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Боголюбовские квазичастицы в системах со связями

Обобщаются преобразования Боголюбова для полевых переменных в голоморфном представлении для систем со связями, где эволюционный параметр редуцированного фазового пространства является одной из динамических переменных расширенного фазового пространства.

Боголюбовские квазичастицы, которые определяются диагонализацией уравнений движения (а не только гамильтониана), дают сохраняющееся «число частиц». Этот подход используется для описания рождения частиц в моделях ранней Вселенной.

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Pervushin V.N., Smirichinski V.I. E2-99-16
Bogoliubov Quasiparticles in Constrained Systems

The Bogoliubov transformations of field variables in the holomorphic representation are generalized to systems with constraints where the evolution parameter in the reduced phase space is one of the dynamical variables of the extended phase space.

The Bogoliubov quasiparticles are determined by the diagonalization of the equations of motion (but not only of the Hamiltonian) to get conserved «numbers of quasiparticles». This approach is applied for the description of particle creation in the models of early Universe.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1 Introduction

The Bogoliubov transformations [1] were applied to get the energy spectrum of the weakly non-ideal Bose gas and they can be considered as an effective mathematical tool to construct integrals of motion of complicate systems.

In this paper, we apply the Bogoliubov transformations to construct integrals of motion of constrained systems invariant with respect to reparametrization of time: $t' = t'(t)$.

This reparametrization-time invariant system plays an important role for the theory of gravitation, in particular, for the description of the early Universe. The main problems of consideration of the constrained systems are to extract the invariant dynamics with respect to an internal evolution parameter, and to get corresponding integrals of motion. Recently, the invariant dynamic description of constrained systems was formulated using explicit resolving the energy constraint [2, 3, 4]. It allows us to remove one of variables (with a negative contribution into the constraint) from the phase space, to convert it to the internal evolution parameter of the corresponding reduced system, and to find the relationship between this evolution parameter and the measurable time interval in gravitation [2, 3, 4] and cosmological models [5].

The problem of the present paper is to construct integrals of motion by the Bogoliubov transformations in the context of the above-mentioned Hamiltonian reduction [2, 3, 4]. In the next section, we formulate the problem using as an example an oscillator-like model of a massive scalar field in the FRW metric [6, 7, 8, 9]. In Section 3, we introduce the Bogoliubov transformations to diagonalize equations of motion. Section 4 is devoted to the description of creation of massive particles and gravitons in the early Universe.

2 Statement of Problem

We first consider an oscillator-like model of a massive particle in the flat FRW metric described by the action [6, 7, 8, 9]

$$W^E = \int_{t_1}^{t_2} dt \left\{ p\dot{q} - P_0\dot{\phi} - N[-\frac{P_0^2}{4V_0} + H(p,q)] \right\}, \tag{1}$$

where ϕ is the cosmic scale factor (R/R_0) multiplied by the Planck constant

$$\phi = \mu \frac{R(t)}{R_0}; \quad \mu = M_{Pl}\sqrt{\frac{3}{8\pi}}, \tag{2}$$

$H(p,q)$ is the Hamiltonian of a massive particle

$$H = \frac{1}{2}(p^2 + q^2\omega^2(\phi)); \quad \omega^2(\phi) = k^2 + \lambda\phi^2, \tag{3}$$

V_0 is a three-dimensional space-volume. Action (1) retains the main peculiarity of GR, the invariance under reparametrization of the coordinate time: $t \to t' = t'(t)$. The problem is to find the evolution of variables P_0, ϕ, p, q with respect to the invariant conformal time

$$Ndt = N'dt' = dT \tag{4}$$

of an observer in the comoving frame of reference.

