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LIE ALGEBRA CONTRACTIONS FOR OVERLAP FUNCTIONS

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Изместьев А.А. и др. E2-99-21
Контракции алгебры Ли и функции перекрытия

Контракции алгебры Ли $o(n+1)$ в $e(n)$ используются для получения асимптотических пределов межбазисных разложений между базисами, соответствующими различным подгрупповым цепочкам для группы $O(n+1)$. Контракции приводят к межбазисным разложениям для различных подгрупповых цепочек евклидовой группы $E(n)$. Рассматриваются только простейшие случаи $n=2$ и $n=3$.

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Lie Algebra Contractions for Overlap Functions

Lie algebra contractions from $o(n+1)$ to $e(n)$ are used to obtain asymptotic limits of interbases expansions between bases corresponding to different subgroup chains for the group $O(n+1)$. The contractions lead to interbases expansions for different subgroup chains of the Euclidean group $E(n)$. The article is restricted to the low dimensional cases $n=2$ and $n=3$.

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1 Introduction

This article is the third in a series [1, 2] devoted to contractions of rotation groups O(n+1) to Euclidean groups E(n) and the separation of variables in Laplace-Beltrami equations. In the first one [1] we considered the sphere S_2 on which the equation

$$\Delta_{LB}\Psi = -\lambda\Psi, \qquad \Delta_{LB} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i}\sqrt{g}\,g^{ik}\frac{\partial}{\partial\xi_k}, \quad g = \det g_{ik} \tag{1.1}$$

allows the separation of variables in two coordinate systems: spherical and elliptic ones. The contraction parameter was the radius R of the sphere. For $R \to \infty$ the sphere $S_n \sim O(n+1)/O(n)$ goes into the Euclidean space $E_n \sim E(n)/O(n)$. For $n = 2$ the two separable coordinate systems on S_2 go into 4 separable coordinate systems on E_2, namely Cartesian, polar, parabolic and elliptic ones. Depending on how the limit is taken, spherical coordinates go into polar, or Cartesian ones. Elliptic coordinates on S_2 go into elliptic, or parabolic coordinates on E_2. Via a two-step procedure, through spherical coordinates, they also contract to Cartesian and polar coordinates on E_2. The contraction was followed through on several levels: the coordinates, the complete sets of commuting operators, the separated equations and the eigenfunctions and eigenvalues.

In the second article [2] the dimension of the space was arbitrary, but only the simplest types of coordinates were considered, namely subgroup type coordinates. These are associated with chains of subgroups of O(n+1), or E(n), respectively.

Vilenkin, Kuznetsov and Smorodinsky [3, 4] developed a graphical method, the "method of trees" to describe subgroup type coordinates on S_n. The corresponding separated eigenfunctions are hyperspherical function (also called polyspherical functions) [5, 6, 7]. Their relation to subgroup chains and subgroup diagrams was analyzed in Ref.2, as were their contractions to subgroup type separated basis functions for the groups $E(n)$.

In many body theories if is often necessary to expand one type of hyperspherical functions in terms of other ones. The expansion coefficients have been called T-coefficients, or overlap functions. The corresponding coefficients for functions on S_n were calculated by Kildyushov [7].

The purpose of this article is a study the $R \to \infty$ contraction limit of the interbases expansions and overlap functions for the different spherical and hyperspherical functions on S_2 and S_3. The mathematical motivation is to obtain asymptotic limits of various expansions and of the overlap functions. These are objects of considerable physical interest: Wigner rotation matrices, Clebsch-Gordan coefficients, Racah coefficients, etc. The physical motivation goes back to the original work of Inönü and Wigner [8]. Typically, a Lie group, or Lie algebra contraction relates two different theories. The contraction parameter in our case is not the speed of light, so we are not relating relativistic and nonrelativistic theories. Rather, we are relating theories in flat and curved spaces, or theories of spherical and highly elongated objects, e.g. nuclei [9].

The contractions we use are analytical ones: the radius of the sphere is built into the infinitesimal operators and into the sets of commuting operators, not only into the structure constants. The contractions can be viewed as singular changes of bases, as was the case of the original Inönü-Wigner ones. They are also "graded contractions" [10,11], in this case corresponding to a Z_2 - grading of o(3), o(4) and more generally o(n+1).

The overall point of view of the separation of variables that we are taking is an operator one [12-17]. Thus, let G be the isometry group of the considered Riemannian or pseudo-Riemannian space and L its Lie algebra. Let $\{X_1, X_2, ..., X_N\}$ be a basis of L and

$$Y_a = \sum_{ik} A^a_{ik} X_i X_k, \qquad [Y_a, Y_b] = 0, \qquad A^a_{ik} = A^a_{ki} \tag{1.2}$$

a complete set of commuting second order operators in the enveloping algebra of L. The separated eigenfunctions will be the common eigenfunctions of such a complete set

$$Y_a \Psi = -\lambda_a \Psi, \qquad \Psi = \prod_i^n f_i(\xi_i), \tag{1.3}$$

where ξ_i are the separable coordinates. For subgroup type coordinates all the operators Y_a are Casimir operators of subalgebras of L (the Laplace-Beltrami operator Δ_{LB} is included in the set $\{Y_a\}$).

2 Contractions of overlap functions for the group O(3)

Two type of tree diagrams exist for the sphere S_2, both shown on the (Fig.1a). Both correspond to the subgroup chain O(3)⊃O(2), however one priviliges the pair (0,1), the other the pair (1,2). In other words, the complete sets of commuting operators consist of the rotation operator L_{01} and L_{12} in the first and second case, respectively (in addition to the Laplace-Beltrami operator that is always present). On the subgroup diagrams (Fig.1c) the circles correspond to O(n) subgroups (with the value of n indicated in the circle). Rectangles correspond to Euclidean subgroups E(n), again with the values of n in the rectangle.

The spherical functions corresponding to the two trees are connected by the interbases expansion

$$Y_{lm_1}\left(\frac{\pi}{2} - \theta'_1, \theta'_2\right) = \sum_{m_2=-l}^{l} D^l_{m_2,m_1}\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right) Y_{lm_2}(\theta_1, \theta_2), \tag{2.1}$$

so that the overlap functions are the Wigner rotation matrices $D^l_{m_2,m_1}(\alpha,\beta,\gamma) = e^{-im_2\alpha} d^l_{m_2,m_1}(\beta)$ $e^{-im_1\gamma}$ [7,18,19]. The angles in both sides of the expansions are connected by the relations

