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THE «FREE» QUANTUM BROWNIAN PARTICLE AS A NON-FOCK LINEAR BOSON SYSTEM

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INTRODUCTION

Quantum Langevin equation has many applications in quantum physics. In particular, the Brownian motion of a quantum oscillator attracted widespread attention (e.g.[1—5]). The present paper treats of the quantization scheme for the «free» quantum Brownian particle. This model may be formally thought to be a limiting case of the harmonic oscillator with friction. However it is complicated by infrared divergences, so the construction of the so-called physical representation, or «dressing-up» of a *-algebra of the system needs a modification of the standard quantization method (whose immediate application gives rise to quantum fields in indefinite metric).

In the model of the «free» Brownian particle, the Langevin equation takes the form

$$m\cdot\partial_t q(t)=p(t),\ \partial_t p(t)=-\gamma\cdot p(t)+\sqrt{2\gamma m}\cdot\xi(t), \tag{1}$$

where $q(t)$ and $p(t)$ are the coordinate and the momentum of the particle with mass $m, \gamma\cdot m>0$ is a viscous friction coefficient, $\sqrt{2\gamma m}\cdot\xi(t)$ is a fluctuating force whose stationary state represents the white noise. Quantization needs a suitable Hamiltonian realization of the equations. The heat bath in the present case (as with a harmonically bound particle) can be modelled by a semi-infinite string [3,4] stretched along the positive x axis and allowed one to oscillate in the y direction. A particle, playing the role of the Brownian particle and attached to the string at $x=0$, may move along the y axis. The string configuration at a time moment t is given by the expression

$$y=(\gamma m)^{-1/2}\ \varphi(t,x). \tag{2}$$

The particle coordinate on the y axis at the time moment t is

$$q(t)=(\gamma m)^{-1/2}\varphi(t,0). \tag{3}$$

Since the system as a whole is a quantum dynamical system, the terminology of quantum fields and quantum processes is relevant here. The connection with the terminology of random (quantum) processes is provided by the fact that dynamical variables of the

Brownian particle as of an open system and the force acting on it from the heat bath are random (possibly noncommuting) variables in any state of a given representation.

The Langrangian of the system is a sum of the Langragian of the free semi-infinite string and the kinetic energy $\frac{1}{2}m(\partial_t q)^2 = \frac{1}{2\gamma}(\partial_t \varphi)^2|_{x=0}$ of the particle:

$$L = \frac{1}{2}\int\left((\partial_t\varphi)^2 - (\partial_x\varphi)^2\right)\theta(x)dx + \frac{1}{2\gamma}\int(\partial_t\varphi)^2\delta(x)dx. \quad (4)$$

(As usual, $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$.) Considering the field configuration $\varphi(t,x)$ as being a «canonical coordinate», we find the conjugate «canonical momentum»

$$j(t,x) = \frac{\delta L}{\delta \partial_t \varphi} = \partial_t\varphi\cdot\left(\theta(x) + \gamma^{-1}\cdot\delta(x)\right). \quad (5)$$

The Hamiltonian of the system is

$$H = \int i(t,x)\cdot\partial_t\varphi(t,x)dx - L = \frac{1}{2}\int\left((\partial_t\varphi)^2 + (\partial_x\varphi)^2\right)\theta(x)dx + \frac{1}{2\gamma}\int(\partial_t\varphi)^2\delta(x)dx, \quad (6)$$

and the equation of motion $\partial_t i + \delta L/\delta\varphi = 0$ means

$$\left(\partial_t^2 - \partial_x^2\right)\varphi\cdot\theta(x) + \gamma^{-1}\cdot\left(\partial_t^2 - \partial_x\right)\varphi\cdot\delta(x) = 0. \quad (7)$$

A consistent treatment of this equation is possible if the field $\varphi(t,x)$ is meant to depend in a smooth (or C^∞) fashion on the spatial variable x as on a parameter, being a usual or generalized function in the temporal variable t. In such a case the equation of motion (7) reduces to the wave equation

$$\left(\partial_t^2 - \partial_x^2\right)\varphi\cdot\theta(x) = 0 \quad (8)$$

and the boundary condition

$$\left(\partial_t^2 - \gamma\cdot\partial_x\right)\varphi|_{x=0} = 0. \quad (9)$$

According to (3), the physical meaning of (9) is that the string acts on the particle with the force $(\gamma/m)^{1/2}\cdot\partial_x\varphi(t,x)|_{x=0}$ along the 0_y axis.

Note that the Lagrangian (4) is invariant under the 1-parameter gauge group with gauge transformations of the first kind

$$\varphi(t,x) \to \varphi(t,x) + g, g \in \mathbb{R}. \quad (10)$$

In this case $i(t,x)$ is the temporal component of the Noether «conserved» current:

$$\partial_t i(t,x) + \partial_x k(t,x) = 0, \text{ where } k(t,x) = -\partial_x\varphi\cdot\theta(x). \quad (11)$$

The corresponding charge is

$$\theta = \int i(t,x)dx = \int \partial_t\varphi\cdot(\theta(x) + \gamma^{-1}\cdot\delta(x))dx. \quad (12)$$

It stands to reason that the charge along with the Hamiltonian is an integral of motion.

