ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2-98-83

A.A.Izmest'ev, G.S.Pogosyan

CONTRACTIONS OF LIE ALGEBRAS AND SEPARATION OF VARIABLES. FROM TWO-DIMENSIONAL HYPERBOLOID TO PSEUDO-EUCLIDEAN PLANE

Submitted to the Conference «Quantum Groups. Deformations and Contractions», Bogazichi University, September 17–24, 1997, Istanbul, Turkey

1998

1 Introduction

In the papers [1, 2] the Inönü-Wigner contractions of the rotation algebra $o(3)$ and Lorentz algebra $o(2,1)$ to the Euclidean algebra $e(2)$ were considered. The two separable coordinate systems on the sphere $S_2 \sim O(3)/O(2)$ and the nine separable coordinate systems on the two-sheeted hyperboloid $L_2 \sim O(2,1)/O(2)$ were related to the four separable systems on the Euclidean plane $E_2 \sim E(2)/O(2)$. Here, we consider the Inönü-Wigner contraction of the Lorentz algebra $o(2,1)$ to the $e(1,1)$ one. In this case, the nine separable coordinate systems on the hyperboloid L_2 are related to the nine orthogonal separable systems on the pseudo-Euclidean plane $E_{1,1}$. Our motivation for present investigation and the results to be expected were discussed in detail in the articles [1, 2].

2 Separable coordinates on the hyperboloid L_2

Consider the hyperboloid L_2

$$u_0^2 - u_1^2 - u_2^2 = R^2; \quad u_0 > 0, \quad R^2 > 0, \tag{1}$$

where u_i $(i = 0, 1, 2)$ are Cartesian coordinates in the ambient space $E_{2,1}$ and R is the radius of curvature of the two-sheeted hyperboloid L_2. The isometry group of L_2 is $O(2,1)$. We choose a standard basis K_1, K_2, M_3 for the Lie algebra $o(1,2)$ [3]

$$K_1 = -\left(u_0\partial_{u_2} + u_2\partial_{u_0}\right), \ K_2 = -\left(u_0\partial_{u_1} + u_1\partial_{u_0}\right), \ M_3 = \left(u_1\partial_{u_2} - u_2\partial_{u_1}\right) \tag{2}$$

with commutation relations

$$[K_1, K_2] = -M_3, \quad [M_3, K_1] = K_2, \quad [K_2, M_3] = K_1. \tag{3}$$

The Laplace - Beltrami operator on L_2 has the form

$$\Delta_{LB} = \frac{1}{R^2}\left(K_1^2 + K_2^2 - M_3^2\right). \tag{4}$$

Following the general method [4] (that has in particular been applied to the hyperboloid [3]) we look for separated eigenfunctions of the Laplace - Beltrami operator satisfying

$$R^2\Delta_{LB}\Psi = l(l+1)\Psi, \quad I\Psi = \lambda^2\Psi; \quad \Psi_{l\lambda}(\zeta_1, \zeta_2) = \Psi_{l\lambda}(\zeta_1)\Psi_{l\lambda}(\zeta_2), \tag{5}$$

where $\lambda = const$ and l, for principal series of unitary irreducible representations, has the form

$$l = -\frac{1}{2} + i\rho, \quad 0 < \rho < \infty. \tag{6}$$

Operator I is a second order operator [5, 6] in the enveloping algebra of $o(2,1)$

$$I = aK_1^2 + b(K_1K_2 + K_2K_1) + cK_2^2 + fM_3^2 + d(K_1M_3 + M_3K_1) + e(K_2M_3 + M_3K_2) \tag{7}$$

(I obviously commutes with the Laplace-Beltrami operator). We list *all* coordinate systems on the hyperboloid L_2 in which the Helmholtz equation (4) permits the separation of the variables [4, 3, 5, 6] and corresponding integrals of motion I are presented in Table 1. There are 9 such systems [4], all are orthogonal, and they are in one to one correspondence with $O(2,1)$ conjugacy classes of operators I. In the notation of coordinate systems we follow [5, 6].