As a consequence of the time-reparametrization invariance, the set of classical equations

$$\frac{dq}{dT} = p; \quad \frac{dp}{dT} + \omega^2 q = 0; \quad \frac{d\phi}{dT} = \frac{P_0}{2V_0}; \quad \frac{dP_0}{dT} = \lambda\phi q^2 \tag{5}$$

is accompanied by the constraint

$$\frac{\delta W}{\delta N} = 0 \quad \rightarrow \quad -\frac{P_0^2}{4V_0} + H = 0 \tag{6}$$

with two solutions

$$(P_0)_\pm = \pm 2\sqrt{V_0 H}. \tag{7}$$

The substitution of this solution into the equation for P_0 (the third equation in (5)) leads to the equation:

$$\frac{\delta W}{\delta P_0} = 0 \quad \rightarrow \quad \frac{d\phi}{dT} = \left(\frac{P_0}{2V_0}\right)_\pm = \pm\sqrt{\rho(\phi)}; \quad \left(\rho = \frac{H}{V_0}\right). \tag{8}$$

The integral form of this equation

$$T_\pm(\phi) = \pm \int_0^\phi \frac{d\phi}{\rho^{1/2}(\phi)} \tag{9}$$

is known [5] as the Friedmann law of the evolution of the measurable time interval with respect to the scale factor (2) of the Universe.

To solve equation (9), we need the dependence of physical variables on the scale factor ϕ, which can be found from (5) with the use of equation (8)

$$\begin{aligned} \tfrac{dq}{dT} &= \tfrac{d\phi}{dT}\tfrac{dq}{d\phi} = \pm\sqrt{\rho(\phi)}\tfrac{dq}{d\phi} = p \\ \tfrac{dp}{dT} &= \tfrac{d\phi}{dT}\tfrac{dp}{d\phi} = \pm\sqrt{\rho(\phi)}\tfrac{dp}{d\phi} = \omega^2(\phi)q. \end{aligned} \tag{10}$$

The scale factor ϕ in this equations represents the internal evolution parameter. We can find the corresponding reduced action for eqs.(10) by the substitution of the explicit solution of the constraint (6),(7) into the initial action (1):

$$W^R = \int\limits_{\phi_1=\phi(t_1)}^{\phi_2=\phi(t_2)} d\phi[p\frac{dq}{d\phi} \mp 2\sqrt{V_0 H}]. \tag{11}$$

Thus, the considered Hamiltonian reduction (11) of the time-reparametrization invariant systems faces three "times":

- The first is the coordinate time (t) in the initial extended action (1).
- The second time is the internal evolution parameter of the reduced system (11) which is one of former variables of the extended system (ϕ).
- Every action of relativistic theory has to be supplemented with a geometrical convention which connects a measurable invariant interval (T) (4) with variables of the extended system.

The Friedmann law of the Universe evolution is a consequence both of the variation principle for action (1) (or (11)) and the convention (4) [2, 3, 4].

The Einstein theory of gravity is mathematically equivalent to a scalar version of the conformal invariant Weyl theory [2, 3, 4] with the scalar ϕ as the measure of integrable change of the length of a vector in its parallel transport. However, the Einstein theory and the Weyl one correspond to different conventions about the measurable interval. The Einstein convention with the measurable (proper) time $dT_f = dT(\phi/\mu)$ leads to the FRW cosmological picture with the expanding Universe, while, the conformal invariant measurable time (T) leads to the Hoyle-Narlikar cosmology where the reason of the "red shift" is alteration of masses of particles formed by the Higgs-Weyl scalar field ϕ and the conformal invariant size of the Universe is an integral of motion.

Three different "times" of Hamiltonian reduction correspond to three different energies:

- constraint (for coordinate time),
- evolution momentum P_0 (for evolution parameter),
- measurable energy (for measurable time)

$$E_m = -\frac{dW^R}{dT}. \tag{12}$$

All these energies depend on the invariant times (ϕ, T).

The problem is to find integrals of motion of the considered time-reparametrization system, and describe creation of "particles" in the evolution of the Universe.