The above treatment of the spatial variable x as a parameter entails a formulation of the canonical formalism in terms of two-times Poisson brackets. Here the use can be made of the fact that the system under consideration is a linear boson one, i.e., that the Langrangian and the Hamiltonian are quadratic functionals of the field. In this case the phase space of the system, identified with a set of «classical» (or c-number) solutions of the equations of motion, is a symplectic space [6]. A symplectic form δ (i.e., a bilinear nondegenerate skew-symmetric form) assigns to any pair φ_1, φ_2 of «classical» solutions of equations of motion an expression

$$\delta(\varphi_1, \varphi_2) = \int (i_1(t,x)\cdot\varphi_2(t,x) - i_2(t,x)\cdot\varphi_1(t,x))dx.$$

With regard to (5), this can be rewritten as

$$\delta(\varphi_1, \varphi_2) = \int (\partial_t\varphi_1\cdot\varphi_2 - \varphi_1\cdot\partial_t\varphi_2)\cdot(\theta(x) + \gamma^{-1}\cdot\delta(x))\cdot dx. \quad (13)$$

Then the Poisson brackets of the «classical» field $\varphi(t,x)$ can be found from the relations

$$\{\delta(\varphi, \varphi_1), \delta(\varphi, \varphi_2)\} = \delta(\varphi_1, \varphi_2). \quad (14)$$

In section 1 we specify the space of «classical» solutions of the equations of motion (8), (9). This turned out to be isomorphic with the phase space of the so-called left component $\varphi^L(t)$ of the free real scalar massless field in the two-dimensional Minkowsky space-time $\mathbb{R}^2$. In this way the equivalence of the given model (as a linear boson system) is established with the model of the free semi-infinite string, whose Lagrangian is gained from (4) in the limit $\gamma \to \infty$. We find two-times Poisson brackets of the «classical» field $\varphi(t,x)$.

In section 2 we construct representations of the quantized field $\varphi(t,x)$. The vacuum two-point function of the field $\varphi(t,x)$ possesses infrared divergences. In the standard treatment of this situation the field $\varphi(t,x)$ is termed «unphysical», and its realization in a pseudo-Hilbert space with indefinite metric may be tolerated. There are gauge invariant quantities, referred to as physical variables, which are generated by the components $i(t,x), k(t,x)$ of the Noether current or, equivalently, by the partial derivatives $\partial_t\varphi, \partial_x\varphi$ of the field. They possess the Fock vacuum state and also (Gibbs) stationary states in case of nonzero temperature T. In this sense the field $\varphi(t,x)$ is considered as a non-Fock linear boson system. The conventional treatment is not wholly satisfactory, since the coordinate $q(t)$ of the Brownian particle turns out to be an unphysical variable and hence only the second of the Langevin equations (1) acquires a physical meaning. Likewise the canonical commutation relations between the coordinate and the momentum of the Brownian particle are devoid of physical meaning. An alternative viewpoint [7] which is adopted here enables one to treat the field $\varphi(t,x)$ as a physical variable in the framework of degenerate Fock linear boson systems. Physical representations of the field algebra are constructed on account of gauge invariance from the vacuum or temperature (for $T>0$) generalized states which in turn are induced by the Fock vacuum or Gibbs states of the algebra of gauge invariant quantities.

1. THE PHASE SPACE OF THE SYSTEM

A phase space of the system «particle+string» is the set χ of all real (or «classical») C^∞ solutions $\varphi(t,x)$ of the equations of motion (8), (9) in $\mathbb{R}\times\overline{\mathbb{R}}+$ * with $\varphi(t,x)$ and $\partial_t\varphi(t,x)$ being in the Schwartz space $\mathcal{S}_r(\overline{\mathbb{R}}_+)$ of rapidly decreasing test functions in the spatial variable x for all $t\in R$. The phase space is a symplectic space with the symplectic form (13) where $\varphi_1,\varphi_2\in\chi$. The right–hand side of (13) does not depend on t since

$$\partial_t\delta(\varphi_1,\varphi_2)=\int\left(\partial_t^2\varphi_1\cdot\varphi_2-\varphi_1\cdot\partial_t^2\varphi_2\right)\cdot\left(\theta(x)+\gamma^{-1}\cdot\delta(x)\right)dx=$$
$$=\int\left(\partial_x^2\varphi_1\cdot\varphi_2-\varphi_1\cdot\partial_x^2\varphi_2\right)\theta(x)dx+\left(\partial_x\varphi_1\cdot\varphi_2-\varphi_1\cdot\partial_x\varphi_2\right)\Big|_{x=0}=0. \qquad (15)$$

The 1-parameter dynamical group acts on the field by time translations

$$\varphi(t,x)\to\varphi_\tau(t,x)=\varphi(t-\tau,x),\tau\in\mathbb{R}. \qquad (16)$$