3 Separable coordinates on the pseudo-euclidean plane $E_{1,1}$

Consider the Lie algebra $e(1,1)$ in the basis [7]

$$p_1 = \partial_t, \quad p_2 = \partial_x, \quad M = (t\partial_x + x\partial_t). \tag{8}$$

Separated eigenfunctions of the Laplace operator $\Delta = p_1^2 - p_2^2$ satisfy the equation

$$\Delta\Phi_{k\mu} = k^2\Phi_{k\mu}, \quad X\Phi_{k\mu} = \mu^2\Phi_{k\mu}; \quad \Phi_{k\mu}(t,x) = \Phi_{k\mu}(t)\Phi_{k\mu}(x), \tag{9}$$

where $\mu = const$ and X is the second order operator

$$X = a\cdot M^2 + b\cdot(Mp_1 + p_1M) + c\cdot(Mp_2 + p_2M) + d\cdot p_1^2 + e\cdot p_2^2 + 2f\cdot p_1p_2. \tag{10}$$

We list *all* orthogonal coordinate systems on the plane $E_{1,1}$ in which the Helmholtz equation permits the separation of the variables and corresponding integrals of motion X are given in Table 2 [7]. There are 9 such orthogonal coordinate systems.

4 The contraction of the Lie algebra

We shall use R^{-1} as the contraction parameter and consider contraction from $o(2,1)$ to $e(1,1)$. To realize the contraction explicitly, let us introduce Beltrami coordinates on the hyperboloid L_2 by putting

$$y_0 = R\frac{u_0}{u_2} = R\frac{u_0}{\sqrt{u_0^2 - u_1^2 - R^2}}, \quad y_1 = R\frac{u_1}{u_2} = R\frac{u_1}{\sqrt{u_0^2 - u_1^2 - R^2}}. \tag{11}$$

The $O(2,1)$ generators (2) can be expressed as

$$-\frac{K_1}{R} \equiv \pi_1 = p_1 - \frac{t}{R^2}(tp_1 + xp_2), \quad -\frac{M_3}{R} \equiv \pi_2 = p_2 + \frac{x}{R^2}(tp_1 + xp_2),$$

$$-K_2 \equiv M = tp_2 + xp_1. \tag{12}$$

The commutators of the $o(2,1)$ algebra (3) in new operators (12) take the form

$$[\pi_1, \pi_2] = \frac{M}{R^2}, \quad [M, \pi_1] = -\pi_2, \quad [\pi_2, M] = \pi_1. \tag{13}$$

so, that for $R \to \infty$ the $o(2,1)$ algebra contracts to the $e(1,1)$ one. The $o(2,1)$ Laplace-Beltrami operator (4) contracts to the $e(1,1)$ one:

$$\Delta_{LB} = \pi_1^2 - \pi_2^2 + \frac{M^2}{R^2} \to \Delta = p_1^2 - p_2^2. \tag{14}$$

5 Contractions of coordinates on L_2 to coordinates on pseudo-euclidean space $E_{1,1}$

5.1. Equidistant coordinates on L_2 to pseudo-polar ones on $E_{1,1}$ plane. For Beltrami coordinates (11) we have:

$$y_0 = R\coth\tau_1\cosh\tau_2, \quad y_1 = R\coth\tau_1\sinh\tau_2. \tag{15}$$

Taking the limit $R \to \infty$, $\tau_1 \to i\frac{\pi}{2} + \frac{r}{R}$ and putting

$$\coth\tau_1 = \tanh\frac{r}{R} \sim \frac{r}{R}, \tag{16}$$

we see that Beltrami coordinates go into pseudo-polar ones on $E_{1,1}$ plane

$$y_0 \to t = r\cosh\tau_2, \quad y_1 \to x = r\sinh\tau_2, \tag{17}$$

where $0 \le r < \infty$, $-\infty < \tau_2 < \infty$. For the integral of motion we get

$$I_{EQ} = K_2^2 \to X_{EQ} = M^2. \tag{18}$$

5.2. Pseudo-spherical coordinates on L_2 to Cartesian coordinates on $E_{1,1}$
For Beltrami coordinates (11) we have

$$y_0 = R \coth\tau \frac{1}{\cos\varphi}, \quad y_1 = R\cot\varphi. \tag{19}$$

Taking the limit $R \to \infty$, $\tau \to i\pi/2$, $\varphi \to \frac{\pi}{2}$ and putting

$$\coth\tau \sim \frac{t}{R}, \quad \cot\varphi \sim \frac{x}{R}, \tag{20}$$

we see that Beltrami coordinates go into Cartesian ones

$$y_0 \to t, \quad y_1 \to x. \tag{21}$$

In the limit $R \to \infty$ for the integral of motion we get

$$\frac{I_S}{R^2} = \frac{M_3^2}{R^2} \to p_2^2 \sim X_C. \tag{22}$$

5.3. Horicyclic coordinates on L_2 to cartesian on $E_{1,1}$ plane
For variables $\tilde{x}$ and $\tilde{y}$ we have

$$\tilde{x} = \frac{u_2}{u_0 - u_1}, \quad \tilde{y} = \frac{R}{u_0 - u_1}. \tag{23}$$