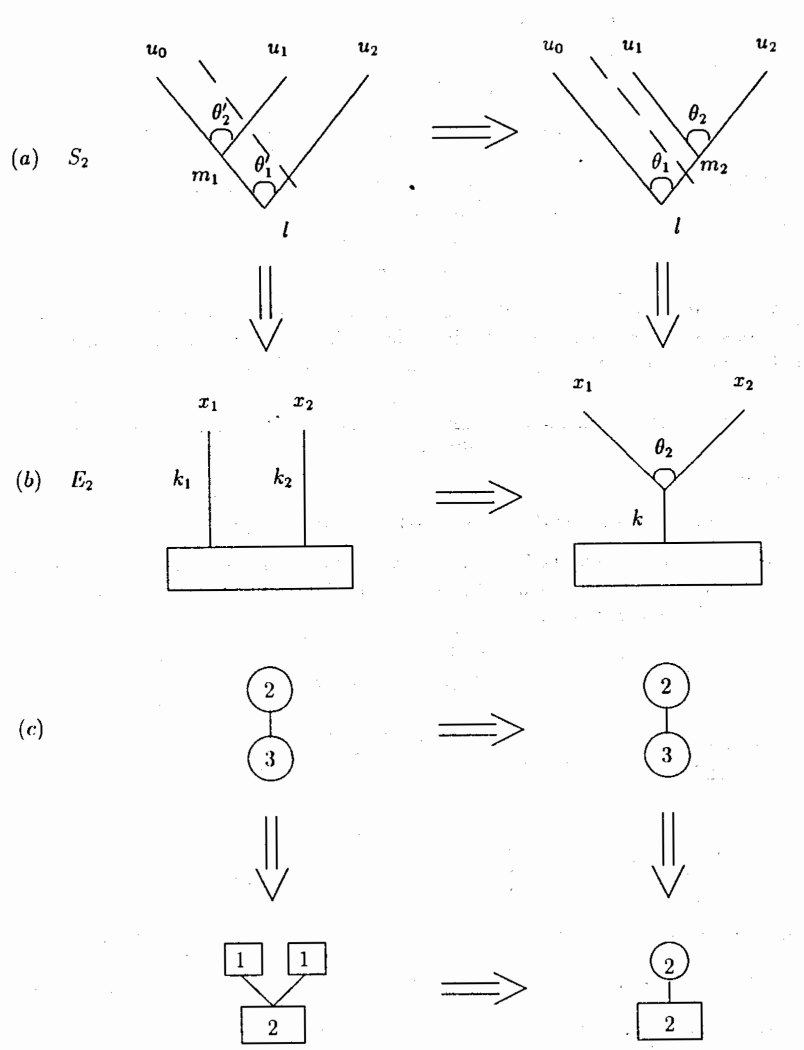
$$\begin{aligned} u_0 &= R\cos\theta_1 && = R\cos\theta'_1 \cos\theta'_2 \\ u_1 &= R\sin\theta_1\cos\theta_2 && = R\cos\theta'_1\sin\theta'_2 \\ u_2 &= R\sin\theta_1\sin\theta_2 && = R\sin\theta'_1. \end{aligned}$$

The expansion (2.1) corresponds to an "elementary" transformation of the O(3) tree diagram on Fig.1: the branch leading to the Cartesian coordinate u_1 is "transplanted" from the u_0 branch to the u_2 one.

As explained in Ref.1, the $R \to \infty$ contraction is realized by first introducing Beltrami coordinates

$$x_\mu = R\frac{u_\mu}{u_0} = \frac{u_\mu}{\sqrt{1 - \sum_{i=1}^n u_i^2/R^2}}, \qquad \mu = 1, 2, ..., n \tag{2.2}$$

*Fig.*1 Tree diagrams and subgroup diagrams illustrating $S_2 \to E_2$ contractions.



(with n=2 for S_2). The o(3) Lie algebra is realized as

$$\begin{aligned} L_{20} &\equiv -R\pi_2 = -Rp_2 - \frac{1}{R}x_2(x_1p_1 + x_2p_2) \\ L_{01} &\equiv R\pi_1 = Rp_1 + \frac{1}{R}x_1(x_1p_1 + x_2p_2) \\ L_{12} &= x_2p_1 - x_1p_2 = x_2\pi_1 - x_1\pi_2, \qquad p_\mu = \frac{\partial}{\partial x_\mu}, \quad \mu = 1,2\,. \end{aligned} \tag{2.3}$$

For $R \to \infty$ the o(3) algebra contracts to the e(2) one, the momenta π_μ contract to the translation operators p_μ and the Laplace-Beltrami operator on S_2 to that on E_2

$$\Delta_{LB} = \pi_1^2 + \pi_2^2 + \frac{L_{12}^2}{R^2} \to \Delta = p_1^2 + p_2^2\,. \tag{2.4}$$

Let us now consider the contraction $R \to \infty$ for the interbasis expansion (2.1). Contractions of basis functions were presented earlier [1]. In order to obtain the corresponding limits of the Wigner D-functions, we use an integral representation for the function $d^l_{m_2,m_1}(\pi/2)$

$$d^l_{m_2,m_1}(\frac{\pi}{2}) = (-1)^{\frac{l-m_1}{2}}\frac{2^l}{\pi}\left\{\frac{(l+m_2)!(l-m_2)!}{(l+m_1)!(l-m_1)!}\right\}^{1/2}\int_0^\pi (\sin\alpha)^{l-m_1}(\cos\alpha)^{l+m_1}e^{2im_2\alpha}d\alpha,$$

and the formulas [20]

$$\cos(2n\alpha) = T_n(\cos 2\alpha), \qquad \sin(2n\alpha) = \sin 2\alpha \cdot U_{n-1}(\cos 2\alpha),$$

where $T_l(x)$ and $U_l(x)$ are Tchebyshev polynomials of the first and second kind. After integrating over α, we obtain a representation of the Wigner D- functions in terms of the hypergeometrical function ${}_3F_2$ (of argument 1):

$$D^l_{m_2,m_1}(\frac{\pi}{2},\frac{\pi}{2},0) = \frac{(-1)^{\frac{l+m_2-m_1}{2}}}{\sqrt{\pi}\,l!}\sqrt{(l+m_2)!(l-m_2)!} \tag{2.5}$$

$$\times\begin{cases} \left\{\frac{\Gamma(\frac{l+m_1+1}{2})\Gamma(\frac{l-m_1+1}{2})}{\Gamma(\frac{l+m_1}{2}+1)\Gamma(\frac{l-m_1}{2}+1)}\right\}^{\frac{1}{2}} {}_3F_2\left(\begin{matrix}-m_2, m_2, \frac{l+m_1+1}{2} \\ \frac{1}{2}, l+1\end{matrix}\middle| 1\right), & (l-m_1) - \text{even}, \\ \frac{2il}{(l+1)}\left\{\frac{\Gamma(\frac{l+m_1}{2}+1)\Gamma(\frac{l-m_1}{2}+1)}{\Gamma(\frac{l+m_1+1}{2})\Gamma(\frac{l-m_1+1}{2})}\right\}^{\frac{1}{2}} {}_3F_2\left(\begin{matrix}-m_2+1, m_2+1, \frac{l+m_1}{2}+1 \\ \frac{3}{2}, l+2\end{matrix}\middle| 1\right), & (l-m_1) - \text{odd}. \end{cases}$$