These are symplectic transformations: $\delta(\varphi_{1,\tau},\varphi_{2,\tau})=\delta(\varphi_1,\varphi_2)$, since (13) is independent of t. They define the Hamiltonian of the linear boson system:

$$H=\frac{1}{2}\partial_\tau\delta(\varphi_\tau,\varphi)\Big|_{\tau=0}.$$

In terms of variables $t\pm x$ and partial derivatives $\partial^\pm$ with respect to these variables, the wave equation $\left(\partial_t^2-\partial_x^2\right)\varphi=0$ can be put in the form $\partial^+\partial^-\varphi=0$, which yields the general formula for solutions of the wave equation in $\mathbb{R}\times\overline{\mathbb{R}}_+$

$$\varphi(t,x)=\frac{1}{\sqrt{2}}\left(\varphi^L(t+x)+\varphi^R(t-x)\right). \qquad (17)$$

*The following notations are used: $\overline{\mathbb{R}}_+=\{x\in\mathbb{R}: x\geq 0\}$, $\overline{\mathbb{R}}_-=\{x\in\mathbb{R}\ x\leq 0\}$. $\mathcal{S}_r(\overline{\mathbb{R}}_\pm)$ and $\mathcal{S}_r(\mathbb{R})$ stand for subsets of all real test functions in the Schwartz spaces $\mathcal{S}(\overline{\mathbb{R}}_\pm)$ and $\mathcal{S}(\mathbb{R})$.

Here $\varphi^L(t)$ and $\varphi^R(t)$ are $\mathscr{C}^\infty$ function on $\mathbb{R}$, called the left and right components of field $\varphi(t,x)$, respectively. They are reproduced from the field uniquely up to a constant c. This is evident from the expressions

$$\varphi^{L,R}(t)=\frac{1}{\sqrt{2}}\left(\varphi(t,0)\mp\left(c+\int_0^t\partial_x\varphi(\tau,x)\Big|_{x=0}d\tau\right)\right), \qquad (18)$$

which follow from (17) after substitution $x=0$ into $\varphi(t,x)$ and $\partial_x\varphi(t,x)$. The arbitrary constant c will be specified further: $c=c(\varphi)$.*

Designating

$$\Phi(t)=\varphi(t,0), \qquad (19)$$

we show that the functions $\Phi(t)$, $\varphi^L(t)$, $\varphi^R(t)$ are antiderivatives of functions from $\mathcal{S}_\tau(\mathbb{R})$. Note that equation (6) implies $|\partial_t\Phi(t)|\leq(2\gamma H)^{1/2}$, hence $|\Phi(t)|$ is bounded by a first-degree polynomial in $|t|$, so that $\Phi(t)\in\mathcal{S}'(\mathbb{R})$. From (17) we find

$$\partial_t\varphi(t,x)\Big|_{t=0}=\frac{1}{\sqrt{2}}\left(\partial_x\varphi^L(x)-\partial_x\varphi^R(-x)\right),$$
$$\partial_x\varphi(0,x)=\frac{1}{\sqrt{2}}\left(\partial_x\varphi^L(x)+\partial_x\varphi^R(-x)\right). \qquad (20)$$

By condition, left-hand sides in (20) belong to $\mathcal{S}(\overline{\mathbb{R}}_+)$ hence restruction of $\partial_t\varphi^L(t)$ on $\overline{\mathbb{R}}_+$ and $\partial_t\varphi^R(t)$ on $\overline{\mathbb{R}}_-$ belong to $\mathcal{S}(\overline{\mathbb{R}}_+)$ and $\mathcal{S}(\overline{\mathbb{R}}_-)$, respectively. The boundary condition (9) can be written in either of the two forms

$$(\gamma+\partial_t)\partial_t\Phi(t)=\sqrt{2}\gamma\partial_t\varphi^L(t),$$
$$(\gamma-\partial_t)\partial_t\Phi(t)=\sqrt{2}\gamma\partial_t\varphi^R(t), \qquad (21)$$

hence

$$\left(\gamma^2-\partial_t^2\right)\partial_t\Phi(t)=\sqrt{2}\gamma(\gamma-\partial_t)\partial_t\varphi^L(t),$$
$$\left(\gamma^2-\partial_t^2\right)\partial_t\Phi(t)=\sqrt{2}\gamma(\gamma+\partial_t)\partial_t\varphi^R(t). \qquad (22)$$

*Although the «classical» fields $\varphi\in\mathscr{X}$ were originally defined on the set $\mathbb{R}\times\overline{\mathbb{R}}_+$, they are force-continued by formula (17) as $\mathscr{C}^\infty$solutions of the wave equation in $\mathbb{R}^2$. This suggests that the semi-infinite string is supplemented by a «virtual» semi-infinite string to form an infinite string, without adding new degrees of freedom. Similarly, the quantum field $\varphi(t,x)$ may be thought of as an operator-valued generalized function in $\mathbb{R}^2$ obeying the wave equation.

Since the restrictions of right-hand sides of (22) on $\overline{\mathbb{R}}_+$ and $\overline{\mathbb{R}}_-$, respectively, belong to $\mathscr{S}(\overline{\mathbb{R}}_+)$ and $\mathscr{S}(\overline{\mathbb{R}}_-)$, and since $\Phi(t)$ is a $\mathscr{C}^\infty$function by condition, we obtain that $\left(\gamma^2-\partial_t^2\right)\partial_t\Phi(t)$ belongs to $\mathscr{S}(\mathbb{R})$. From this it is inferred by applying Fourier transformation to the generalized function $\partial_t\Phi(t)\in\mathscr{S}'(\mathbb{R})$ that $\partial_t\Phi(t)$ belongs to $\mathscr{S}(\mathbb{R})$. Invoking (22) we obtain that $\partial_t\varphi^L(t)$ and $\partial_t\varphi^R(t)$ also belong to $\mathscr{S}(\mathbb{R})$.