In the limit $R \to \infty$ we get

$$\tilde{x} \simeq \frac{R}{t-x} + \frac{t+x}{2R}, \quad \tilde{y} \simeq -\frac{iR}{t-x}. \tag{24}$$

Beltrami coordinates go into Cartesian ones

$$y_0 \to t, \quad y_1 \to x. \tag{25}$$

For the integral of motion we have:

$$\frac{I_{HO}}{R^2} = \frac{1}{R^2}(K_1 + M_3)^2 \to (p_1 + p_2)^2 \sim X_C. \tag{26}$$

5.4. Elliptic coordinates on L_2 to elliptic coordinates on $E_{1,1}$ plane.
For elliptic variables $\rho_{1,2}$ (see Table 1) we have

$$\frac{\rho_{1,2} - a_2}{a_1 - a_2} = \frac{1}{2}\left[\frac{u_0^2 - u_1^2}{R^2} + \frac{a_2 - a_3}{a_1 - a_2}\left(\frac{u_0^2}{R^2} - 1\right)\right] \pm$$
$$\frac{1}{2}\sqrt{\left[\frac{u_0^2 - u_1^2}{R^2} + \frac{a_2 - a_3}{a_1 - a_2}\left(\frac{u_0^2}{R^2} - 1\right)\right]^2 - 4\frac{a_2 - a_3}{a_1 - a_2}\frac{u_1^2}{R^2}}. \tag{27}$$

5.4a. Putting

$$\frac{R^2}{a_1 - a_2} = \frac{D^2}{a_2 - a_3}. \tag{28}$$

and writting coordinates as

$$\rho_1 = a_2 - (a_2 - a_3)\cosh^2\eta, \quad \rho_2 = a_2 + (a_2 - a_3)\sinh^2\zeta \tag{29}$$

in the limit $R \to \infty$ we obtain that Beltrami coordinates go into elliptic coordinates Type I on the $E_{1,1}$ plane

$$y_0 \to t = D\sinh\eta\cosh\zeta, \quad y_1 \to x = D\cosh\eta\sinh\zeta. \tag{30}$$

where $2D$ is the focal distance. For the integral of motion in the contraction limit we obtain

$$\frac{D^2}{R^2}I_E = \frac{D^2}{R^2}(M_3^2 + \sinh^2 f K_2^2) = M^2 + D^2\pi_2^2 \to M^2 + D^2 p_2^2 = X_E. \tag{31}$$

5.4b. Putting

$$\frac{R^2}{a_1 - a_2} = \frac{d^2}{a_3 - a_2} \tag{32}$$

and writting coordinates as

$$\rho_1 = a_2 + (a_2 - a_3)\sinh^2\eta, \quad \rho_2 = a_2 + (a_2 - a_3)\sinh^2\zeta \tag{33}$$

in the limit $R \to \infty$ we obtain that Beltrami coordinates go into elliptic coordinates Type II on the $E_{1,1}$ plane

$$y_0 \to t = d\cosh\eta\cosh\zeta, \quad y_1 \to x = d\sinh\eta\sinh\zeta, \tag{34}$$

where $2d$ is the focal distance. For the integral of motion in the contraction limit we obtain

$$-\frac{d^2}{R^2}I_E = -\frac{d^2}{R^2}(M_3^2 + \sinh^2 f K_2^2) = M^2 - d^2\pi_2^2 \to M^2 - d^2 p_2^2 = X_E. \tag{35}$$

5.4c. As in previous case, using formulas (32), (27) and writting coordinates as

$$\rho_1 = a_2 - (a_2 - a_3)\sin^2\eta, \quad \rho_2 = a_2 - (a_2 - a_3)\sin^2\zeta, \tag{36}$$

we see that Beltrami coordinates in the contraction limit go into elliptic coordinates Type III on the $E_{1,1}$ plane

$$y_0 \to t = d\cos\eta\cos\zeta, \quad y_1 \to x = d\sin\eta\sin\zeta. \tag{37}$$

The integral of motion in the contraction limit is given by (35).

