be derived later.

Besides the involutions (31–32) the flows (26–28) possess an infinite-dimensional group of discrete Darboux transformations (see eq. (A.14) of appendix)

$$\begin{aligned}
&(v,\ u)^\ddagger = (\ v(QD\ln v - uv),\ \frac{1}{v}\ ), \\
&(z, \theta)^\ddagger = (z, \theta), \quad (t_p, U_p, D_p, Q_p)^\ddagger = (t_p, U_p, -D_p, -Q_p),
\end{aligned} \quad (35)$$

$$\mathcal{M}^{\ddagger} = -\mathcal{T}\mathcal{M}\mathcal{T}^{-1}, \quad \mathcal{T} \equiv vDv^{-1}. \tag{36}$$

Let us remark that formula (36) represents the Darboux-Bäcklund transformation[4] of the Lax operator $\mathcal{M}$ (17).

Applying involutions (31–32) and the discrete group (35) to the Lax operator $\mathcal{M}$ (17) one can derive other consistent Lax operators

$$\mathcal{M}^{*} = Q - uD^{-1}v \equiv -\mathcal{M}^{T}, \qquad \mathcal{M}^{\dagger} = D + uQ^{-1}v, \tag{37}$$

$$\begin{aligned} &\mathcal{M}^{((j+1)\ddagger)} = D_{-} + v^{((j+1)\ddagger)} D_{+}^{-1} u^{((j+1)\ddagger)} \equiv -\mathcal{T}^{(j\ddagger)} \mathcal{M}^{(j\ddagger)} \mathcal{T}^{(j\ddagger)^{-1}}, \\ &\mathcal{T}^{(j\ddagger)} \equiv v^{(j\ddagger)} D_{+} v^{(j\ddagger)^{-1}} \end{aligned} \tag{38}$$

which generate isomorphic flows. $\mathcal{M}^{(j\ddagger)}$ is obtained from $\mathcal{M}$ by applying j times the discrete transformation (36), e.g. $\mathcal{M}^{(3\ddagger)} \equiv ((\mathcal{M}^{\ddagger})^{\ddagger})^{\ddagger}$, $\mathcal{M}^{(0\ddagger)} \equiv \mathcal{M}$.

Generalizing results obtained in [8, 5] one can construct an infinite class of solutions for the reduced hierarchy under consideration. We briefly present this construction and refer to [8, 5] for details.

The simplest solution of the hierarchy corresponds to

$$u = 0, \tag{39}$$

in which case the bosonic and fermionic flows for the remaining superfield $v \equiv -\tau_0$ are linear and have the following form:

$$\frac{\partial}{\partial t_k}\tau_0 = \partial^k \tau_0, \quad D_k\tau_0 = -D\partial^{k-1}\tau_0, \quad Q_k\tau_0 = Q\partial^{k-1}\tau_0. \quad U_k\tau_0 = \theta D\partial^k\tau_0. \tag{40}$$

To derive these equations it suffices to take into account the length dimensions (7) of the evolution derivatives, their algebra (8–9) and the invariance of all flows (24) with respect to the $U(1)$ transformations

$$(\ v, u\ ) \quad \Longrightarrow \quad (\ \exp(+i\beta)\ v, \ \exp(-i\beta)\ u\ ) \tag{41}$$

which is obvious due to the invariance of the reduction constraint (19).

[4] For the reduced Manin-Radul $N=1$ supersymmetric KP hierarchy the Darboux-Bäcklund transformations were discussed in [6] (see also references therein).

For technical reasons, we restrict the analysis of the hierarchy to the case when only the flows $\frac{\partial}{\partial t_k}$, D_k and Q_k (but not U_k) are considered. Then, using the realization (10), the solution of eqs. (40) is

$$\begin{aligned} \tau_0 \ = \ & \int d\lambda\, d\eta\, \varphi(\lambda,\eta)\ \exp\Big\{x\lambda - \eta\theta + \sum_{k=1}^{\infty}\Big[t_k + \eta(\theta_k - \rho_k)\lambda^{-1} + \theta(\theta_k+\rho_k) \\ & - \ \theta_k \sum_{n=1}^{\infty} \rho_n \lambda^{n-1}\Big]\lambda^k\Big\}, \end{aligned} \tag{42}$$

where φ is an arbitrary fermionic function of the bosonic (λ) and fermionic (η) spectral parameters with length dimensions

$$[\lambda] = -1, \qquad [\eta] = -\frac{1}{2}. \tag{43}$$

Applying the discrete group (35) to the solution constructed $\{u = 0,\ v = -\tau_0\}$, an infinite class of new solutions of the hierarchy is generated through an obvious iterative procedure [8]

$$\begin{aligned} v^{((2j+1)\ddagger)} &= +(-1)^j \frac{\tau_{2j}}{\tau_{2j+1}}, & v^{(2(j+1)\ddagger)} &= (-1)^j \frac{\tau_{2(j+1)}}{\tau_{2j+1}}, \\ u^{((2j+1)\ddagger)} &= -(-1)^j \frac{\tau_{2j-1}}{\tau_{2j}}, & u^{(2(j+1)\ddagger)} &= (-1)^j \frac{\tau_{2j+1}}{\tau_{2j}}, \end{aligned} \tag{44}$$

where the τ_j are[5]

$$\begin{aligned} \tau_{2j} \ &= \ \mathrm{sdet}\begin{pmatrix} (-1)^q\partial^{p+q}\tau_0 & (-1)^m\partial^{p+m}Q\tau_0 \\ (-1)^q\partial^{k+q}D\tau_0 & (-1)^m\partial^{k+m}DQ\tau_0 \end{pmatrix}^{0\le p,q\le j}_{0\le k,m\le j-1}, \\ \tau_{2j+1} \ &= \ \mathrm{sdet}\begin{pmatrix} (-1)^q\partial^{p+q}\tau_0 & (-1)^m\partial^{p+m}Q\tau_0 \\ (-1)^q\partial^{k+q}D\tau_0 & (-1)^m\partial^{k+m}DQ\tau_0 \end{pmatrix}^{0\le p,q\le j}_{0\le k,m\le j}. \end{aligned} \tag{45}$$

3.3 Hamiltonian structure

In this subsection we construct local and nonlocal Hamiltonians, first two Hamiltonian structures and the recursion operator of the reduced hierarchy.