This entails, in particular, the existence of limits $\Phi(\pm\infty)$, $\varphi^L(\pm\infty)$, $\varphi^R(\pm\infty)$, which may be subjected to relations

$$\Phi(-\infty)=\sqrt{2}\varphi^L(-\infty)=\sqrt{2}\varphi^R(-\infty). \tag{23}$$

The former of these equations deleates the arbitrariness of the constant c in (18) and the latter follows from equation (17). Now equations (21) can be rewritten in the form

$$(\gamma+\partial_t)\Phi(t)=\sqrt{2}\gamma\varphi^L(t), \qquad (\gamma-\partial_t)\Phi(t)=\sqrt{2}\gamma\varphi^R(t). \tag{24}$$

Invoking that $\partial_t\Phi(t)\in\mathscr{S}(\mathbb{R})$ we obtain also

$$\Phi(+\infty)=\sqrt{2}\varphi^L(+\infty)=\sqrt{2}\varphi^R(+\infty). \tag{25}$$

At last, equation (17) in the limit $x\to+\infty$ implies $\varphi^L(+\infty)+\varphi^R(-\infty)=0$, so in addition to (23), (25) we have

$$\Phi(-\infty)=-\Phi(+\infty), \varphi^L(-\infty)=-\varphi^L(+\infty), \varphi^R(-\infty)=-\varphi^R(+\infty). \tag{26}$$

These relations mean that all of the functions $\Phi(t), \varphi^L(t), \varphi^R(t)$, belong to the space $\varepsilon*\mathscr{S}_r(\mathbb{R})$, consisting of functions of the form $g(t)=(\varepsilon*f)(t)\equiv\int\varepsilon(t-\tau)f(\tau)d\tau$ for $f\in\mathscr{S}_r(\mathbb{R})$; as usual, $\varepsilon(t)=1$ for $t\geq0$ and $\varepsilon(t)=-1$ for $t<0$.

Define Fourier transformation by the formula

$$\overline{\Phi}(\lambda)=\int\Phi(t)\cdot\exp(i\lambda t)dt.$$

Now the relations (24) between the functions $\Phi(t), \varphi^L(t), \varphi^R(t)$ become

$$\overline{\varphi}^R(\lambda)=\frac{\gamma+i\lambda}{\gamma-i\lambda}\overline{\varphi}^L(\lambda), \overline{\Phi}(\lambda)=\frac{1}{\sqrt{2}}\left(\overline{\varphi}^L(\lambda)+\overline{\varphi}^R(\lambda)\right)=\frac{\sqrt{2}\cdot\gamma}{\gamma-i\lambda}\overline{\varphi}^L(\lambda). \tag{27}$$

Thus, we obtained isomorphisms $\Phi(t)\leftrightarrow\varphi^L(t)\leftrightarrow\varphi^R(t)$ in the space $\varepsilon*\mathscr{S}_r(\mathbb{R})$. Any of the functions $\Phi(t), \varphi^L(t), \varphi^R(t)$ can be used as a way of parametrization of elements of the phase space $\mathscr{X}$ of the system. It is convenient to choose the field $\varphi^L(t)$ as such parametrization. On account of formula (17), an expression for the symplectic

form in terms of φ_1^L, φ_2^L is obtained as t tends to $-\infty$ in the formula (13) (which makes nondegeneracy of the form σ explicit):

$$\sigma(\varphi_1,\varphi_2)=\frac{1}{2}\int\left(\partial_t\varphi_1^L(t)\cdot\varphi_2^L(t)-\varphi_1^L(t)\cdot\partial_t\varphi_2^L(t)\right)dt. \tag{28}$$

Similarly, formulas (6) and (12) yield the Hamiltonian

$$H=\frac{1}{2}\int\left(\partial_t\varphi^L(t)\right)^2dt \tag{29}$$

and the charge

$$Q=\frac{1}{\sqrt{2}}\int j^L(t)dt \tag{30}$$

in terms of $\varphi^L(t)$, where $j^L(t)=\partial_t\varphi^L(t)$ is a left current. Note that the dynamical groups act as before by translations of the variable t:

$$\varphi^L(t)\to\varphi_\tau^L(t)=\varphi^L(t-\tau),\ \tau\in\mathbb{R}. \tag{31}$$

These formulas establish an equivalence between the present linear boson system and the model of the free semi-infinite string described by the field

$$\varphi^{(0)}(t,x)=\frac{1}{\sqrt{2}}\left(\varphi^L(t+x)+\varphi^L(t-x)\right).$$