Consider now the contraction limit $R \to \infty$ in the expansion (2.1). For large R we put

$$l \sim kR, \qquad m_1 \sim k_1R, \qquad \theta_1 \sim \frac{r}{R}, \qquad \theta_1' \sim \frac{y}{R}, \qquad \theta_2' \sim \frac{x}{R}, \qquad R \to \infty, \tag{2.6}$$

where $k^2 = k_1^2 + k_2^2$, and have [2]

$$\lim_{R\to\infty}\frac{1}{\sqrt{R}}Y_{lm_2}(\theta_1,\theta_2) = (-1)^{\frac{m_2+|m_2|}{2}}\sqrt{k}J_{|m_2|}(kr)\frac{e^{im_2\theta_2}}{\sqrt{2\pi}}, \tag{2.7}$$

$$\lim_{R\to\infty}(-1)^{-\frac{l-|m_1|}{2}}Y_{lm_1}\left(\frac{\pi}{2}-\theta_1',\theta_2'\right) = \sqrt{\frac{k}{k_2}}\frac{e^{ik_1x}}{\pi}\begin{cases}\cos k_2y, & (l-|m_1|) - \text{even}, \\ -i\sin k_2y, & (l-|m_1|) - \text{odd}.\end{cases} \tag{2.8}$$

Using known asymptotic formulas [20] for the ${}_3F_2$ functions and Γ- functions in eq. (2.5) we obtain:

$$\lim_{R\to\infty}(-1)^{-\frac{l-|m_1|}{2}}\sqrt{R}\,D^l_{m_2,m_1}(\frac{\pi}{2},\frac{\pi}{2},0) = (-1)^{\frac{m_2}{2}}\sqrt{\frac{2}{\pi k}}$$

$$\times\begin{cases} \left(\frac{k^2}{k_2}\right)^{\frac{1}{4}}\ {}_2F_1\left(-m_2, m_2; \frac{1}{2}; \frac{k+k_1}{2k}\right), & (l-m_1) - \text{even}, \\ -im_2\left(\frac{k_2}{k^2}\right)^{\frac{1}{4}}\ {}_2F_1\left(-m_2+1, m_2+1; \frac{3}{2}; \frac{k+k_1}{2k}\right), & (l-m_1) - \text{odd}. \end{cases}$$

$$= (-1)^{\frac{3m_2}{2}}\sqrt{\frac{2}{\pi k_2}}\begin{cases}\cos m_2\varphi, & (l-m_1) - \text{even}, \\ i\sin m_2\varphi, & (l-m_1) - \text{odd},\end{cases} \tag{2.9}$$

where $\cos\varphi = k_1/k$.

Multiplying the interbases expansion (2.1) by the factor $(-1)^{-\frac{l-|m_1|}{2}}$ and taking the contraction limit $R \to \infty$ we obtain ($\theta \equiv \theta_2$, $m \equiv m_2$)

$$e^{ik_1x}\left\{\begin{matrix}\cos k_2y \\ \sin k_2y\end{matrix}\right\} = \sum_{m=-\infty}^{\infty}(i)^{|m|}\left\{\begin{matrix}\cos m\varphi \\ -\sin m\varphi\end{matrix}\right\}J_{|m|}(kr)\,e^{im\theta}, \tag{2.10}$$

or in exponential form

$$e^{ikr\cos(\theta-\varphi)} = \sum_{m=-\infty}^{\infty}(i)^m J_m(kr)\,e^{im(\theta-\varphi)}. \tag{2.11}$$

The inverse expansion is

$$J_m(kr)e^{im\theta} = \frac{(-i)^m}{2\pi}\int_0^{2\pi}e^{im\varphi-ikr\cos(\theta-\varphi)}\,d\varphi. \tag{2.12}$$

For $\theta = 0$ the two last formulas are equivalent to well known formulas in the theory of Bessel functions [20], namely expansions of plane waves in terms of cylindrical ones and vice versa.

The entire procedure is illustrated on Fig.1. The vertical arrows correspond to the contraction (2.6). The O(3) interbasis expansion (2.1) has contracted to the E(2) interbasis expansion (2.11) and its inverse (2.12), i.e. the relations between plane and spherical waves. The contraction of the overlap functions is given by eq. (2.9): an asymptotic formula for Wigner D-functions.

We recall [2] that the E_n "cluster" diagrams are obtained from the S_n tree diagrams by cutting along the dotted lines on Fig.1. The dotted line becomes the basis for the E_n (in this case E_2) diagram. Thus two topologically equivalent tree diagrams go into inequivalent cluster diagrams. The first contracts to Cartesian coordinates, the second to polar ones. In terms of subgroup diagrams the situation is illustrated in the Fig.1c.

3 Contractions for the group O(4)

Five types of tree diagrams exist for the sphere $S_3 \sim O(4)/O(3)$. Two of them are shown on Fig.2a, two more on Fig.3a, fifth on Fig.4a. Transitions exist between the basis functions that correspond to all of them. However, only the tree transitions shown on the diagrams are "elementary", i.e. correspond to the "transplanting" of one twig to a neighboring branch. All other transitions between bases are obtained by composing the elementary ones and making use of the O(3) overlap functions for transitions that are inside an O(3) subgroup of O(4).

1. Contractions of Clebsch-Gordan coefficients.

The tree on the left-hand side of Fig.2a corresponds to the subgroup chain O(4)⊃O(2)⊗O(2), as indicated on Fig.2c. The one on the right-hand side corresponds to the chain O(4)⊃O(3)⊃O(2).

The interbases expansions no longer correspond to a rotation of the sphere, but to a recoupling of some of the angular momenta involved. The overlap functions are expressed [1] in terms of Clebsch-Gordan coefficients of the O(3) group and we have

$$\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \sum_{l=|m|}^{J} (i)^{l-|m|}(-1)^{\frac{J-|m|-n}{2}} C^{l,|m|}_{\frac{J}{2},\frac{|m|+n}{2};\frac{J}{2},\frac{|m|-n}{2}} \Psi_{Jlm}(\theta_1',\theta_2',\theta_3), \qquad (3.1)$$

where

$$\begin{aligned} u_0 &= R\cos\theta_1\cos\theta_2 = R\cos\theta_1' \\ u_1 &= R\cos\theta_1\sin\theta_2 = R\sin\theta_1'\cos\theta_2' \\ u_2 &= R\sin\theta_1\cos\theta_3 = R\sin\theta_1'\sin\theta_2'\cos\theta_3 \\ u_3 &= R\sin\theta_1\sin\theta_3 = R\sin\theta_1'\sin\theta_2'\sin\theta_3, \end{aligned} \qquad (3.2)$$

