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In the present note we discuss some problems related to derivative analyticity relations which, under certain conditions, give point-to-point correlations between the real and the imaginary part of a scattering amplitude on the energy cut. In the case of the symmetric amplitude $F_S(s,t)$, for instance, Bronzan et al.[1] have derived the relation

$$\mathrm{Re}\,\widetilde{F}_S(s,t) = s^{\alpha}\tan\left[\frac{\pi}{2}\left(\alpha - 1 + \frac{\partial}{\partial \ln s}\right)\right]\frac{\mathrm{Im}\,\widetilde{F}_S(s,t)}{s^{\alpha}} \qquad (1)$$

(α being a real number), assuming that the imaginary part can be expanded in powers of $\ln s$ on the energy cut. Here $\widetilde{F}_S(s,t)$ means the amplitude without poles and subtraction constants, s and t denoting the usual Mandelstam variables.

Suppressing higher branch points given by the unitarity condition, this derivation restricts severely the validity of the result to certain mathematical models. Indeed, it has been proved by Eichmann and Dronkers[2] that relation (1) is exactly valid only on some class of entire functions of $\ln s$. Further, Heidrich and Kazes[3] show that relation (1) is violated even in the energy intervals between two branch points.

In the present note we examine the derivative analyticity relations in the asymptotic regime.

Retaining only one derivative in (1), we obtain the following relation

$$\frac{\mathrm{Re}\,\widetilde{F}_S(s,t)}{s} = \frac{\pi}{2}\frac{\partial}{\partial \ln s}\frac{\mathrm{Im}\,\widetilde{F}_S(s,t)}{s} \qquad (2)$$

which has been derived by Gribov and Migdal[4] in the Regge theory for the contribution of the Pomeron and its cuts. We shall show that relations of type (2) are, in the region of asymptotic energies, an immediate consequence of the general, model-independent principles of the S-matrix theory provided that the high-energy limits (finite or infinite) of certain quantities are assumed to exist.

The symmetric and the antisymmetric hadron-hadron elastic scattering amplitude will be denoted by $F_S(E,t)$ and $F_A(E,t)$, respectively, E being defined as $(s-u)/4M$, where M is the mass of the target particle. Further, we use the following notation for convenience:

$F_{S,A}(E) \equiv F_{S,A}(E,0)$, $\sigma_S(E) = \frac{1}{E}$ Im $F_S(E)$ (the optical theorem),

$$\rho_{S,A}(E,t) \equiv \frac{\mathrm{Re}\, F_{S,A}(E,t)}{\mathrm{Im}\, F_{S,A}(E,t)} \quad \text{and} \quad \rho_{S,A}(E) \equiv \rho_{S,A}(E,0).$$

We recall the following general properties[5] of $F_{S,A}(E,t)$:
(i) There exist two positive numbers, $R(t)$ and t_0, such that $F_{S,A}(E,t)$ are analytic functions of E in the upper half of the complex E-plane for $|E| > R(t)$, provided that $|t| < t_0$. This region will be denoted by $\mathcal{D}_t$.
(ii)

$$F_S(E,t) = F_S^*(-E^*,t)$$
$$F_A(E,t) = -F_A^*(-E^*,t)$$

for every $E \in \mathcal{D}_t$ and $|t| < t_0$.
(iii) $F_{S,A}(E,t)$ on the energy cut have been obtained by regularizing the corresponding distributions and are continuous in the closure of $\mathcal{D}_t$.
(iv) $F_{S,A}(E,t)$ are polynomially bounded for $|E| \to \infty$, $E \in \mathcal{D}_t$, $|t| < t_0$.
(v) For the real values of E, $F_{S,A}(E,t)$ satisfy the Froissart-Martin bound

$$\left| F_{S,A}(E,t)/E \ln^2 E \right| < \text{const}$$

for any $E > E_1$ (the bound for $t < 0$ being even stronger).
(vi)

$$\mathrm{Im}\, F_S(E, t=0) \geqq 0 \quad \text{for any} \quad E > E_2 .$$

These properties allow us to derive the following statements.