We are now ready to derive Poisson brackets of the «classical» field. Formula (28) implies for $\varphi_1,\varphi_2\in\mathscr{X}$ with $\varphi_1^L(t)=\frac{1}{2}\varepsilon*f, f\in\mathscr{S}_r(\mathbb{R})$:

$$\int\varphi^L(t)f(t)dt=\sigma(\varphi,\varphi_1).$$

Putting $\varphi_j=\frac{1}{2}\varepsilon*f_j$ with $f_j\in\mathscr{S}_r(\mathbb{R})$ for $j=1,2$, we see that (14) becomes now

$$\{\int\varphi^2(t_1)f_1(t_1)dt_1, \int\varphi^L(t_2)f_2(t_2)dt_2\}=\sigma(\varphi_1,\varphi_2)=$$
$$\int D^L(t_1-t_2)f_1(t_1)f_2(t_2)dt_1dt_2,$$

where

$$D^L(t_1-t_2)=\frac{1}{2}\varepsilon(t_1-t_2)=i\int P\frac{1}{\lambda}\cdot\exp(-i\lambda(t_1-t_2))\frac{d\lambda}{2\pi}; \tag{32}$$

consequently

$$\{\int\varphi^L(t_1),\varphi^L(t_2)\}=D^L(t_1-t_2). \tag{33}$$

By virtue of (17), (27) this suffices for derivation of two-times Poisson brackets of the «classical» fields φ:

$$\{\varphi(t_1,x_1),\varphi(t_2,x_2)\}=D(t_1,x_1;t_2,x_2), \tag{34}$$

where

$$D(t_1,x_1;t_2,x_2)=$$
$$=\frac{i}{2}\int P\frac{1}{\lambda}(\exp(-i\lambda(t_1-t_2+x_1-x_2))+\exp(-i\lambda(t_1-t_2-x_1+x_2))+$$
$$+\frac{\gamma+i\lambda}{\gamma-i\lambda}\exp(-i\lambda(t_1-t_2-x_1-x_2))+$$
$$+\frac{\gamma-i\lambda}{\gamma+i\lambda}\exp(-i\lambda(t_1-t_2+x_1+x_2)))\frac{d\lambda}{2\pi}. \tag{35}$$

2. QUANTIZATION

The quantized field $\varphi(t,x)$ of the system «particle+string» obeys the same equations of motion as does the «classical» field. It is reasonable to regard $\varphi(t,x)$ as an operator-valued generalized function over the space $\mathscr{S}(\mathbb{R}^2)$ of test functions, with the wave equation $\left(\partial_t^2-\partial_x^2\right)\varphi=0$ satisfied. By analogy, we may assume that $\varphi(t,x)$ is expressed by (13) in terms of the left and right components. Now $\varphi^L(t)$ and $\varphi^R(t)$ are operator-valued generalized functions over the space $\mathscr{S}(\mathbb{R})$ of test functions, interrelated by the formula

$$\overline{R}^R(\lambda)=\frac{\gamma-i\lambda}{\gamma+i\lambda}\overline{\varphi}^L(\lambda). \tag{36}$$

The canonical commutation relations (CCR) corresponding to (33), (34) are

$$\left[\varphi^L(t_1),\varphi^L(t_2)\right]=\frac{1}{i}D^L(t_1-t_2), \tag{37}$$

$$[\varphi(t_1,x_1),\varphi(t_2,x_2)]=\frac{1}{i}D(t_1,x_1;t_2,x_2). \tag{38}$$

Of the two components $\varphi^L(t)$ and $\varphi^R(t)$ only one is an independent field, so it suffices to construct physical representations only for the left component $\varphi^L(t)$.

We use the construction [7] of the vacuum (or zero-temperature) representation of the field $\varphi^L(t)$, with suitable modification in case of nonzero temperature representations. Let $\mathscr{A}$ be an abstact *-algebra of CCR of the field $\varphi^L(t)$, which contains unitary elements $\mathscr{E}_f$ with $f\in\mathscr{S}_r(\mathbb{R})$ satisfying the relations

$$\mathscr{E}_{f_1}\mathscr{E}_{f_2}=\exp\left(\frac{i}{2}\int D^L(t_1-t_2)f_1(t_1)f_2(t_2)dt_1dt_2\right)\cdot\mathscr{E}_{f_1+f_2} \tag{39}$$

We suppose that $\mathscr{A}$ consists of elements of the form*

$$\int_{\subset\mathscr{M}}\mathscr{E}_f d\mu(f), \tag{40}$$

with $\mathscr{M}$ — an arbitrary finite-dimensional subspace in $\mathscr{S}_r(\mathbb{R})$, μ — an arbitrary complex Borel measure on $\mathscr{M}$ having compact support. Quantization of the system means the construction of a physical representation (or «dressing-up») of the *-algebra $\mathscr{A}$ in a Hilbert space. Here we are interested in the vacuum and the temperature representations. In this case unitary operators $E_{f,T}$ with continuous dependence of f (in the weak operator topology) correspond to unitary elements $\mathscr{E}_f$. They can be written as

$$E_{f,T}=\exp\left(i\int\varphi^L(t)f(t)dt\right), \tag{41}$$

with $\varphi^L(t)$ the quantized field in a concrete (vacuum or temperature) representation.