[5] The superdeterminant is defined as $\mathrm{sdet}\begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \det(A - BD^{-1}C)(\det D)^{-1}$.

Let us first present our notations for the $N=1$ superspace measure and delta function

$$dZ \equiv dzd\theta, \quad \delta^{N=1}(Z) \equiv \theta\delta(z), \tag{46}$$

as well as the realization of the inverse derivatives

$$D^{-1} \equiv D\partial_z^{-1}, \quad Q^{-1} \equiv -Q\partial_z^{-1}, \quad \partial_z^{-1} \equiv \frac{1}{2}\int_{-\infty}^{+\infty} dx\epsilon(z-x),$$
$$\epsilon(z-x) = -\epsilon(x-z) \equiv 1, \quad if \quad z > x \tag{47}$$

which we use in what follows. We also use the correspondence:

$$\frac{\partial}{\partial\tau_l^a} \equiv \{\frac{\partial}{\partial t_l}, U_l, D_l, Q_l\} \quad \Leftrightarrow \quad \mathcal{H}_l^a \equiv \{\mathcal{H}_l^t, \mathcal{H}_l^U, \mathcal{H}_l^D, \mathcal{H}_l^Q\} \tag{48}$$

between the evolution derivatives $\frac{\partial}{\partial\tau_l^a}$ and Hamiltonian densities $\mathcal{H}_l^a$, and, consequently, the latter ones have the length dimensions

$$[\mathcal{H}_l^t] = [\mathcal{H}_l^U] = -l, \quad [\mathcal{H}_l^D] = [\mathcal{H}_l^Q] = -l + \frac{1}{2}. \tag{49}$$

Let us remark that the length dimensions of the Hamiltonian H_l^a,

$$H_l^a \equiv \int dZ\mathcal{H}_l^a, \tag{50}$$

and its density $\mathcal{H}_l^a$ are different. They are related as

$$[H_l^a] = [\mathcal{H}_l^a] + \frac{1}{2} \tag{51}$$

because the length dimension of the $N=1$ superspace measure is not equal to zero, $[dZ] = \frac{1}{2}$. Moreover, their Grassmann parities, d_{H^a} and $d_{\mathcal{H}^a}$, are opposite,

$$d_{H^a} = d_{\mathcal{H}^a} + 1, \tag{52}$$

due to the fermionic nature of the $N=1$ superspace measure (46).

We define the residue of a pseudo-differential operator Ψ with respect to the fermionic covariant derivative D_+ according to the rule:

$$\Psi \equiv ... + (Qres(\Psi))D^{-1} + ... \tag{53}$$

which will be justified a posteriori. Then, bosonic Hamiltonian densities[6] can be defined as

$$\mathcal{H}_l^t \equiv res(\mathcal{L}_{2l}). \tag{54}$$

Using these formulae and the relation (18) one can derive the general formulae for the Hamiltonians H_l^t in terms of the Lax operator $\mathcal{M}$ (17)

$$H_{l-1}^t = \int dZ D^{-1}\sum_{k=0}^{2l-3}(-1)^k(\mathcal{M}^{2l-3-k}v)((\mathcal{M}^k)^T u). \tag{55}$$

We present, for example, the explicit expressions for the first few bosonic Hamiltonians[7],

$$H_1^t = \int dZuv, \quad H_2^t = \int dZuv\,',$$
$$H_3^t = \int dZ\ [\ uv\,'' - vu[u(DQv) - v(DQu)] - \frac{2}{3}(uv)^3\ ], \tag{56}$$

$$H_0^U = \int dZ\ u\theta Dv, \tag{57}$$

and the first few fermionic ones,

$$H_1^D = H_1^Q = \int dZD^{-1}(uv), \tag{58}$$

$$H_2^D = \int dZ\ [\ vDu + uvQ^{-1}(uv)\ ], \quad H_2^Q = \int dZ\ [\ vQu - uvD^{-1}(uv)\ ],$$
$$H_3^D = \int dZ\ [\ vDu\,' + 2vu\,'Q^{-1}(uv) + v(Du)[\theta, D](uv)\ ],$$
$$H_3^Q = \int dZ\ [\ vQu\,' + 2vu\,'D^{-1}(uv) + v(Qu)[\theta, D](uv)\ ]. \tag{59}$$

The Hamiltonians (57–59) were found manually by requiring that they are conserved with respect to the flows $\frac{\partial}{\partial t_l}$ (26).

[6]Let us recall that Hamiltonian densities are defined up to terms which are fermionic or bosonic total derivatives of an arbitrary functional $f(Z)$ of the initial superfields subjected to the constraint: $f(+\infty,\theta) - f(-\infty,\theta) = 0$.

[7]When deriving eqs. (55-59) we integrated by parts and made essential use of realizations (47) for the inverse derivatives and of the relationship $Q \equiv D - 2\theta\partial$. We also used the following definition of the superspace integral: $\int dZ f(Z) \equiv \int dz(Df)(z,0)$.