Statement 1. Let $F_S(E)$ be a function of complex E satisfying conditions (i) to (vi). Let further $\lim_{E\to\infty} \rho_S(E)$ exist and

$$\lim_{E\to\infty} \sigma_S(E) = \infty . \qquad (3)$$

If the limit

$$\lim_{E\to\infty} \frac{\pi}{2} \frac{E \frac{d}{d\ln E} \sigma_s(E)}{Re\, F_s(E)} \tag{4}$$

exists, then its value is 1 .

In the remainder of the present paper, t is kept fixed in the interval $-t_0 < t \leqq 0$.

The following statement includes both forward and non-forward scattering and allows $\mathrm{Im}\, F_S(E,t)/E$ to tend to infinity or to zero.

<u>Statement 2</u>. Let $F_S(E,t)$ satisfy the conditions (i) to (vi). Let $\mathrm{Im}\, F_S(E,t)$ not change the sign above some energy on the cut and the conditions

$$\left| \int_{E_0}^{\infty} \mathrm{Im}\, F_s(E,t) \frac{dE}{E} \right| = \infty , \quad \text{for some } E_0 \tag{5}$$

and

$$\left| \lim_{E\to\infty} \ln \left| \frac{F_s(E,t)}{E} \right| \right| = \infty \tag{6}$$

be satisfied. If the limit

$$\lim_{E\to\infty} \frac{\pi}{2} \frac{E \frac{\partial}{\partial \ln E} \left| F_s(E,t)/E \right|}{\left| F_s(E,t) \right| \arctan \rho_s(E,t)} \tag{7}$$

exists, then its value is 1 .

The proof is given in the Appendix.

As a consequence, we find for the antisymmetric amplitude $F_A(E,t)$ the following result:

<u>Corollary</u>. Let $F_A(E,t)$ satisfy conditions (i) to (v) and let further

a) $\lim_{E\to\infty} \left| \frac{F_A(E,t)}{E} \right| = 0$; (8)

b) $\operatorname{Im} F_A(E,t)$ not change the sign above some energy;

c)

$$\left| \int_{E_0}^{\infty} \operatorname{Im} F_A(E,t)\, dE \right| = \infty \qquad \text{for some } E_0 \tag{9}$$

d)

$$\lim_{E\to\infty} \left| \ln \left| F_A(E,t) \right| \right| = \infty \tag{10}$$

If the limit

$$\lim_{E\to\infty} \frac{\pi}{2} \frac{\frac{\partial}{\partial \ln E} \left| F_A(E,t) \right|}{\left| F_A(E,t) \right| \arctan \rho_A(E,t)} \tag{11}$$

exists, then its value is 1.

An interesting high-energy result can be obtained for the difference

$$D(E,t) = \frac{\pi}{2} \frac{\partial}{\partial \ln E} \left| \frac{F_S(E,t)}{E} \right| - \left| \frac{F_S(E,t)}{E} \right| \arctan \rho_S(E,t) \tag{12}$$

which, provided that $\rho_S(E,t)$ is sufficiently smooth and tends to zero, transforms into

$$\frac{\pi}{2} \frac{\partial}{\partial \ln E} \left(\frac{\operatorname{Im} F_S(E,t)}{E} \right) - \frac{\operatorname{Re} F_S(E,t)}{E} \tag{13}$$

Statement 3. Let $F_S(E,t)$ be a function satisfying (i) to (vi). Let $\operatorname{Im} F_S(E,t)$ not change the sign above some energy on the cut and condition (5) be satisfied. If the limit $\lim_{E\to\infty} D(E,t)$ exists then its value is 0.

The proof is sketched in the Appendix.

Similar theorems for the difference (13) and for $F_A(E,t)$ can also be obtained; details will be published elsewhere.