5.5. Elliptic coordinates on L_2 to Cartesian on $E_{1,1}$ plane

We make the special choice $a_1 - a_2 = a_2 - a_3 \equiv a$. Then variables $\xi_{1,2}$ are determined by the formula

$$\xi_{1,2} = \frac{\rho_{1,2} - a_2}{a} = \frac{u_0^2 + u_2^2}{2R^2} \pm \frac{1}{2}\sqrt{\left(\frac{u_0^2 + u_2^2}{R^2}\right)^2 - 4\frac{u_1^2}{R^2}}. \tag{38}$$

Considering the limit $R \to \infty$ we obtain

$$\xi_1 \to \frac{x^2}{R^2}, \quad \xi_2 \to -\left(1 + 2\frac{t^2}{R^2}\right) \tag{39}$$

and Beltrami coordinates go into Cartesian ones. The operator I_E goes into Cartesian one

$$\frac{I_E}{R^2} = \frac{M_3^2}{R^2} + \pi_2^2 \to p_2^2 \sim X_C. \tag{40}$$

5.6. Hyperbolic coordinates on L_2 to elliptic ones on $E_{1,1}$ plane

The variables $\rho_{1,2}$ are determined by formula

$$\frac{\rho_{1,2} - a_2}{a_1 - a_2} = \frac{1}{2}\left[\frac{u_0^2 - u_1^2}{R^2} - \frac{a_2 - a_3}{a_1 - a_2}\frac{u_0^2 - u_2^2}{R^2}\right] \pm$$

$$\frac{1}{2}\sqrt{\left[\frac{u_0^2 - u_1^2}{R^2} - \frac{a_2 - a_3}{a_1 - a_2}\frac{u_0^2 - u_2^2}{R^2}\right]^2 - 4\frac{a_2 - a_3}{a_1 - a_2}\frac{u_0^2}{R^2}}. \tag{41}$$

Putting

$$\frac{R^2}{a_1 - a_3} = \frac{d^2}{a_3 - a_2}, \tag{42}$$

and writting coordinates as

$$\rho_1 = a_2 + (a_2 - a_3)\cosh^2\eta, \quad \rho_2 = a_2 + (a_2 - a_3)\cosh^2\zeta \tag{43}$$

we see that Beltrami coordinates in the contraction limit $R \to \infty$ go into elliptic coordinates Type II

$$y_0 \to t = d\cosh\eta\cosh\zeta, \quad y_1 \to x = d\sinh\eta\sinh\zeta. \tag{44}$$

For the integral of motion in the contraction limit we obtain

$$I_H = K_2^2 - M_3^2\sin^2\gamma \to M^2 - d^2p_2^2 = X_E. \tag{45}$$

5.7. Semi-hyperbolic coordinates on L_2 to Cartesian coordinates on $E_{1,1}$ plane

In this case variables $\mu_{1,2}$ are determined by formulae

$$\mu_{1,2} = \sqrt{\frac{u_0^2u_1^2}{R^4} + \frac{u_2^2}{R^2} \pm \frac{u_0u_1}{R^2}}. \tag{46}$$

In the contraction limit $R \to \infty$ we have

$$\mu_{1,2} = i + \frac{i}{2R^2}(t \pm ix)^2. \tag{47}$$

For Beltrami coordinates we obtain

$$y_0 \to t, \quad y_1 \to x. \tag{48}$$

The integral of motion in the contraction limit takes the form:

$$I_{SH} = \{K_1, M_3\} \to 2p_1p_2 = X_C. \tag{49}$$

5.8. Elliptic-parabolic coordinates on L_2 to hyperbolic ones on $E_{1,1}$ plane

Choosing new variables $\xi_{1,2}$ as

$$\xi_{1,2} = \rho_{1,2} + a = \frac{1}{2}\left[\frac{(u_0 - u_1)^2}{R^2} + a\frac{u_0^2 - u_1^2}{R^2}\right]$$

$$\pm\frac{1}{2}\sqrt{\left[\frac{(u_0 - u_1)^2}{R^2} + a\frac{u_2^2 - R^2}{R^2}\right]^2 + 4a^2\frac{u_2^2}{R^2}}. \tag{50}$$

In the contraction limit $R \sim a \to \infty$ so that $\frac{a}{R^2} \sim \frac{1}{l^2}$ we obtain

$$\xi_1 \sim -e^{2\eta}, \quad \xi_2 \sim e^{2\zeta}. \tag{51}$$

Beltrami coordinates in such limit go into hyperbolic coordinates of Type II

$$y_0 \to t = l(\sinh(\eta - \zeta) + e^{\eta+\zeta}), \quad y_1 \to x = l(\sinh(\eta - \zeta) - e^{\eta+\zeta}), \tag{52}$$