We should add that the Hamiltonians in eqs. (54) for $l \geq 4$ are only conjectured to be conserved under the bosonic flows $\frac{\partial}{\partial t_l}$ (24). This conjecture was checked explicitly for a few of them.

It is well known that a bi-Hamiltonian system of evolution equations can be represented as:

$$\frac{\partial}{\partial \tau_l^a}\begin{pmatrix} v \\ u \end{pmatrix} = J_1 \begin{pmatrix} \delta/\delta v \\ \delta/\delta u \end{pmatrix} H_{l+1}^a = J_2 \begin{pmatrix} \delta/\delta v \\ \delta/\delta u \end{pmatrix} H_l^a, \tag{60}$$

where J_1 and J_2 are the first and second Hamiltonian structures. In terms of these the Poisson brackets of the superfields v and u are given by the formula:

$$\{\begin{pmatrix} v(Z_1) \\ u(Z_1) \end{pmatrix} \overset{\otimes}{,} \begin{pmatrix} v(Z_2), u(Z_2) \end{pmatrix}\}_l = J_l(Z_1)\delta^{N=1}(Z_1 - Z_2). \tag{61}$$

An important remark is in order. In $N = 1$ superspace the variational derivatives $\frac{\delta}{\delta v}$ and $\frac{\delta}{\delta u}$ are Grassmann odd because of the definition $\frac{\delta}{\delta v(Z_1)}v(Z_2) \equiv \delta^{N=1}(Z_1 - Z_2)$ and the fermionic nature of the $N = 1$ delta function (46).

Using flows (26–28) and Hamiltonians (56–59), we have found the Hamiltonian structures to be

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad and \quad J_2 = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \tag{62}$$

with

$$\begin{aligned}
J_{11} &\equiv +vD^{-1}vQ - (Qv)D^{-1}v - 2vD^{-1}uvD^{-1}v \\
&\quad +vQ^{-1}vD - (Dv)Q^{-1}v + 2vQ^{-1}uvQ^{-1}v, \\
J_{22} &\equiv -uQ^{-1}uD + (Du)Q^{-1}u + 2uQ^{-1}uvQ^{-1}u \\
&\quad -uD^{-1}uQ + (Qu)D^{-1}u - 2uD^{-1}uvD^{-1}u, \\
J_{12} &\equiv -\partial + \{Q, vD^{-1}u\} + 2vD^{-1}uvD^{-1}u + \{D, vQ^{-1}u\} - 2vQ^{-1}uvQ^{-1}u, \\
J_{21} &\equiv -\partial - \{D, uQ^{-1}v\} - 2uQ^{-1}uvQ^{-1}v - \{Q, uD^{-1}v\} + 2uD^{-1}uvD^{-1}v.
\end{aligned} \tag{63}$$

We would like to note that, other than for the $N = 4$ Toda chain hierarchy [5], the Hamiltonian structures (61–63) are Grassmann odd due to the bosonic character of the matrices J_1 and J_2 (62–63) and the fermionic nature of the $N = 1$ delta function (46). Odd Hamiltonian structures were also used earlier in the description of some supersymmetric integrable systems (for recent papers, see [12, 13, 14, 15, 16] and references therein).

The second Hamiltonian structure J_2 (63) is rather complicated, nonlinear and nonlocal. It becomes linear and local in terms of the original Toda-chain superfields b and f (see eq. (A.13) of appendix)

$$b \equiv uv, \quad f \equiv D\ln v. \tag{64}$$

The corresponding Hamiltonian structures $J_1^{(b,f)}$ and $J_2^{(b,f)}$ can be expressed via J_1 and J_2 (62–63) by the following standard relation:

$$J_l^{(b,f)} = FJ_lF^T, \quad F \equiv \begin{pmatrix} u & v \\ D\frac{1}{v} & 0 \end{pmatrix}, \tag{65}$$

where F is the matrix of Frechet derivatives corresponding to the transformation $\{b, f\} \Rightarrow \{v, u\}$ (64). One finds:

$$J_1^{(b,f)} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, \quad J_2^{(b,f)} = \begin{pmatrix} J_{11}^{(b,f)} & J_{12}^{(b,f)} \\ J_{21}^{(b,f)} & J_{22}^{(b,f)} \end{pmatrix},$$

$$\begin{aligned}
J_{11}^{(b,f)} &\equiv -\partial b - b\partial, \\
J_{12}^{(b,f)} &\equiv \partial D + Qb + Db[\theta, D] - (Df)D, \\
J_{21}^{(b,f)} &\equiv -\partial D + bQ - [\theta, D]bD - D(Df), \\
J_{22}^{(b,f)} &\equiv -2QD + 2b - 2(Qf) - [(Df), [\theta, D]] + 2[\theta, D]b[\theta, D].
\end{aligned} \tag{66}$$

Using equations (58), (60) and (62), we obtain, for example, the 0-th fermionic flow,

$$Q_0 v = v\theta, \quad Q_0 u = -u\theta. \tag{67}$$

Knowledge of the first and second Hamiltonian structures allows us to construct the recursion operator of the hierarchy,

$$R = J_2 J_1^{-1} \equiv \begin{pmatrix} J_{12}, & -J_{11} \\ J_{22}, & -J_{21} \end{pmatrix}, \quad \frac{\partial}{\partial \tau_{l+1}^a}\begin{pmatrix} v \\ u \end{pmatrix} = R\frac{\partial}{\partial \tau_l^a}\begin{pmatrix} v \\ u \end{pmatrix}, \tag{68}$$

$$\frac{\partial}{\partial \tau_{l+1}^a}\begin{pmatrix} v \\ u \end{pmatrix} = R^l \frac{\partial}{\partial \tau_1^a}\begin{pmatrix} v \\ u \end{pmatrix}, \quad J_{l+1} = R^l J_1. \tag{69}$$