$C^{l,\gamma}_{a,\alpha;b,\beta}$ - are the Clebsch-Gordan coefficients for the O(3) group. The corresponding hyperspherical functions have the form:

$$\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \frac{\sqrt{2J+2}}{2\pi}\sqrt{\frac{(\frac{J+|m|+|n|}{2})!(\frac{J-|m|-|n|}{2})!}{(\frac{J+|m|-|n|}{2})!(\frac{J-|m|+|n|}{2})!}}\, e^{in\theta_2}e^{im\theta_3}$$

$$\times(\sin\theta_1)^{|m|}(\cos\theta_1)^{|n|}P^{(|m|,|n|)}_{\frac{J-|m|-|n|}{2}}(\cos 2\theta_1), \qquad (3.3)$$

$$\Psi_{Jlm}(\theta_1',\theta_2',\theta_3) = \frac{\sqrt{(2J+1)(J+l+1)!(J-l)!}}{2^{l+1}\Gamma(J+\frac{3}{2})}$$

$$\times(\sin\theta_1')^l P^{(l+\frac{1}{2},l+\frac{1}{2})}_{J-l}(\cos\theta_1')\, Y_{lm}(\theta_2',\theta_3), \qquad (3.4)$$

where $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials. We again make use of the Beltrami coordinates (2.2) (with n=3). In the contraction limit $R\to\infty$ and

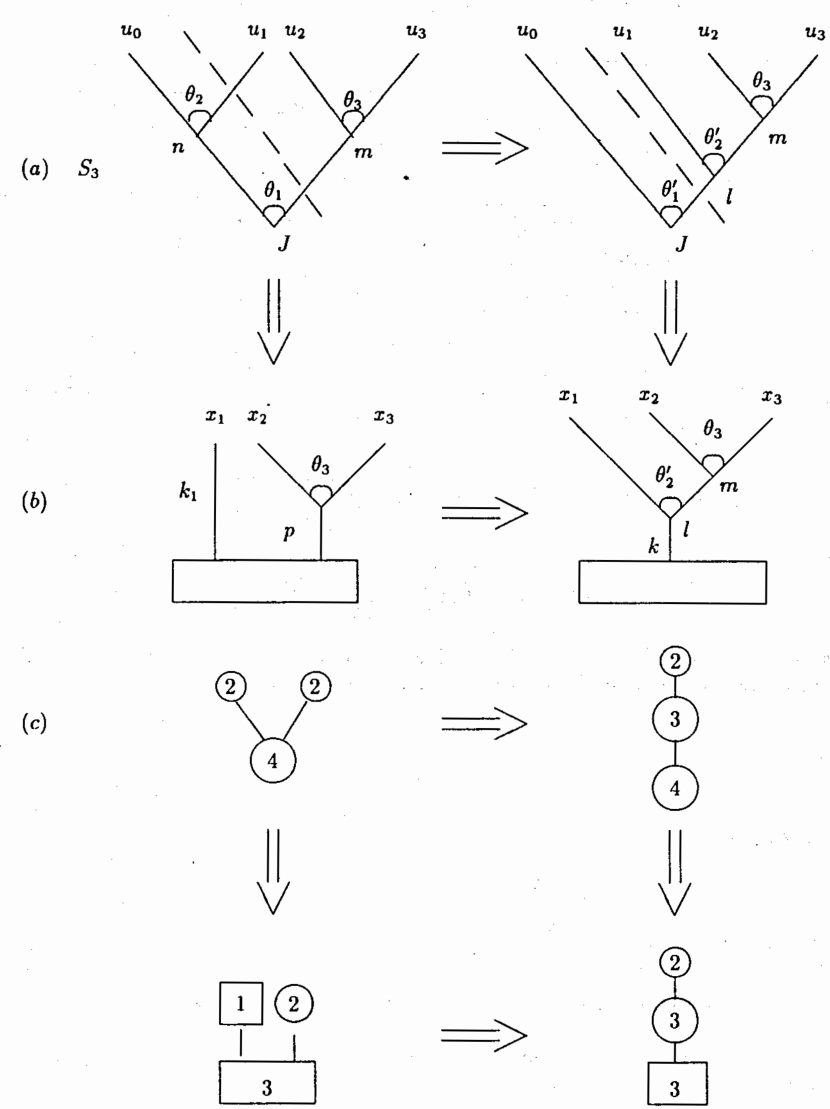
$$\theta_1' \to \frac{r}{R}, \quad \theta_1 \to \frac{\rho}{R}, \quad \theta_2 \to \frac{x_1}{R}, \quad J \sim kR, \quad n \sim k_1 R,$$

where $r=\sqrt{x_1^2+\rho^2}=\sqrt{x_1^2+x_2^2+x_3^2}$, $k=\sqrt{k_1^2+p^2}=\sqrt{k_1^2+k_2^2+k_3^2}$. We have [2]

$$\lim_{R\to\infty}\frac{1}{\sqrt{R}}\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \Phi_{kk_1m}(x_1,\rho,\theta_3) = \sqrt{\frac{k}{\pi}}\, J_{|m|}(p\rho)e^{ik_1x_1}\frac{e^{im\theta_3}}{\sqrt{2\pi}}, \qquad (3.5)$$

*Fig.*2 Elementary interbases expansions contracted from $O(4)$ to $E(3)$.

Contractions of rotation matrices.



and

$$\lim_{R\to\infty} \frac{1}{R} \Psi_{Jlm}(\theta_1', \theta_2', \theta_3) = \Phi_{klm}(r, \theta_2', \theta_3) = \sqrt{\frac{k}{r}}\, J_{l+\frac{1}{2}}(kr)\, Y_{lm}(\theta_2', \theta_3). \tag{3.6}$$

We take the Clebsch-Gordan coefficients in the form

$$C^{l,|m|}_{\frac{J}{2},\frac{|m|+n}{2};\frac{J}{2},\frac{|m|-n}{2}} = (-1)^{\frac{J-|m|-n}{2}} \frac{(J)!}{(|m|)!} \sqrt{\frac{(2l+1)(l+|m|)!}{(J-l)!(J+l+1)!(l-|m|)!}}$$

$$\times \sqrt{\frac{(\frac{J+|m|-|n|}{2})!(\frac{J-|m|+|n|}{2})!}{(\frac{J+|m|+|n|}{2})!(\frac{J-|m|-|n|}{2})!}}\; {}_3F_2 \left\{ \begin{matrix} -\frac{J-n-|m|}{2},\ -l,\ l+1, \\ -J,\ |m|+1, \end{matrix} \Big| 1 \right\}. \tag{3.7}$$