Gauge transformations (10) in terms of the left component $\varphi^L(t)$ take the form

$$\varphi^L(t)\to\varphi^L(t)+g^L,\ \text{where } g^L=\frac{1}{\sqrt{2}}g. \tag{42}$$

The corresponding *-automorphisms $A\to A^g$ act on the algebra $\mathscr{A}$ by

$$\mathscr{E}_f\to(\mathscr{E}_f)^g=\exp\left(ig^L\int(t)dt\right)\cdot\mathscr{E}_f. \tag{43}$$

Elements $\mathscr{E}_f$ for $\int f(t)dt=0$, i.e., for $f=-\partial_tF$ with $F\in\mathscr{S}_r(\mathbb{R})$, are gauge invariant; in a concrete representation of $\mathscr{A}$ such elements can be expressed in terms of the left current $j^L(t)=\partial_t\varphi^L(t)$:

$$E_{-\partial F,T}=\exp\left(i\int j^L(t)F(t)dt\right). \tag{44}$$

Admitting in (40) only subspaces $\mathscr{M}\subset\mathscr{S}_r(\mathbb{R})$ consisted of functions $f\in\mathscr{S}_r(\mathbb{R})$ with $f\in\mathscr{S}_r(\mathbb{R})$, we obtain a *-subalgebra $\mathscr{B}$ of gauge invariant elements of $\mathscr{A}$.

Let us assume, for the moment, that the field $\varphi^L(t)$ is a Fock system, then $\overline{\varphi}^L(\lambda)\cdot\theta(-\lambda)$ and $\overline{\varphi}^L(\lambda)\cdot\theta(\lambda)$ would be creation and annihilation operators (operator-valued generalized functions), respectively, and there would exist vacuum $W_0^L(t_1-t_2)$ and temperature (or Gibbs with temperature T) $W_T^L(t_1-t_2)$ two-point

*Completion of the *-algebra $\mathscr{A}$ in a corresponding norm topology is a kind of an abstract C*-algebra of CCR.

functions, which could be restored as generalized functions by the commutator function $D^L(t_1-t_2)$ according to

$$\overline{W}_0^L(\lambda)=\theta(\lambda)\frac{1}{i}\overline{D}^L(\lambda),$$
$$\overline{W}_T^L(\lambda)=\left(1-\exp(-\beta\lambda)\right)^{-1}\frac{1}{i}\overline{D}_T^L(\lambda)\text{ for }T>0 \qquad (45)$$

(in connection with the latter of these equations see, e.g., [2]). Here $\beta=\frac{1}{kT}$ with k the Boltzmann constant. The general requirements for the two-point functions $W_T^L(t_1-t_2)$ are that they be hermitian Im $(\overline{W}_T^L(\lambda)=0)$, positive definite $(\overline{W}_T^L(\lambda)\geq 0)$ and properly related to $D^L(t_1-t_2)\left(D^L(t_1-t_2)=-2\,\text{Im}\,W_T^L(t_1-t_2)\right)$.

$W_0^T(t_1-t_2)$ is distinguished by additional spectrum condition (support of $\overline{W}_0^L(\lambda)$ is contained in $\overline{R}_+$).

However, the product $\theta(\lambda)\cdot\frac{1}{i}\overline{D}^L(\lambda)=\theta(\lambda)\cdot P\frac{1}{\lambda}$ is ill defined (i.e., infrared divergent) and needs further definition as a generalized function from $\mathcal{S}'(\mathbb{R})$. The result of such definition contains an arbitrary dimensional parameter $\kappa>0$ and can be written as

$$W_0^L(t_1-t_2)=\int\theta(\lambda)\cdot P\frac{1}{\lambda}\exp(-i\lambda(t_1-t_2))\frac{d\lambda}{2\pi}=$$
$$=-\frac{1}{2\pi}\ln(0+i\kappa(t_1-t_2)). \qquad (46)$$

It is fairly obvious, $W_0^L(t_1-t_2)$ satisfies all the requirments apart from positive definiteness. This indicates that Fock representation of the field $\varphi^L(t)$ consistent with the action (31) of the dynamical group does not exist. In other words, the field $\varphi^L(t)$ is a non-Fock linear boson system.

Nevertheless, the positive definiteness property is restored partly if the functions f are restricted by the condition $f=-\partial_t F$ with $F\in\mathcal{S}_r(\mathbb{R})$:

$$\int W_0^L(t_1-t_2)\partial_{t_1}\overline{F(t_1)}\partial_{t_2}F(t_2)dt_1dt_2=\int\theta(\lambda)\lambda\cdot\left|\overline{F}(\lambda)\right|^2\frac{d\lambda}{2\pi}\geq 0,$$

so $W_0^L(t)$ belongs to the class of the so-called conditionally positive definite generalized functions of the first order. Hence the left current $j^L(t)=\partial_t\varphi^L(t)$ and the *-subalgebra $\mathcal{B}$ of gauge invariant variables of the algebra $\mathcal{A}$ of the field $\varphi^L(t)$ possess Fock representation consistent with the action of the dynamical group. In this case we call the field $\varphi^L(t)$ (or, equivalently, the field $\varphi(t,x)$) degenerate Fock linear boson system*. The Fock vacuum state s_0 of the left current $j^L(t)$ is defined by a characteristic functional

$$s_0(\mathcal{E}_f)=\exp\left(-\frac{1}{2}\int W_0^L(t_1-t_2)f(t_1)f(t_2)dt_1dt_2\right)$$
$$\text{for }f=-\partial_t F, F\in\mathcal{S}_r(R). \qquad (47)$$