The Hamiltonian structures $J_1^{(b,f)}$ and $J_2^{(b,f)}$ (66) (and, consequently, the original Hamiltonian structures J_1 and J_2 (62–63)) are obviously mutually compatible: a deformation of the superfield f to $f+\gamma\theta$, where γ is an arbitrary parameter, transforms $J_2^{(b,f)}$ into the Hamiltonian structure defined by the algebraic sum

$$J_2^{(b,f+\gamma\theta)} = J_2^{(b,f)} + \gamma J_1^{(b,f)}. \tag{70}$$

Therefore, the recursion operator R (68) is hereditary as the operator obtained from the compatible pair of the Hamiltonian structures [17].

Applying formulae (68) we obtain the following recurrence relations for the flows:

$$\begin{aligned}
\frac{\partial}{\partial\tau^a_{l+1}}v = +\frac{\partial}{\partial\tau^a_l}v\,' &+ (-1)^{d_{\tau^a}}vD^{-1}\frac{\partial}{\partial\tau^a_l}\mathcal{H}^Q_2 - [(Qv)+v(D^{-1}uv)]D^{-1}\frac{\partial}{\partial\tau^a_l}\mathcal{H}^t_1\\
&+ (-1)^{d_{\tau^a}}vQ^{-1}\frac{\partial}{\partial\tau^a_l}\mathcal{H}^D_2 - [(Dv)-v(Q^{-1}uv)]Q^{-1}\frac{\partial}{\partial\tau^a_l}\mathcal{H}^t_1,\\
\frac{\partial}{\partial\tau^a_{l+1}}u = -\frac{\partial}{\partial\tau^a_l}u\,' &- (-1)^{d_{\tau^a}}uQ^{-1}\frac{\partial}{\partial\tau^a_l}\mathcal{H}^D_2 - [(Du)+u(Q^{-1}uv)]Q^{-1}\frac{\partial}{\partial\tau^a_l}\mathcal{H}^t_1\\
&- (-1)^{d_{\tau^a}}uD^{-1}\frac{\partial}{\partial\tau^a_l}\mathcal{H}^Q_2 - [(Qu)-u(D^{-1}uv)]D^{-1}\frac{\partial}{\partial\tau^a_l}\mathcal{H}^t_1\\
&+ 2u[\theta,D]\frac{\partial}{\partial\tau^a_l}\mathcal{H}^t_1,
\end{aligned}\tag{71}$$

where d_{τ^a} is the Grassmann parity of the evolution derivative $\frac{\partial}{\partial\tau^a_l}$ and

$$\mathcal{H}^t_1 \equiv uv,\quad \mathcal{H}^D_2 \equiv vDu + uvQ^{-1}(uv),\quad \mathcal{H}^Q_2 \equiv vQu - uvD^{-1}(uv) \tag{72}$$

are the densities of the Hamiltonians H^t_1 (56) as well as H^D_2 and H^Q_2 (59), respectively.

Taking into account the involution properties

$$\begin{aligned}
-(\mathcal{H}^t_1)^* = (\mathcal{H}^t_1)^\dagger &= \mathcal{H}^t_1,\\
(\mathcal{H}^Q_2)^* = -Q\mathcal{H}^t_1 + \mathcal{H}^Q_2,&\quad (\mathcal{H}^D_2)^* = -D\mathcal{H}^t_1 + \mathcal{H}^D_2,\\
(\mathcal{H}^Q_2)^\dagger = D\mathcal{H}^t_1 - \mathcal{H}^D_2,&\quad (\mathcal{H}^D_2)^\dagger = Q\mathcal{H}^t_1 - \mathcal{H}^Q_2
\end{aligned}\tag{73}$$

of the Hamiltonian densities (72), one can verify that the recurrence relations (71) possess the involutions (31–32). Together with the already verified fact that the first flows (26–28) also admit these involutions one concludes that the all other flows of the hierarchy under consideration admit them as well.

Using eqs. (71) and (27), we obtain, for example, the third bosonic flow

$$\begin{aligned}
\frac{\partial}{\partial t_3}v &= v\,''' + 3(Dv)\,'(Quv) - 3(Qv)\,'(Duv) + 3v\,'(Du)(Qv)\\
&\quad -3v\,'(Qu)(Dv) + 6vv\,'(DQu) - 6(uv)^2v\,',\\
\frac{\partial}{\partial t_3}u &= u\,''' + 3(Qu)\,'(Duv) - 3(Du)\,'(Quv) + 3u\,'(Qv)(Du)\\
&\quad -3u\,'(Dv)(Qu) + 6uu\,'(QDv) - 6(uv)^2u\,'
\end{aligned}\tag{74}$$

which coincides with the corresponding flow that can be derived from the Lax-pair representation (24). Let us underline that all U_l flows for $l\geq 1$ can also be derived in this way starting from the zero flow U_0 (28) as an input.