3 The Bogoliubov transformations

We define "particle"-like variables as the holomorphic representations of the standard variables [10]

$$q(t) = \frac{1}{\sqrt{2\omega}}(a^+(t) + a(t))\,; \quad p(t) = i\sqrt{\frac{\omega}{2}}(a^+(t) - a(t)). \tag{13}$$

There is the quantum field theory convention that we measure the "number of particles"

$$\hat{\mathcal{N}}_a(t) = \{a^+(t)a(t)\} \equiv \frac{1}{2}(a^+a + aa^+). \tag{14}$$

In the considered case this quantity is not conserved

$$\frac{d\hat{\mathcal{N}}_a(t)}{dt} \neq 0. \tag{15}$$

In particular, just this quantity forms the observable density ρ and evolution of the Universe (9)

$$\rho = \frac{H_0}{V_0}; \quad H_0 = \omega\hat{\mathcal{N}}_a \equiv \frac{1}{2}\omega(a^+a + aa^+). \tag{16}$$

The main goal of our paper is to show that there are the Bogoliubov transformations of the "particle" variables

$$b^+ = \alpha^* a^+ + \beta^* a, \quad b = \alpha a + \beta a^+, \tag{17}$$

which diagonalize the corresponding classical equations expressed in terms of "particles" (a^+, a), so that the "number of quasiparticles" is conserved

$$\frac{d\mathcal{N}_b(t)}{dt} \equiv \frac{d(b^+b)}{dt} = 0. \tag{18}$$

In terms of the "particle" variables (13) the action (1) has the form

$$W^E = \int_{t_1}^{t_2} dt[-P_0\dot{\phi} - N\frac{P_0^2}{4V_0} + \frac{i}{2}\bar{\chi}_a\partial_t\chi_a - N\frac{1}{2}\bar{\chi}_a\hat{H}_a\chi_a], \tag{19}$$

where

$$\bar{\chi}_a = (a, -a^+); \;\; \chi_a = \begin{pmatrix} a^+ \\ a \end{pmatrix}; \;\; \hat{H}_a = \begin{vmatrix} \omega & , & -i\Delta \\ -i\Delta & , & -\omega \end{vmatrix}, \tag{20}$$

$$\boxed{\Delta = \frac{d}{Ndt}\log\sqrt{\frac{\omega(t)}{\omega(0)}},} \tag{21}$$

where $\omega(0)$ is defined by initial data. The classical equations (5) can be written as

$$i\frac{d}{dT}\chi_a = \hat{H}_a\chi_a. \tag{22}$$

After Bogoliubov transformations (17)

$$\chi_b = \hat{O}\chi_a; \;\; \hat{O} = \begin{pmatrix} \alpha^*, & \beta^* \\ \beta, & \alpha \end{pmatrix}; \;\; \hat{O}^{-1} = \begin{pmatrix} \alpha^*, & -\beta^* \\ -\beta, & \alpha \end{pmatrix}; \tag{23}$$

This equation has the form

$$i\frac{d}{dT}\chi_b = [-i\hat{O}^{-1}\frac{d}{dT}\hat{O} + \hat{O}^{-1}\hat{H}_a\hat{O}]\chi_b \equiv \hat{H}_b\chi_b. \tag{24}$$

Let us require that $\hat{H}_b$ to be diagonal

$$\hat{H}_b = \begin{pmatrix} h, & 0 \\ 0, & -h \end{pmatrix}. \tag{25}$$

This means that α and β satisfy the equations

$$h = (|\alpha|^2 + |\beta|^2)\omega - i(\beta^*\alpha - \beta\alpha^*)\Delta - i(\beta^*\partial_T\beta - \alpha\partial_T\alpha^*), \tag{26}$$

$$0 = 2\beta\alpha\omega - i(\alpha^2 + \beta^2)\Delta - i(\alpha\partial_T\beta - \beta\partial_T\alpha). \tag{27}$$

For

$$\alpha = \cosh(r)e^{i\theta}\;; \;\; \beta = i\sinh(r)e^{-i\theta} \tag{28}$$

these equations convert into

$$\boxed{\begin{aligned} h =& \;\; \omega\cosh 2r - \Delta\sinh 2r\cos 2\theta - \cosh 2r\partial_T\theta \\ 0 =& \;\; \omega\sinh 2r - \Delta\cosh 2r\cos 2\theta - \sinh 2r\partial_T\theta \\ 0 =& \;\; \Delta\sin 2\theta + \partial_T r \end{aligned}} \tag{29}$$