In the contraction limit $R \to \infty$, we get

$$\lim_{R\to\infty} \sqrt{R}(-1)^{\frac{J-|m|-n}{2}} C^{l,|m|}_{\frac{J}{2},\frac{|m|+|n|}{2};\frac{J}{2},\frac{|m|-|n|}{2}} = W^l_{k|m|}(\cos\phi) = \sqrt{\frac{(2l+1)(l+|m|)!}{k(l-|m|)!}}\frac{(\sin\phi)^{|m|}}{2^{|m|}|m|!}$$

$$\times\, {}_2F_1\left(-l+|m|, l+|m|+1;\ |m|+1;\ \frac{1-\cos\phi}{2}\right) = \sqrt{\frac{2}{k}}\, \mathcal{P}_l^{|m|}(\cos\phi), \tag{3.8}$$

where

$$\mathcal{P}_l^{|m|}(x) = \sqrt{\frac{(2l+1)(l-|m|)!}{2(l+|m|)!}}\, P_l^{|m|}(x)$$

are the orthonormalized Legendre polynomials and $\cos\phi = p/k$. Thus the interbases expansion (3.1) transforms to the expansion between the cylindrical and spherical bases for the Helmholtz equation

$$\Phi_{kk_1m}(x_1, \rho, \theta_3) = \sum_{l=|m|}^{\infty} W^l_{k|m|}(\cos\phi)\, \Phi_{klm}(r, \theta_2', \theta_3). \tag{3.9}$$

We use the formula

$$\int_0^\pi W^l_{k|m|}(\cos\phi)\, W^{*l'}_{k|m|}(\cos\phi)\, \sin\phi d\phi = 2\,\delta_{l,l'} \tag{3.10}$$

to obtain the inverse expansion

$$\Phi_{klm}(r, \theta_2', \theta_3) = \frac{1}{2}\int_0^\pi W^{*l}_{k,|m|}(\cos\phi)\Phi_{kk_1m}(x_1, \rho, \theta_3)\sin\phi\, d\phi. \tag{3.11}$$

Putting the functions (3.5)-(3.6) and interbases coefficients (3.8) into the expansions (3.9) and (3.11), we obtain

$$\frac{1}{\sqrt{2\pi}} e^{ikr\cos\phi\cos\theta_2'} J_{|m|}(kr\sin\phi\sin\theta_2') = \sum_{l=|m|}^{\infty} (i)^{l+m} \frac{1}{\sqrt{kr}} J_{l+\frac{1}{2}}(kr)\, \mathcal{P}_l^{|m|}(\cos\phi)\, \mathcal{P}_l^{|m|}(\cos\theta_2') \tag{3.12}$$

$$\frac{1}{\sqrt{kr}} J_{l+\frac{1}{2}}(kr)\mathcal{P}_l^{|m|}(\cos\theta_2') = \frac{(-i)^{l+m}}{\sqrt{2\pi}} \int_0^\pi e^{ikr\cos\phi\cos\theta_2'} J_{|m|}(kr\sin\phi\sin\theta_2')\mathcal{P}_l^{|m|}(\cos\phi)\sin\phi\, d\phi$$

The last two expansions coincide with well known formulas in the theory of the Bessel functions [20].

2. Contraction of Racah coefficients

In this case both trees correspond to isomorphic subgroup chains $O(4)\supset O(3)\supset O(2)$. The twig leading to the Cartesian coordinates (u_1, u_2) is transplanted to the neighbouring branch, so an $O(2)$ subgroup is moved from the $O(3)$ subgroup (012) to the (123) one. In the contraction the (012) subgroup is destroyed, the (123) one survives (see the "cut" lines on Fig.3a).

The $O(4)$ interbases expansion in this case is

$$\Psi_{Jnm}(\theta_1, \theta_2, \theta_3) = \sum_{l=|m|}^{J} T^l_{Jnm} \Psi_{Jlm}(\theta_1', \theta_2', \theta_3), \tag{3.13}$$

where

$$\begin{aligned} u_0 &= R\cos\theta_1\cos\theta_2 &&= R\cos\theta_1' \\ u_1 &= R\cos\theta_1\sin\theta_2\cos\theta_3 &&= R\sin\theta_1'\cos\theta_2'\cos\theta_3 \\ u_2 &= R\cos\theta_1\sin\theta_2\sin\theta_3 &&= R\sin\theta_1'\cos\theta_2'\sin\theta_3 \\ u_3 &= R\sin\theta_1 &&= R\sin\theta_1'\sin\theta_2', \end{aligned} \tag{3.14}$$

[see Fig.3(a)]. The hyperspherical wave functions corresponding to these two trees are

$$\Psi_{Jnm}(\theta_1, \theta_2, \theta_3) = \frac{\sqrt{(2J+1)(J+n+1)!(J-n)!}}{2^{n+1}\Gamma(J+\frac{3}{2})}$$

$$\times(\cos\theta_1)^n\, P_{J-n}^{(n+\frac{1}{2},n+\frac{1}{2})}(\sin\theta_1)\, Y_{nm}(\theta_2, \theta_3), \tag{3.15}$$

and

$$\Psi_{Jlm}(\theta_1', \theta_2', \theta_3) = \frac{\sqrt{(2J+1)(J+l+1)!(J-l)!}}{2^{l+1}\Gamma(J+\frac{3}{2})}$$

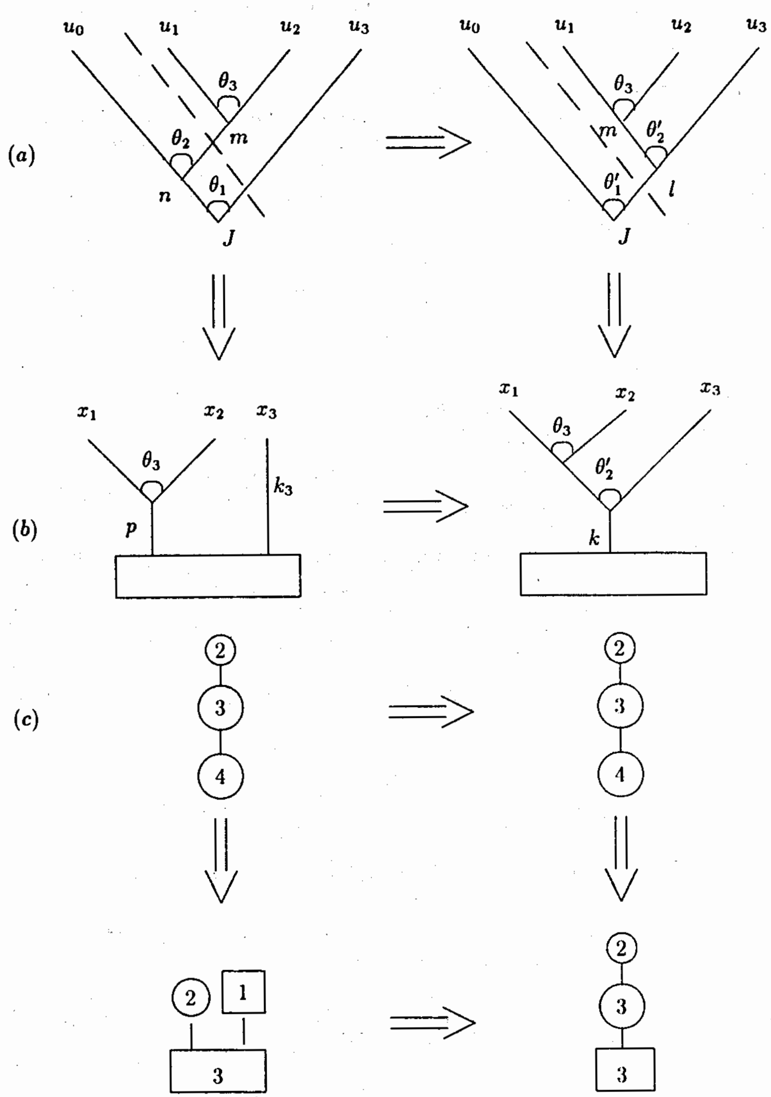
$$\times(\sin\theta_1')^l\, P_{J-l}^{(l+\frac{1}{2},l+\frac{1}{2})}(\cos\theta_1')\, Y_{lm}(\frac{\pi}{2}-\theta_2', \theta_3), \tag{3.16}$$