Finally, let us transform the first bosonic and fermionic flows from eqs. (26–28) and recurrence relations (71) to the basis (12), where they become

$$\begin{aligned}
&\frac{\partial}{\partial t_1}\begin{pmatrix} v\\ u\end{pmatrix} = \partial\begin{pmatrix} v\\ u\end{pmatrix},\quad U_0\begin{pmatrix} v\\ u\end{pmatrix} = -\frac{i}{\sqrt{2}}\theta\mathcal{D}\begin{pmatrix} v\\ u\end{pmatrix},\\
&\mathcal{D}_1v = -\mathcal{D}v - 2v\partial^{-1}\mathcal{D}(uv),\quad \mathcal{D}_1u = -\mathcal{D}u + 2u\partial^{-1}\mathcal{D}(uv),\\
&\overline{\mathcal{D}}_1v = -\overline{\mathcal{D}}v + 2v\partial^{-1}\overline{\mathcal{D}}(uv),\quad \overline{\mathcal{D}}_1u = -\overline{\mathcal{D}}u - 2u\partial^{-1}\overline{\mathcal{D}}(uv),
\end{aligned}\tag{75}$$

$$\begin{aligned}
&\frac{\partial}{\partial\tau^a_{l+1}}v = +\frac{\partial}{\partial\tau^a_l}v\,' + (-1)^{d_{\tau^a}}[v\partial^{-1}\mathcal{D}\frac{\partial}{\partial\tau^a_l}(v\overline{\mathcal{D}}u - uv\partial^{-1}\overline{\mathcal{D}}uv)\\
&-v\partial^{-1}\overline{\mathcal{D}}\frac{\partial}{\partial\tau^a_l}(v\mathcal{D}u + uv\partial^{-1}\mathcal{D}uv)] - [(\overline{\mathcal{D}}v) + v(\partial^{-1}\overline{\mathcal{D}}uv)]\partial^{-1}\mathcal{D}\frac{\partial}{\partial\tau^a_l}(uv)\\
&+[(\mathcal{D}v) - v(\partial^{-1}\mathcal{D}uv)]\partial^{-1}\overline{\mathcal{D}}\frac{\partial}{\partial\tau^a_l}(uv),\\
&\frac{\partial}{\partial\tau^a_{l+1}}u = -\frac{\partial}{\partial\tau^a_l}u\,' - (-1)^{d_{\tau^a}}[u\partial^{-1}\overline{\mathcal{D}}\frac{\partial}{\partial\tau^a_l}(u\mathcal{D}v - uv\partial^{-1}\mathcal{D}uv)\\
&-u\partial^{-1}\mathcal{D}\frac{\partial}{\partial\tau^a_l}(u\overline{\mathcal{D}}v + uv\partial^{-1}\overline{\mathcal{D}}uv)] + [(\mathcal{D}u) + u(\partial^{-1}\mathcal{D}uv)]\partial^{-1}\overline{\mathcal{D}}\frac{\partial}{\partial\tau^a_l}(uv)\\
&-[(\overline{\mathcal{D}}u) - u(\partial^{-1}\overline{\mathcal{D}}uv)]\partial^{-1}\mathcal{D}\frac{\partial}{\partial\tau^a_l}(uv).
\end{aligned}\tag{76}$$

These equations are obviously invariant under the $U(1)$ transformation (16). Consequently, all higher flows admit this automorphism as well. Despite of this, the Lax operator $\mathcal{M}$ (17) is not invariant with respect to the $U(1)$ transformation. Hence, applying it to $\mathcal{M}$ one can derive a one-parameter family of consistent Lax operators,

$$\mathcal{M}\quad\Longrightarrow\quad \mathcal{M}^\phi = \cos\phi\;\mathcal{M} + \sin\phi\;(\mathcal{M}^\dagger)^T \tag{77}$$

with $\mathcal{M}^\dagger$ defined in eq. (37). The flows generated in this way are all isomorphic. We remark that the superfields v and u have trivial transformation properties under the $U(1)$ transformation (16), while the superfunctions w_n^f and w_n^b (20) expressed in terms of these transform in a rather complicated manner.

4 Secondary reduction: a new N=2 supersymmetric modified KdV hierarchy

In this section we derive a new $N = 2$ supersymmetric modified KdV hierarchy by means of the secondary reduction.

Let us investigate the secondary reduction of the hierarchy considered in the preceding sections. We impose the following secondary constraint[8] on the Lax operator $\mathcal{L}$ (17):

$$\mathcal{M}^T = D\mathcal{M}D^{-1} \tag{78}$$

which can easily be resolved in terms of the superfield v entering $\mathcal{M}$,

$$v = 1. \tag{79}$$

Then, the reduced Lax operator $\mathcal{M}^{red}$ becomes

$$\mathcal{M}^{red} = D_- + D_+^{-1}u. \tag{80}$$

Condition (78) by means of eq. (19) induces the secondary constraint

$$(\mathcal{W}^{-1})^T = D\mathcal{W}D^{-1} \tag{81}$$

on the dressing operator $\mathcal{W}$ (20) which in turn induces the following secondary constraints on the operators L_l (4):

$$(\mathcal{L}_{2l})^T = (-1)^l D\mathcal{L}_{2l}D^{-1}, \quad (\mathcal{L}_{2l-1})^T = (-1)^l D\mathcal{L}_{2l-1}D^{-1} \tag{82}$$

which are identically satisfied if constraint (78) (or (81)) is imposed. Importantly, eqs. (82) imply that

$$(\mathcal{L}_{2(2k-1)})_0 = (\mathcal{L}_{2(2k)-1})_0 = 0, \quad k = 1, 2 \ldots, \tag{83}$$

[8]See also refs. [18, 19, 20], where a similar reduction of the Manin-Radul [1] and Mulase-Rabin [2, 3] $N = 1$ supersymmetric KP and KdV hierarchies has been discussed.

where the subscript 0 refers to the constant part of the operators. Consequently, the equations:

$$((\mathcal{L}_{2(2k-1)})_+1) = ((\mathcal{L}_{2(2k)-1})_+1) = 0 \tag{84}$$

are identically satisfied as well. Using these relations, the involution (32), and the algebraic structure (8–9) we are led to the conclusion that only half of the flows (25) are consistent with the reduction (78–79), namely

$$\{ \tfrac{\partial}{\partial t_{2k-1}}, \; U_{2k}, \; D_{2k}, \; Q_{2k} \}. \tag{85}$$

In order to understand better what kind of reduced hierarchy we have in fact derived, one might analyze its Hamiltonian structure via Hamiltonian reduction of the first and second Hamiltonian structures (62–63) we started with. However, it is easier to reduce the less complicated expressions (66). In this basis, the constraint (79) becomes

$$f = 0, \tag{86}$$

and the superfield b coincides with the superfield u on the constraint surface.