In the case of constant ω, Δ we get the result of Bogoliubov paper [1]

$$\theta = 0\;; \;\; \cosh 2r = \frac{\omega}{h} \;\;\; h = \sqrt{\omega^2 - \Delta^2}. \tag{30}$$

Finally , we obtain the classical equation in terms of "quasiparticles"

$$\frac{d}{dT}b^+ = ihb^+; \;\; \frac{d}{dT}b = -ihb; \;\;\; (h \equiv \frac{d}{dT}Q), \tag{31}$$

with the solution

$$b^+ = \exp(iQ)b_0^+\;; \;\; b = \exp(-iQ)b_0; \;\;\;\; Q = \int^T dT'h(T') \tag{32}$$

and the conserved number of quasiparticles

$$\hat{\mathcal{N}}_b(t) = \{b^+(t)b(t)\} = \{b_0^+b_0\} \tag{33}$$

where b_0^+ and b_0 are initial data. To close equation (31), we should recall that the evolution of the Universe is determined by the density of "observable particles"

$$\frac{d\phi}{dT} = \sqrt{\rho(\phi)}; \;\; \rho(\phi) = \frac{H}{V_0} = \frac{\omega(\phi)\{a^+a\}}{V_0}; \tag{34}$$

$$\boxed{\{a^+a\} = \{b^+b\}\cosh 2r - \tfrac{i}{2}(b^{+2} - b^2)\sinh 2r} \tag{35}$$

Equations (29)-(35) represent a complete set of equations of classical theory.

4 Creation of Particles

As ϕ has left the phase space to convert into the internal evolution parameter, in quantum theory, we can quantize only the particle sector

$$[a, a^+] = 1\;; \;\; [b, b^+] = 1 \tag{36}$$

In the following, we restrict ourselves by the Universe in the state of vacuum of quasiparticles :

$${}_b < 0|b^+b|0 >_b = 0\;; \;\; < 0|\{a^+a\}|0 >= \frac{1}{2}\cosh 2r = \frac{1}{2}\mathcal{N}_0(\phi). \tag{37}$$

We can rewrite eqs. (29) in terms of the number of particles $N_0(\phi)$

$$
\begin{aligned}
&h = \omega N_0 - \Delta[\sqrt{N_0^2-1}\cos 2\theta + N_0 2\tfrac{d\theta}{d\tau}] \\
&\omega\sqrt{N_0^2-1} = \Delta[N_0\cos 2\theta + \sqrt{N_0^2-1}\ \ 2\tfrac{d\theta}{d\tau}] \qquad (38)\\
&\sqrt{N_0^2-1}\sin 2\theta = -\tfrac{dN_0}{d\tau};
\end{aligned}
$$

where

$$\Delta = \sqrt{\rho}\frac{d\tau(\phi)}{2d\phi};\ \ \tau(\phi) = \log\left(\frac{\omega(\phi)}{\omega_0}\right);\ \ \omega(\phi) = \sqrt{k^2+\lambda\phi^2} \qquad (39)$$

where ω_0 is the initial data.

For $\Delta = 0$ and $N_0 = 1$ particles are not created. For $\Delta \to \infty$ the solution to these equations has the functional form

$$N_0(\phi) = \frac{1}{2}\left(\frac{\omega_0}{\omega(\phi)} + \frac{\omega(\phi)}{\omega_0}\right) \qquad (40)$$

with the density

$$<\rho> = \frac{\omega(\phi)}{V_0}\frac{N_0(\phi)}{2} = \frac{1}{4V_0}\left(\omega_0 + \frac{\omega^2(\phi)}{\omega_0}\right). \qquad (41)$$