respectively. The interbases coefficients T^l_{Jnm} are expressed [7] in terms of Racah coefficients, in turn expressed in terms of the ${}_4F_3$ hypergeometric function:

$$T^l_{Jnm} = \left[\frac{1+(-1)^{J-n+l-m}}{2}\right] \sqrt{\frac{(2l+1)(2n+1)(n+|m|)!(l+|m|)!(J-l)!(J-n)!}{(n-|m|)!(l-|m|)!(J+n+1)!(J+l+1)!}}$$

$$\times(-1)^{\frac{J-n+l-m}{2}} \frac{2^{l+n-2m}}{|m|!} \frac{\Gamma(\frac{J-n-l+|m|}{2}+1)}{\Gamma(\frac{J+n+l-|m|}{2}+1)}\, {}_4F_3\left\{ \begin{matrix} -\frac{n-|m|}{2},\ -\frac{n-|m|-1}{2},\ -\frac{l-|m|}{2},\ -\frac{l-|m|-1}{2} \\ |m|+1,\ -\frac{J+n+l-|m|}{2},\ \frac{J-n-l+|m|}{2}+1 \end{matrix} \Big| 1 \right\}.$$

*Fig.*3 Elementary interbases expansions contracted from $O(4)$ to $E(3)$.
Contractions of Racah coefficients.



In the contraction limit $R \to \infty$ and

$$\theta_1 \sim \frac{x_3}{R}, \qquad \theta_2 \sim \frac{\rho}{R}, \qquad \theta_1' \sim \frac{r}{R}, \qquad J \sim kR, \qquad n \sim pR,$$

where $r = \sqrt{\rho^2 + x_3^2}$ and $k = \sqrt{p^2 + k_3^2}$, we obtain [2]

$$\lim_{R\to\infty} \frac{(-1)^{-\frac{J-n}{2}}}{\sqrt{R}} \Psi_{Jnm}(\theta_1, \theta_2, \theta_3) = \Phi_{kpm}(\rho, x_3, \theta_3)$$

$$= \sqrt{k}\, J_m(p\rho)\, \frac{e^{im\theta_3}}{\pi} \begin{cases} \cos k_3 x_3, & (J-n) - \text{even} \\ -i \sin k_3 x_3, & (J-n) - \text{odd} \end{cases}$$

$$\lim_{R\to\infty} \frac{1}{R} \Psi_{Jlm}(\theta_1', \theta_2', \theta_3) = \Phi_{klm}(r, \theta_2', \theta_3) = \sqrt{\frac{k}{r}}\, J_{l+\frac{1}{2}}(kr)\, Y_{lm}(\frac{\pi}{2} - \theta_2', \theta_3).$$

For the contractions of interbases coefficients T^l_{Jnm} we get

$$\lim_{R\to\infty} (-1)^{-\frac{J-n}{2}} \sqrt{R}\, T^l_{Jnm} = W^l_{k|m|}(\cos\phi) = \frac{(-1)^{\frac{l-m}{2}}}{|m|!} \sqrt{\frac{(2l+1)(l+|m|)!}{2k(l-|m|)!}}$$

$$\times (\cot\phi)^{|m|+\frac{1}{2}} (\sin\phi)^l\, {}_2F_1\left(-\frac{l-|m|}{2}, -\frac{l-|m|-1}{2}; |m|+1; -\cot^2\phi\right)$$

$$= (-1)^{\frac{l+|m|}{2}} \sqrt{\frac{2}{k}}\, (\cot\phi)^{\frac{1}{2}}\, \mathcal{P}_l^{|m|}(\sin\phi), \tag{3.17}$$

where $\cos\phi = p/k$. The interbases expansion in eq. (3.13) transforms to the expansion between the cylindrical and spherical bases for the Helmholtz equation

$$\frac{1}{\sqrt{\pi}} J_m(p\rho) \left\{ \begin{array}{c} \cos k_3 x_3 \\ \\ -i \sin k_3 x_3 \end{array} \right\} = \sum_l \frac{(-1)^{\frac{l-|m|}{2}}}{\sqrt{kr}} J_{l+\frac{1}{2}}(kr)\, (\cot\phi)^{\frac{1}{2}}\, \mathcal{P}_l^{|m|}(\sin\phi)\, \mathcal{P}_l^{|m|}(\sin\theta_2'), \tag{3.18}$$

where the top line on the left-hand side corresponds to a summation over $l = |m|, |m|+2, |m|+4, \ldots$ and the bottom one to a summation over $l = |m|+1, |m|+3, \ldots$ on the right-hand side. The E(3) expansion (3.18) is related to the expansion (3.12) by the substitution $k_1 = k\cos\phi \to k_3$, $x_1 = r\cos\theta_2' \to x_3$, $\phi \to \pi/2 - \phi$ and $\theta_2' \to \pi/2 - \theta_2'$.