Let us start with the first Hamiltonian structure $J_1^{(b,f)}$ (66). In this case, the constraint (86) is a gauge constraint, and a gauge can be fixed by the condition $b = 0$. As the result, the trivial reduced Hamiltonian structure is generated.

In the case of the second Hamiltonian structure $J_2^{(b,f)}$ (66), the constraint (86) is second class, and we can use Dirac brackets in order to obtain the second Hamiltonian structure for the reduced system. The result is

$$\begin{aligned} J_{11}^{(Dirac)} &= J_{11}^{(u,0)} - J_{12}^{(u,0)} {J_{22}^{(u,0)}}^{-1} J_{21}^{(u,0)} \\ &\equiv \frac{1}{2}(\partial DQ + QuQ - DuD + 2\partial u + 2u\partial), \end{aligned} \tag{87}$$

where we have exploited the relations

$$J_{12}^{(b,0)}Q = \frac{1}{2}QJ_{22}^{(b,0)}Q = QJ_{21}^{(b,0)} = -\partial DQ - QbQ + DbD \tag{88}$$

which can easily be read off eqs. (66).

From eqs. (87) we see that the second Hamiltonian structure of the secondary reduced hierarchy displays the reduced $N = 2$ superconformal structure, and its flows (85) possess a global $N = 2$ supersymmetry with an unusual length dimensions of its generators,

$$[\tfrac{\partial}{\partial t_3}] = -3, \quad [U_0] = 0, \quad [D_2] = [Q_2] = -\frac{3}{2}. \tag{89}$$

Substituting the constraint (79) into the third flow equations (74) of the reduced hierarchy, they become

$$\frac{\partial}{\partial t_3} u \;=\; (u\,'' - 3(Du)(Qu) + 2u^3)\,', \tag{90}$$

and one can easily recognize that this equation reproduces the modified KdV equation in the bosonic limit when the fermionic component is put equal to zero. Equation (90) does not coincide with any of the three known $N = 2$ extensions [21] of the modified KdV equation. Therefore, we summarize that the secondary reduced hierarchy gives a new type of $N = 2$ supersymmetric generalization of the modified KdV hierarchy.

5 Generalizations, Conclusion and Outlook

In this section we discuss possible generalizations of the reduced $N = 2$ KP hierarchy to the matrix case and some open problems.

The hierarchies discussed in the preceding sections admit a natural generalization to the non-abelian case. One may consider the $N = 2$ supersymmetric matrix KP hierarchy generated by a matrix-valued dressing operator W in $N = 1$ superspace,

$$W \equiv I + \sum_{n=1}^{\infty} (w_n^{(b)} + w_n^{(f)} D)\, \partial^{-n}, \tag{91}$$

which can be treated as a reduction of the analogous operator in $N = 2$ superspace considered in [5]. Its consistent reductions are characterized by the reduced operator

$$\mathcal{M}_1 = IQ + vD^{-1}u. \tag{92}$$

Here, $w_n \equiv (w_n)_{AB}(Z)$, $v \equiv v_{Aa}(Z)$ and $u \equiv u_{aA}(Z)$ $(A, B = 1, \ldots, k;$ $a, b = 1, \ldots, n+m)$ are rectangular matrix-valued superfields, and I is the

identity matrix, $I \equiv \delta_{A,B}$. In (92) the matrix product is understood, for example $(vu)_{AB} \equiv \sum_{a=1}^{n+m} v_{Aa}u_{aB}$. The matrix entries are bosonic superfields for $a = 1, \ldots, n$ and fermionic superfields for $a = n+1, \ldots, n+m$, i.e., $v_{Aa}u_{bB} = (-1)^{d_a \overline{d}_b} u_{bB}v_{Aa}$, where d_a and $\overline{d}_b$ are the Grassmann parities of the matrix elements v_{Aa} and u_{bB}, respectively, $d_a = 1$ $(d_a = 0)$ for fermionic (bosonic) entries. The grading choosen guarantees that the Lax operator $\mathcal{M}_1$ is Grassmann odd [22].

A detailed analysis of the emerging hierarchies is, however, beyond the scope of the present paper. Without going into more details, let us only present a few non-trivial bosonic and fermionic flows in this noncommutative case (compare with the abelian flows (26–27)):

$$\begin{aligned}
&\frac{\partial}{\partial t_2} v = +v\,'' - 2\{Q, vDu\}v - 2v(uv)^2, \\
&\frac{\partial}{\partial t_2} u = -u\,'' + 2\{D, uQv\}u + 2(uv)^2 u, \\
&D_1 v = -Dv + 2(Q^{-1}v\mathcal{I}u)v, \quad D_1 u = -Du - 2uQ^{-1}(vu), \\
&Q_1 v = -Qv - 2vD^{-1}(uv), \quad Q_1 u = -Qu + 2\mathcal{I}(D^{-1}uv)u
\end{aligned} \tag{93}$$

which are derived using Lax-pair representations (24) with $\mathcal{M}_1$ (92) and

$$(\mathcal{L}_1)_+ = ID - 2(Q^{-1}(v\mathcal{I}u)), \tag{94}$$

and the matrix $\mathcal{I}$ is defined as

$$\mathcal{I} \equiv (-1)^{d_a}\delta_{ab}. \tag{95}$$

It is crucial that the existence of these two different fermionic first flows, D_1 and Q_1 (93), guarantees the $N = 2$ supersymmetry of the corresponding hierarchies.