This solution has the zero energy of the Bogoliubov quasiparticles ($h = 0$) of the type of the Landau sound in theory of superfluid liquid. The corresponding proper time dynamics

$$T(\phi) = \int_0^{\phi} d\bar{\phi}\frac{2\sqrt{\omega_0 V_0}}{\sqrt{\omega_0^2+k^2+\lambda\bar{\phi}^2}} = 2\sqrt{\frac{\omega_0 V_0}{\lambda}}\log\left(\frac{\lambda^{1/2}\phi + \sqrt{\omega_0^2+k^2+\lambda\bar{\phi}^2}}{\sqrt{\omega_0^2+k^2}}\right) \qquad (42)$$

determines the red shift and the Hubble parameter of evolution of the Universe with this excitation.

Equations (29) and (38) can be applyed for gravitons [8] which are described by the action of the type of (1) with the Hamiltonian

$$H = \frac{1}{2}\sum_k\left(\frac{p_k^2}{\phi^2} + \phi^2 q_k^2 k^2\right). \qquad (43)$$

In this case, in eqs (38) we have

$$\Delta = \frac{\sqrt{\rho}}{\phi};\ \ \tau(\phi) = 2\log\left(\frac{\phi}{\phi_0}\right);\ \ \omega(k) = \sqrt{k^2}, \qquad (44)$$

where ϕ_0 is the initial data.

For $\Delta \to \infty$ the "Landau sound" solution ($h = 0$) to eqs. (38) has the functional form

$$N_0(\phi) = \frac{1}{2}\left(\frac{\phi_0^2}{\phi^2} + \frac{\phi^2}{\phi_0^2}\right) \qquad (45)$$

with density

$$<\rho> = \rho_0\frac{N_0(\phi)}{2}, \qquad (46)$$

where ρ_0 is the vacuum density.

The proper time dynamics is determined by the integral

$$T(\phi) = \int_0^{\phi} d\bar{\phi}\frac{1}{\sqrt{\rho_0(\bar{\phi})}} \qquad (47)$$

which leads to the red shift of the Universe evolution

$$\frac{\phi^2(T)}{\phi_0^2} = \sinh\left(\frac{2T}{T_0}\right);\quad T_0 = \phi_0\sqrt{\frac{2}{\rho_0}}. \qquad (48)$$

We can see that there can be the period of inflation-like evolution of the scale factor in GR, or the Higgs field in Conformal Unified Theory [2, 3, 4], with respect to the conformal world time measurable in CUT [2, 3, 4] by a Weyl observer with relative standard of length. While, an Einstein-Friedmann observer (with absolute standard of the length) sees the linear dependence of the measured proper time on the scale factor. Thus, ten billion years for an observer with the absolute standard of length can convert in the "biblical" short period of several thousand years for an observer with relative standard.

5 Conclusion

The conceptions of "particle" and "quasiparticle" were considered in constrained systems with the time-reparametrization invariance. The main peculiarity of such systems is the internal evolution parameter as one of variables of the extended phase space. After the Hamiltonian reduction all equations of motion of the constrained system are converted into the reparametrization invariant equations for variables in the reduced phase space with respect to the evolution parameter. Relativistic systems are defined also the proper (measurable) time of an observer in a comoving frame of reference. The dependence of the measurable time on the internal evolution parameter determines the Universe evolution law in the form of the red shift or the Hubble parameter.

Accordingly, there are two different energies: the evolution energy (as the momentum with negative contribution to the energy constraint) and the measurable energy as the variation of the reduced action with respect to the measurable time with negative sign.

We define "particles" as variables in the holomorphic representation which diagonalizes the evolution energy. Just this energy forms the observable Hubble parameter. Therefore, these "particles" can be treated as "observable" ones. As a number of "particles" is not conserved, we construct the Bogoliubov quasiparticles which digonalize the equations of motion. A number of quasiparticles is conserved, and they are required to find a set of integrals of motion to describe the measurable time in the parametric form depending on initial data.

These definitions strongly differ from the conventional approach [6, 7, 8, 9] which goes from the conserved "particles" as initial data to unconserved "quasiparticles" with diagonalization only of the Hamiltonian.

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