In an analogous way $\overline{W}_T^L(\lambda)$ is ill-defined (infrarer divergent), and can be defined as a generalized function from $\mathcal{S}'(\mathbb{R})$ by the formula* (with a suitable choice of a real parameter a_T and $P\frac{1}{\lambda^2}=-\partial_\lambda P\frac{1}{\lambda}$):

$$W_T^L(t_1-t_2)=\int\left(\left(\frac{1-\exp(-\beta\lambda)}{\lambda}\right)^{-1}\cdot P\frac{1}{\lambda^2}+a_T\cdot\delta(\lambda)\right)\times$$
$$\times\exp(-i\lambda(t_1-t_2))\frac{d\lambda}{2\pi}=-\frac{1}{2\pi}\ln\left(0+i\kappa\frac{\beta}{\pi}\sinh\left(\frac{\pi}{\beta}(t_1-t_2)\right)\right). \qquad (48)$$

Of the requirements on the two-point functions, the positive definiteness condition is not fulfilled. Rather $W_T^L(t)$ belongs to the class of conditionally positive definite generalized functions of the first order (i.e., $\lambda^2\overline{W}_T^L(\lambda)\geq 0$). Therefore there is the Gibbs state s_T of the left current $j^L(t)$ and of the subalgebra $\mathcal{B}$ with a characteristic functional obtained by substitution $W_T^L(t_1-t_2)$ instead of $W_0^L(t_1-t_2)$ into (47). The Fock and the Gibbs representations of the algebra $\mathcal{B}$ are constructed uniquely (up to unitary equivalence) by the Gel'fand-Naimark-Segal (GNS [9]) construction.

The state s_T $(T\geq 0)$ of subalgebra $\mathcal{B}$ induces a generalized state s_T of the algebra $\mathcal{A}$ with a characteristic functional

$$s_T(\mathcal{E}_f)=2\pi\cdot\delta\left(\int f(t)dt\right)\cdot s_T(\mathcal{E}_f),\ f\in\mathcal{S}_r(\mathbb{R}). \qquad (49)$$

This is a «state» with an infinite norm. In contradistinction to a state it is defined on the *-ideal $\mathcal{J}_{\mathcal{E}}$ of $\mathcal{A}$ consisting of elements (40) with a measure $d\mu(f)$, which we call smooth in $\overline{f}(0)$. This means that $\overline{f}(0)\left(=\int f(t)dt\right)$ is not identical zero on a finite-di-

*In the general case an arbitrary finite dimension of gauge group of the system of this class is allowable.

*See, e.g., [8], 4.116.3

mensional subspace $\mathcal{M} \subset \mathcal{S}_r(\mathbb{R})$, and that with linear coordinates $r_1 = \overline{f}(0), r_2, \ldots, r_n$ chosen on $\mathcal{M}$ the measure $d\mu(f)$ is representable in the form $d\mu(f) = \nu(r_1, \ldots, r_n) dr_1 \ldots dr_n$, where $\nu(r_1, \ldots, r_n)$ is a generalized function in $\mathbb{R}^n$ smooth in r_1. The generalized state s_T is obtained from the state s_T by integration over the gauge group R:

$$s_T(A) = s_T\left(\int A^g dg\right), A \in \mathcal{J}. \tag{50}$$

(This is a correct formula in lieu of the previous definition (49).)

The GNS construction is easily extended to the generalized states s_T which enables one to obtain vacuum and temperature representations of the algebra $\mathcal{A}$. Hilbert space $\mathcal{H}_T$ is generated by generalized vectors $X_{f,T}$ $(f \in \mathcal{S}_r(\mathbb{R}))$ with scalar product

$$\left\langle X_{f_1,T}, X_{f_2,T} \right\rangle = 2\pi \cdot \delta(-\overline{f}_1(0) + \overline{f}_2(0)) \cdot s_T(\mathcal{E}_{-f_1+f_2}). \tag{51}$$

Smoothing $X_{f,T}$ with an arbitrary measure $d\mu(f)$ smooth in $\overline{f}(0)$ (entering in the definition of the ideal $\mathcal{J}$) gains a subspace of vectors dense in $\mathcal{H}_T$. The vacuum or the temperature representation $\mathcal{E}_f \to E_{f,T}$ is defined by generalized matrix elements

$$\left\langle X_{f_1,T}, E_{f,T} X_{f_2,T} \right\rangle = 2\pi \cdot \delta(-\overline{f}_1(0) + \overline{f}(0) + \overline{f}_2(0)) \cdot s_T(\mathcal{E}_{-f_1+f+f_2}), \tag{52}$$

and quantum fields $\varphi^L(t)$ and $\varphi(t,x)$ in the vacuum or the temperature representation are defined by formulas (41), (36), (17). These are operator-valued distributions over the test function spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R}^2)$, respectively. The dynamical group of time translations and the gauge group are implemented by unitary operators $U_T(\tau)$ and $V_T(g)$ in $\mathcal{H}_T$:

$$U_T(\tau) X_{f,T} = X_{h,T} \text{ with } h(t) = f(t+\tau);$$
$$V_T(g) X_{f,T} = \exp(i g^L \overline{f}(0)) \cdot X_{f,T}. \tag{53}$$