For the particular case when the index A takes only the value $A = 1$, the matrix reduced Lax operator (92) becomes a scalar operator generating a reduced hierarchy with $n + m$ pairs of scalar superfields v_a, u_a. In the more special case $A = 1, a = 1$ and $n = 1, m = 0$, the Lax operator (92) reproduces the Lax operator (17).

The results described in the previous sections can also be generalized to the case of some other known reductions of the supersymmetric KP hierarchy in $N = 1$ superspace. For example, a wide class of the following reductions

$$L_1 = \mathcal{L} \equiv D + \sum_{a=1}^{m} v_a D^{-1} u_a, \quad m \in \mathbb{N} \tag{96}$$

was proposed in [6]. It is quite obvious that one can generalize them by replacing the superfunctions v_a and u_a by supermatrices with the above–described grading. A less obvious fact is that one can consistently extend in an $N=2$ supersymmetric fashion the number of bosonic and fermionic flows of the reduced hierarchies obtained in [6]. To simplify the consideration let us concentrate on the simplest example of the reduced hierarchy, characterized by the Lax operator (96) at $m=1$. Then, as it was shown in [6], the Lax operator $\mathcal{L}$ (96) satisfies an equation which can be read from eq. (18) by replacing the operator $\mathcal{M}$ by $\mathcal{L}$ there, and the flows

$$\frac{\partial}{\partial t_l}\mathcal{W} = -(\mathcal{L}_{2l})_-\mathcal{W}, \quad D_l\mathcal{W} = -((\mathcal{L}_{2l-1})_- - \widetilde{\mathcal{L}}_{2l-1})\mathcal{W} \tag{97}$$

can consistently be introduced. Here, the operator $\widetilde{\mathcal{L}}_{2l-1}$ can be read off eq. (23) by replacing the operators $\widetilde{\mathcal{M}}_{2l-1}$ and $\mathcal{M}$ there by the operators $\widetilde{\mathcal{L}}_{2l-1}$ and $\mathcal{L}$, respectively. Now, one can easily observe that the operators $\mathcal{L}$ (97) and $\mathcal{M}$ (17) possess the same properties in spite of their different appearance, and for this reason one can construct the same set of consistent Sato equations (22) for each of them. Comparing equations (97) with (22) shows that at least one more series of fermionic flows, namely

$$Q_l\mathcal{W} = -(\mathcal{M}_{2l-1})_-\mathcal{W} \tag{98}$$

can consistently be added to the Sato equations (97) and, consequently, the extended hierarchy of the flows are indeed $N=2$ supersymmetric.

The hierarchies proposed in this paper may appear to have come out of the blue. It is time to explain how we were lead to their construction by relating them to previously known hierarchies. Forerunners of the present paper are refs. [4, 7, 8] and especially refs. [6, 5]. As one might suspect, there is a correspondence between the $N=2$ supersymmetric hierarchies defined above and the $N=4$ supersymmetric hierarchies proposed in [5], but this correspondence is rather non-trivial and indirect. The heuristic analysis of the $N=4$ flows constructed in [5] shows that among them exist flows which contain only the operators D_+ and D_- (and not Q_+ and Q_-) and which are in some sense $N=2$ like. Restricting the whole hierarchy to only these flows, one can consistently reduce them by the constraint $\theta_+ = i\theta_- \equiv \theta$ which leads to the correspondence $D_+ \equiv D$ and $D_- \equiv iQ$ with the fermionic derivatives of the present paper, where i is the imaginary unity and $\theta_\pm$ are the Grassmann coordinates of $N=2$ superspace. This constraint is consistent for the algebra of the fermionic

derivatives $D_\pm$, but it is surely inconsistent for the algebra extended by any of the two fermionic derivatives $Q_\pm$. Without going into details we would like to stress that this reduction is a rather nontrivial one, and the whole construction given in [5] must properly be adjusted. For illustrative purposes consider the product $D_+D_-^{-1}$ which appears when constructing the consistent Sato equations. This product has no differential piece before the reduction, but it becomes a purely *differential(!)* operator by virtue of the reduction constraint, drastically changing the construction. Moreover, our U_l flows cannot be derived by reducing the $N=4$ flows, so they must be added by hand in order to complete the hierarchy. To close this discussion let us state two unsolved questions whose answering should yield a deeper understanding of the proposed hierarchies:

1. What is the consistent Lax-pair representation of the U_l flows?

2. What are proper general formulae for the Hamiltonians H_l^U, H_l^D and H_l^Q analogous to formula (54) for the Hamiltonian H_l^t?

We hope to return to this questions elsewhere.

Finally, we would like to briefly comment on some unusual properties of our hierarchy.

1. Our hierarchy flows in $N=1$ superspace contain both the $N=1$ fermionic derivative D and the $N=1$ supersymmetry generator Q, nevertheless they are $N=2$ supersymmetric. The resolution of this sophism is hidden in the nonlocal character of the $N=2$ transformations.

2. The equations for the bosonic components of our bosonic flows $\frac{\partial}{\partial t_l}$ do not contain the fermionic components at all. Nevertheless, the supersymmetrization of these equations is non-trivial[9] because it involves the fermionic operators D and Q.