We should like to draw the reader's attention to some interesting differences among the statements derived. The symmetric total cross-section $\sigma_S(E)$ is required to tend to infinity in Statement 1. If $\sigma_S(E)$ approaches zero in the high-energy limit, expression (4) is modified to (7) because $\rho_S(E)$ may be non-vanishing at $E\to\infty$ in this case[6]. In contradistinction to this no explicit restriction is imposed on the high-energy behaviour of $\sigma_S(E)$ in Statement 3.

Let us note that the results obtained have the form of high-energy limits and no a priori claim can be laid to their validity at finite energies. On the other hand, referring to the measurements of small-angle hadron-hadron scattering at the CERN ISR and at Fermilab above 100 GeV (lab), one can infer that the available energies would be already appropriate for the numerical verification of the Statements presented.

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APPENDIX

Firstly, we give the proof of Statement 2. Statement 1 is proved in an analogous manner.

Since the limit (7) exists, the limit

$$\lim_{E\to\infty}\left[\left(\frac{2}{\pi}\int_{E_0}^{E}\arctan\rho_S(E',t)\frac{dE'}{E'}-\ln\left|\frac{F_S(E,t)}{E}\right|\right)\Big/\ln\left|\frac{F_S(E,t)}{E}\right|\right] \tag{14}$$

also exists because of l'Hôpital's rule. The Cauchy theorem for $\ln(-F_S(E,t))/E$ implies [6,7]

$$\ln\left|\frac{F_S(E,t)}{E}\right| = \frac{2}{\pi}\int_{E_0}^{E}\arctan\rho_S(E',t)\frac{dE'}{E'} + C - \mu(E,t) \tag{15}$$

where $\mu(E,t) = \frac{1}{\pi}\int_0^{\pi}\ln\left|\frac{F_S(Ee^{i\varphi},t)}{F_S(E,t)}\right|d\varphi$

and C is a constant depending on E_0 and t (t fixed $-t_0 < t \leq 0$). Combining (14) with (15) and using (6) we easily see that the limit (14) is equal to

$$\lim_{E\to\infty}\frac{\mu(E,t)}{\ln|F_S(E,t)/E|} \tag{16}$$

This limit can be found by using a method which is due to Vernov[7]. Indeed, it follows that

$$\left|\int_{E_0}^{E}\mu(E',t)\frac{dE'}{E'}\right| < \text{const}. \tag{17}$$

for every $E > E_0$ (note that E_0 must be sufficiently large [6] and, obviously, $E_0 > \max(R(t), E_1, E_2)$). Consequently, (16) is equal to zero and (7) equals 1.

We prove now Statement 3. Since $\lim\limits_{E\to\infty} D(E,t)$ is assumed to exist, it follows from (12) and (15) that

$$\lim_{E\to\infty} \frac{\partial\sigma(E,t)}{\partial \ln E} \left|\frac{F_s(E,t)}{E}\right|$$

also exists and

$$\lim_{E\to\infty} D(E,t) = -\frac{\pi}{2} \lim_{E\to\infty} \frac{\partial\sigma(E,t)}{\partial \ln E} \left| F_s(E,t)/E \right|$$

To prove that $\lim\limits_{E\to\infty} D(E,t) = 0$, let us assume that this limit is different from zero. Then the integral

$$\int_{\ln E_0}^{\infty} \frac{\partial\sigma(E',t)}{\partial \ln E'} \left|\frac{F_s(E',t)}{E'}\right| \frac{d\ln E'}{\ln E'}$$

diverges and, because of the Froissart-Martin bound, also

$$\int_{\ln E_0}^{\infty} \frac{\partial\sigma(E',t)}{\partial \ln E'} \ln E' \, d\ln E'$$

is divergent. Integrating by parts, we transform it to

$$\lim_{E\to\infty} \left(\sigma(E',t) \ln E' \Big|_{E_0}^{E} - \int_{E_0}^{E} \sigma(E',t) \frac{dE'}{E'} \right).$$

But the divergence of this expression contradicts (17). Thus, $\lim\limits_{E\to\infty} D(E,t) = 0$.

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