It is easy to pass from the fields $\varphi^L(t)$ and $\varphi(t,x)$ to dynamical variables of the Brownian particle. The coordinate $q(t)$ of the Brownian particle in the vacuum or temperature representation as an operator-valued generalized function over the space $\mathcal{S}(\mathbb{R})$ is defined by (3) or, equivalently, by

$$q(t) = (\gamma m)^{-1/2} \Phi(t) \tag{54}$$

with $\Phi(t)$ defined by (19), (27). The former of the Langevin equations (1) serves as a definition of momentum $P(t)$ of the particle, and the latter of the equations (1) reduces

to the first of the equations (21), when the field in the definition of the fluctuating force is identified with the left current:

$$\xi(t) = \partial_t \varphi^L(t) = j^L(t). \tag{55}$$

Note that the fluctuating force in the vacuum s_0 or the Gibbs state s_T of the subalgebra $\mathcal{B}$ is a Gaussian quantum field with mean 0 and with a two-point function defined by the relations

$$\left\langle \xi(t_1)\xi(t_2) \right\rangle_0 = \int \theta(\lambda) \cdot \lambda \exp(-i\lambda(t_1 - t_2)) \frac{d\lambda}{2\pi},$$

$$\left\langle \xi(t_1)\xi(t_2) \right\rangle_T = \int (1 - \exp(-\beta\lambda))^{-1} \lambda \cdot \exp(-i\lambda(t_1 - t_2)) \frac{d\lambda}{2\pi} \text{ for } T > 0. \tag{56}$$

When $T \to \infty$, the «classical» white noise is obtained in the asymptotics

$$\left\langle \left(\xi(t_1)\xi(t_2)\right)_{\text{sym}} \right\rangle_T \sim \frac{1}{\beta} \delta(t_1 - t_2).$$

The fact that $q(t)$ and $p(t)$ are operator-valued distributions is displayed in an interpretation of quantum-mechanical commutation relations. With the help of (52) it can be checked that (for $\alpha \in \mathbb{R}$), $h(t) \in \mathcal{S}_r(\mathbb{R})$) there exists the limit of $\exp(i\alpha \int q(t) h(t) dt)$ when $h(t)$ tends to $\delta(t - t_0)$ defining thereby $\exp(i\alpha q(t_0))$ whereas such a limit does not exist for $\exp(i\alpha \int p(t) h(t) dt)$ (because of ultraviolet divergences). This is an indication that the operator-valued distribution $q(t)$ permits a restriction $q(t)\big|_{t=t_0}$ defining the coordinate $q(t_0)$ at a fixed time. For the momentum $p(t)$ such a restriction is not allowable. In this regard commutation relations between the coordinate and the momentum inferred from (38) are of interest:

$$[q(t_1), p(t_2)] = i \int 2\gamma(\gamma^2 + \lambda^2)^{-1} \exp(i\lambda(t_1 - t_2)) \frac{d\lambda}{2\pi} =$$
$$= i \exp(-\gamma |t_1 - t_2|). \tag{57}$$

This suggests that, after smearing the momentum $p(t)$ by means of convolution with a δ-shaped sequence of functions $h_n(t)$, the equal-time canonical commutation relation between the coordinate and the momentum is reproduced in the limit $h(t) \to \delta(t)$:

$$[q(t_0), p_h(t_0)] \to i, \text{ where } p_h(t) = p(t) * h(t). \tag{58}$$

The peculiarity of the model of Brownian motion under consideration is that the Langevin equations cannot be formulated in the frame of the algebra generated by the fluctuating force. It is necessary to extend this algebra incorporating an antiderivative of the fluctuating force. Indeed, as a gauge-invariant field the fluctuating force generates only a gauge-invariant subalgebra of the field algebra rather than the whole algebra, so only the momentum of the Brownian particle can be restored by the fluctuating force. It is the formula (27) that implies $\overline{p}(\lambda)=\sqrt{2\gamma m}\cdot(\gamma-i\lambda)^{-1}\,\xi(\lambda)$, so $p(t)$ is a convolution of the form

$$p(t)=\sqrt{2\gamma m}\int\theta(t-\tau)\exp(-\gamma(t-\tau))\xi(\tau)\,d\tau=$$
$$=\sqrt{2\gamma m}\,(\theta(t)\exp(-\gamma t))*\xi(t). \tag{59}$$

Formula (27) suggests also $\overline{q}(\lambda)=\sqrt{2\gamma/m}(\gamma-i\lambda)^{-1}\,\overline{\varphi}^{L}(\lambda)$,
hence

$$q(t)=\sqrt{2\gamma/m}\int\theta(t-\tau)\exp(-\gamma(t-\tau))\varphi^{L}(\tau)\,d\tau=$$
$$=\sqrt{2\gamma/m}\cdot(\theta(t)\cdot\exp(-\gamma t))*\varphi^{L}(t). \tag{60}$$

The coordinate $q(t)$, as a not gauge-invariant variable which does not commute with the group of gauge transportation operators $V_T(g)$, cannot be expressed in terms of the fluctuating force. This is a distinguishing feature of the «free» quantum Brownian particle as opposed to the quantum oscillator with friction.

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