3. The residue (53) we used for pseudo-differential operators in $N=1$ superspace is not the usual $N=1$ residue which is the coefficient of the operator D^{-1}. We obtained this unusual definition for the residue by the above–explained reduction of the residue introduced in [5].

4. Grassmann-odd Hamiltonian structures appear at the Hamiltonian description of our supersymmetric hierarchy. To our knowledge, this is the first example of a non-trivial supersymmetrized hierarchy with odd bi-Hamiltonian structure. It is interesting to speculate whether an even

[9] By trivial supersymmetrization of bosonic equations we mean just replacing functions by superfunctions. In this case the resulting equations are $N=2$ supersymmetric as well, but they do not contain the fermionic derivatives at all.

bi-Hamiltonian structure exists as well.

5. The secondary reduced hierarchy is a new $N=2$ supersymmetric modified KdV hierarchy with unusual length dimensions of the $N=2$ supersymmetry generators (see, eqs. (89)).

All these peculiarities once more demonstrate the rich structure encoded in supersymmetry.

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Appendix. A new N=2 supersymmetric Toda chain

In this appendix we present a new version of the supersymmetric Toda chain equation and derive its zero-curvature representation which lies at the origin of the reduction constraint (17).

Let us introduce the new equation

$$QD\ln b_i = b_{i+1} - b_{i-1} \tag{A.1}$$

written in terms of the bosonic $N=1$ superfields $b_i \equiv b_i(z,\theta)$ defined on the chain, $i \in \mathbb{Z}$. This equation represents a one-dimensional $N=1$ generalization of the two-dimensional $N=(1|1)$ superconformal Toda lattice equation. It can be rewritten as a system of two equations

$$Qf_i = b_i + b_{i+1}, \quad D\ln b_i = f_i - f_{i-1} \tag{A.2}$$

which admits the zero-curvature representation

$$\{D - A^{\theta}_D\ , \ Q - A^{\theta}_Q\} = 0 \tag{A.3}$$

with the fermionic connections

$$(A^{\theta}_D)_{ij} \equiv f_i\delta_{i,j} + \delta_{i,j-1}, \quad (A^{\theta}_Q)_{ij} \equiv -b_i\delta_{i,j+1}, \tag{A.4}$$

where $f_i \equiv f_i(z,\theta)$ are fermionic $N=1$ chain superfields. One can define the bosonic connections $A_{z^{\pm}}$ by

$$\partial + A^z_D \equiv (D - A^{\theta}_D)^2, \quad \partial + A^z_Q \equiv -(Q - A^{\theta}_Q)^2. \tag{A.5}$$

More explicitly, they read

$$\begin{aligned}(A^z_Q)_{ij} &\equiv -Qb_i\delta_{i,j+1} + b_ib_{i-1}\delta_{i,j+2},\\ (A^z_D)_{ij} &\equiv -D_+f_i\delta_{i,j} + (f_i - f_{i+1})\delta_{i,j-1} - \delta_{i,j-2}\end{aligned} \tag{A.6}$$

and due to (A.3), obviously satisfy the zero-curvature condition

$$[\partial + A^z_Q\ , \ \partial + A^z_D] = 0 \tag{A.7}$$

which is a consistency condition for the linear system

$$(\partial + A^z_Q)\Psi = \lambda\Psi, \tag{A.8}$$

$$(\partial + A^z_D)\Psi = 0, \tag{A.9}$$

where $\Psi \equiv \Psi_i$ is the chain wave function and λ is a spectral parameter. Taking into account the first relation of eqs. (A.5), equation (A.9) can equivalently be rewritten in the form

$$(D - A^{\theta}_D)\Psi = 0. \tag{A.10}$$

The linear system (A.8), (A.10) is a key object in our consideration.

In order to derive the Lax operator we are looking for, we follow a trick proposed in [23] and express each chain function entering the spectral equation (A.8) in terms of chain functions defined at the single chain point i, using eqs. (A.2) and (A.10). In this manner we obtain the new spectral equation

$$(Q + \frac{1}{D - f_i}b_i)^2\Psi_i = \lambda\Psi_i. \tag{A.11}$$

For each fixed value of i, it represents the spectral equation of the differential hierarchy, i.e. of the hierarchy of equations involving only the superfields b_i, f_i at a single lattice point. Applying the discrete chain shift (i.e., the system of eqs. (A.2)) to the differential hierarchy generates the discrete hierarchy. Thus, the discrete hierarchy appears as a collection of an infinite number of isomorphic differential hierarchies [23].

It is well known that a spectral equation is just an equation for a Lax operator. For a fixed value of i one can completely omit the chain index in the spectral equation (A.11), and it is obvious that the operator

$$M = (Q + \frac{1}{D-f} b)^2 \tag{A.12}$$

is just the Lax operator which is responsible for the bosonic flows of the differential hierarchy. In the new superfield basis $\{v_i, u_i\}$ defined by

$$b_i \equiv u_i v_i, \quad f_i \equiv D \ln v_i, \tag{A.13}$$

in which the system (A.2) becomes an $N = 1$ supersymmetric generalization of the Darboux transformation (35)

$$u_{i+1} = \frac{1}{v_i}, \quad QD \ln v_i = u_{i+1} v_{i+1} + u_i v_i, \tag{A.14}$$

the Lax operator (A.12) simplifies to

$$M = (Q + vD^{-1}u)^2. \tag{A.15}$$

Let us remark that the operator $\mathcal{M}$ (17) is just the square root of the operator M (A.15). This Lax operator has been used in section 3 for constructing a consistent reduction of all other flows of the $N = 2$ supersymmetric KP hierarchy.

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