3. Further contractions of Clebsch-Gordan coefficients

As on Fig.2, the two O(4) trees on Fig.4 correspond to two different subgroup reductions: O(4)⊃O(3)⊃O(2) on the left and O(4)⊃ O(2)⊗O(2) on the right. Since a recoupling of momenta is involved the overlap functions are again expressed in terms of O(3) Clebsch-Gordan coefficients [7]. The corresponding interbases expansion is

$$\Psi_{Jlm}(\theta_1, \theta_2, \theta_3) = \sum_{n=-(J-|m|)}^{J-|m|} (-i)^{l-|m|}\, C^{l,|m|}_{\frac{J}{2}, \frac{|m|+n}{2}; \frac{J}{2}, \frac{|m|-n}{2}}\, \Psi_{Jmn}(\theta_1', \theta_2', \theta_3), \tag{3.19}$$

where n has the same parity as $(J-|m|)$ and

$$
\begin{aligned}
u_0 &= R\cos\theta_1\cos\theta_2\cos\theta_3 = R\cos\theta_1'\cos\theta_3 \\
u_1 &= R\cos\theta_1\cos\theta_2\sin\theta_3 = R\cos\theta_1'\sin\theta_3 \\
u_2 &= R\cos\theta_1\sin\theta_2 \quad\;\; = R\sin\theta_1'\cos\theta_2' \\
u_3 &= R\sin\theta_1 \qquad\qquad\;\; = R\sin\theta_1'\sin\theta_2',
\end{aligned}
\tag{3.20}
$$

(see Fig.4). The corresponding hyperspherical function is:

$$
\Psi_{Jlm}(\theta_1,\theta_2,\theta_3) = \frac{\sqrt{(2J+1)(J+l+1)!(J-l)!}}{2^{l+1}\Gamma(J+\frac{3}{2})}
\times(\cos\theta_1)^l\, P_{J-l}^{(l+\frac{1}{2},l+\frac{1}{2})}(\sin\theta_1)\, Y_{lm}\left(\frac{\pi}{2}-\theta_2,\theta_3\right),
$$

and the wave function $\Psi_{Jmn}(\theta_1',\theta_2',\theta_3)$ is given by eq. (3.3) (with n replaced by m).

The contraction in this case (see Fig.4 and eq. (3.22) below) will involve 3 quantum numbers J, l and m. Eq. (3.7) expressing Clebsch-Gordan coefficients in terms of the ${}_3F_2$ function is not convenient for taking this limit. Instead, we use the following integral representation [7]

$$
C^{l,|m|}_{\frac{J}{2},\frac{|m|+n}{2};\frac{J}{2},\frac{|m|-n}{2}} = (i)^{l-|m|}(-1)^{\frac{J-|m|-n}{2}} \left\{\frac{(l+|m|)!(\frac{J-|m|-|n|}{2})!(\frac{J-|m|+|n|}{2})!}{(l-|m|)!(\frac{J+|m|-|n|}{2})!(\frac{J+|m|+|n|}{2})!}\right\}^{1/2}
\times\frac{\sqrt{(2l+1)(J-l)!(J+l+1)!}}{2^{l+|m|+2}\Gamma(J+3/2)}\frac{1}{\sqrt{\pi}}\int_0^{2\pi}(\sin\phi)^{l-|m|}\,P_{J-l}^{(l+\frac{1}{2},l+\frac{1}{2})}(\cos\phi)e^{-in\phi}d\phi
\tag{3.21}
$$

and the formulas [20]

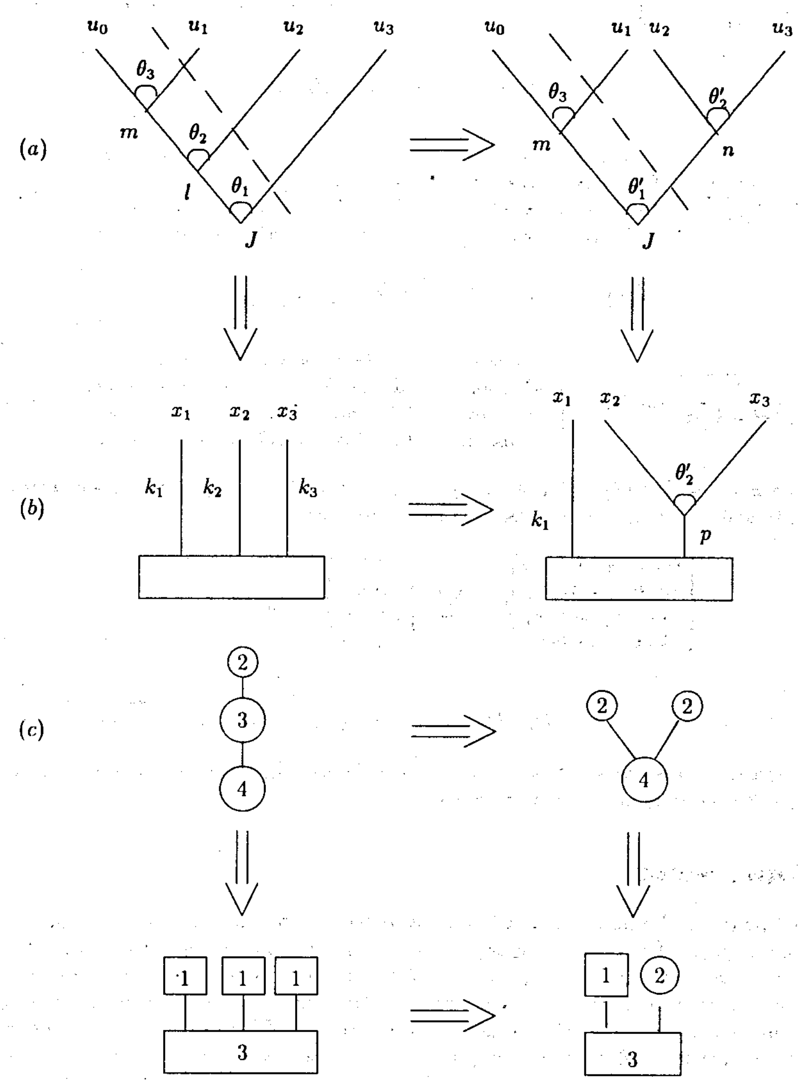
$$
P_n^{(\alpha,\alpha)}(\cos\phi) = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)n!}
\times\begin{cases} {}_2F_1(-\frac{n}{2},\frac{n+1}{2}+\alpha;\alpha+1;\sin^2\phi), & n-\text{even}, \\ \cos\phi\,{}_2F_1(-\frac{n-1}{2},\frac{n}{2}+\alpha+1;\alpha+1;\sin^2\phi), & n-\text{odd}. \end{cases}
$$

After integrating over ϕ, we obtain a representation of the Clebsch-Gordan coefficients in terms of the hypergeometrical function ${}_4F_3$ (of argument 1):

$$
C^{l,|m|}_{\frac{J}{2},\frac{|m|+n}{2};\frac{J}{2},\frac{|m|-n}{2}} = (i)^{l-|m|}(-1)^{\frac{J-|m|}{2}-n}\frac{\sqrt{2l+1}}{2^{2l}}\sqrt{\frac{(J+l+1)!(\frac{J-|m|-|n|}{2})!(\frac{J+|m|-|n|}{2})!}{(J-l)!(\frac{J-|m|+|n|}{2})!(\frac{J+|m|+|n|}{2})!}}
$$

$$
\times\begin{cases}
\dfrac{\sqrt{(l-|m|)!(l+|m|)!}}{\Gamma(1+\frac{l+|m|-n}{2})\Gamma(1+\frac{l-|m|-n}{2})}\dfrac{\Gamma(\frac{J-l+1}{2})}{\Gamma(\frac{J+l+3}{2})}\,{}_4F_3\left(\left.\begin{matrix}-\frac{n}{2},-\frac{n-1}{2},\frac{J+l}{2}+1,-\frac{J-l}{2};\\ \frac{1}{2},1+\frac{l-|m|-n}{2},1+\frac{l+|m|-n}{2}\end{matrix}\right|1\right), & (J-l)-\text{even}, \\[2ex]
\dfrac{-in\sqrt{(l-|m|)!(l+|m|)!}}{\Gamma(\frac{l+|m|-n+3}{2})\Gamma(1+\frac{l-|m|-n+3}{2})}\dfrac{\Gamma(\frac{J-l}{2})}{\Gamma(\frac{J+l+2}{2})}\,{}_4F_3\left(\left.\begin{matrix}-\frac{n-1}{2},-\frac{n-2}{2},\frac{J+l+3}{2},-\frac{J-l-1}{2};\\ \frac{3}{2},\frac{l-|m|-n+3}{2},\frac{l+|m|-n+3}{2}\end{matrix}\right|1\right), & (J-l)-\text{odd}.
\end{cases}
$$

*Fig.*4 Elementary interbases expansions contracted from $O(4)$ to $E(3)$.
Contractions of Clebsch-Gordan coefficients.



(To our knowledge, this expression is new). In the contraction limit $R \to \infty$ and

$$\theta_1 \sim \frac{x_3}{R}, \; \theta_2 \sim \frac{x_2}{R}, \; \theta_3 \sim \frac{x_1}{R}, \quad \theta'_1 \sim \frac{\rho}{R}, \quad J \sim kR, \quad l \sim pR, \quad m \sim k_1 R, \tag{3.22}$$

we get

$$\lim_{R\to\infty} (-1)^{-\frac{J-|m|}{2}} \Psi_{Jlm}(\theta_1, \theta_2, \theta_3)$$

$$= \sqrt{\frac{2kp}{\pi k_2 k_3}} \frac{e^{ik_1x_1}}{\pi} \begin{cases} \cos k_2x_2 \, \cos k_3x_3 & (J-|m|) - \text{even}, \quad (l-|m|) - \text{even}, \\ -i\sin k_2x_2 \, \cos k_3x_3 & (J-|m|) - \text{odd}, \quad (l-|m|) - \text{even}, \\ -i\cos k_2x_2 \, \sin k_3x_3 & (J-|m|) - \text{even}, \quad (l-|m|) - \text{odd}, \\ -\sin k_2x_2 \, \sin k_3x_3 & (J-|m|) - \text{odd}, \quad (l-|m|) - \text{odd}, \end{cases} \tag{3.23}$$

$$\lim_{R\to\infty} (-i)^{l-|m|} (-1)^{-\frac{J-|m|}{2}} \sqrt{R}\, C^{l,|m|}_{\frac{J}{2},\frac{|m|-n}{2};\frac{J}{2},\frac{|m|+n}{2}}$$

$$= \sqrt{\frac{8p}{(k^2-k_1^2)\pi}} (\sin 2\phi)^{-\frac{1}{2}} \begin{cases} \cos n\phi, & (J-|m|) - \text{even}, \quad (l-|m|) - \text{even}, \\ -i\sin n\phi, & (J-|m|) - \text{odd}, \quad (l-|m|) - \text{even}, \\ -i\sin n\phi, & (J-|m|) - \text{even}, \quad (l-|m|) - \text{odd}, \\ -\cos n\phi, & (J-|m|) - \text{odd}, \quad (l-|m|) - \text{odd}, \end{cases} \tag{3.24}$$

where $\cos\phi = (p^2 - k_1^2)/(k^2 - k_1^2)$ and $k^2 = p^2 + k_3^2 = k_1^2 + k_2^2 + k_3^2$. Substituting the formulas (3.5), (3.23) and (3.24) into the expansion (3.19) we obtain

$$\begin{Bmatrix} \cos k_2x_2 \, \cos k_3x_3 \\ \sin k_2x_2 \, \cos k_3x_3 \\ \cos k_2x_2 \, \sin k_3x_3 \\ \sin k_2x_2 \, \sin k_3x_3 \end{Bmatrix} = \sum_{n=-\infty}^{\infty} \begin{Bmatrix} \cos n\phi \\ \sin n\phi \\ \sin n\phi \\ \cos n\phi \end{Bmatrix} J_{|n|}(q\rho) e^{in\theta'_2}.$$

where

$$\tan\theta'_2 = \frac{x_3}{x_2}, \qquad q^2 = k_2^2 + k_3^2, \qquad \rho^2 = x_2^2 + x_3^2, \qquad \cos^2\phi = \frac{k_2^2}{k_2^2 + k_3^2}.$$

Thus the interbases expansion (3.19) contracts to the expansion between Cartesian and cylindrical bases for the Helmholtz equation on E_3.

4 Conclusions

The "method of trees" was introduced [4,5] in order to describe the separation of variables on homogeneous spaces of compact Lie groups, more specifically O(n+1) and SU(n). The "trees" turned out to be related to subgroup chains and it is useful to complement the tree diagrams by subgroup diagrams [2]. Moreover, the method of trees has been extended in a very simple and straighforward way to Euclidean spaces [2], where instead of trees we have clusters of trees. The S_n tree diagrams and E_n cluster diagrams are very helpful in the study of contractions. They tell us, at least for subgroup type coordinates, how coordinates on S_n and E_n can be related by contractions.

The contribution of this paper is to treat contractions of interbases expansions and hence of overlap function. Overlap functions for different bases corresponding to isomorphic subgroup

chains involve rotation matrices. If the subgroup chains are not isomorphic, the overlap functions will be expressed in terms of Clebsch-Gordan coefficients, Racah coefficients, or higher order recoupling coefficients. For all of these we obtain asymptotic expressions.

For O(3) the contraction breaks the equivalence of the two types of subgroup chains. One O(3)⊃O(2) basis contracts to an E(2)⊃O(2) basis, the other to an E(2)⊃E(1)⊗E(1) one. Thus we obtain the well known relations between plane and cylindrical waves in E_2.

For O(4) the contraction provides relations between spherical and cylindrical bases and also between cylindrical and Cartesian ones. The interbases expansions relating E(3) Cartesian and spherical bases are obtained by composing the elementary transitions.

Work is in progress in two directions. The first is contraction of interbases expansions for arbitrary values of n. The other is an extension to spaces of negative constant curvature, generalizing the results on the two-dimensional hyperboloid, obtained earlier [21].

In this article we restricted ourselves to subgroup type coordinates only and moreover to the lowest dimensional spheres S_2 and S_3. Two earlier articles were devoted to contractions of separated basis functions that correspond to nonsubgroup type coordinates, in particular elliptic coordinates on S_2 and on the hyperboloid H_2 [1, 21]. It would also be possible to obtain interbases expansions for other types of bases, though so far this has not been done.

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