For the integral of motion we have

$$I_{EP} = aK_2^2 + (K_1 + M_3)^2 \to \frac{I_{EP}}{R^2} = \frac{M^2}{l^2} + (p_1 + p_2)^2 = X_H^{II}. \tag{53}$$

5.9. Hyperbolic-parabolic coordinates on L_2 to hyperbolic ones on $E_{1,1}$ plane

This coordinate system is quite analogous to the previous one. In the contraction limit $R \sim a \to \infty$, and $\frac{a}{R^2} \sim \frac{1}{l^2}$ Beltrami coordinates take the form

$$y_0 \to t = l(\cosh(\eta - \zeta) + e^{\eta+\zeta}), \quad y_1 \to x = l(\cosh(\eta - \zeta) - e^{\eta+\zeta}) \tag{54}$$

and we have the hyperbolic coordinates Type III. For the integral of motion we obtain

$$I_{HP} = -aK_2^2 + (K_1 + M_3)^2 \to \frac{I_{HP}}{R^2} = -\frac{M^2}{l^2} + (p_1 + p_2)^2 = X_H^{III}. \tag{55}$$

6 Contraction of bases functions

Using the contraction properties of separable coordinates, we shall now consider the contraction limits for the two simplest eigenfunctions - equidistant and pseudo-spherical bases.

6.1 Equidistant basis on L_2 to pseudo-polar basis on $E_{1,1}$

The equidistant normalized eigenfunctions $\Psi^{EQ}_{\rho\lambda}(\tau_1,\tau_2)$ have the form

$$\Psi^{EQ}_{\rho\lambda}(\tau_1,\tau_2) = \sqrt{\frac{\rho\sinh\pi\rho}{\cosh^2\pi\lambda+\sinh^2\pi\rho}}\cdot(\cosh\tau_1)^{-1/2}P^{i\rho}_{i\lambda-1/2}(\tanh\tau_1)\cdot e^{i\lambda\tau_2}. \quad (56)$$

To perform the contraction we write the Legendre function in terms of hypergeometric functions [8]

$$P^{i\rho}_{i\lambda-1/2}(\tanh\tau_1) = \frac{1}{\sqrt{2\pi}}(-i\sinh\tau_1)^{i\rho}\frac{\Gamma(-i\lambda)}{\Gamma(\frac{1}{2}-i(\rho+\lambda))}\cdot$$

$$\Bigg\{2^{-i\lambda}(\coth\tau_1)^{i\lambda+1/2}{}_2F_1\left(\frac{1}{4}-\frac{i(\rho-\lambda)}{2},\frac{3}{4}-\frac{i(\rho-\lambda)}{2};1+i\lambda;\coth^2\tau_1\right)+$$

$$2^{i\lambda}(\coth\tau_1)^{-i\lambda+1/2}\frac{\Gamma(\frac{1}{2}-i(\rho+\lambda))\Gamma(i\lambda)}{\Gamma(\frac{1}{2}-i(\rho-\lambda))\Gamma(-i\lambda)}$$

$${}_2F_1\left(\frac{1}{4}-\frac{i(\rho+\lambda)}{2},\frac{3}{4}-\frac{i(\rho+\lambda)}{2};1-i\lambda;\coth^2\tau_1\right)\Bigg\}. \quad (57)$$

In the contraction limit $R\to\infty$ we put

$$\rho\sim kR,\quad \cosh\tau_1\sim\frac{r}{R}. \quad (58)$$

Using the asymptotic formulae for hypergeometric functions

$$\lim_{R\to\infty}{}_2F_1\left(\frac{1}{4}-\frac{i(\rho-\lambda)}{2},\frac{3}{4}-\frac{i(\rho-\lambda)}{2};1+i\lambda;\coth^2\tau_1\right)$$

$$=\Gamma(1+i\lambda)\left(\frac{kr}{2}\right)^{-i\lambda}J_{i\lambda}(kr),$$

$$\lim_{R\to\infty}{}_2F_1\left(\frac{1}{4}-\frac{i(\rho+\lambda)}{2},\frac{3}{4}-\frac{i(\rho+\lambda)}{2};1-i\lambda;\coth^2\tau_2\right)$$

$$=\Gamma(1-i\lambda)\left(\frac{kr}{2}\right)^{+i\lambda}J_{-i\lambda}(kr),$$

where $J_\nu(z)$ is the Bessel function [8], and

$$\lim_{|z|\to\infty}\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left(1+\frac{1}{2z}(\alpha-\beta)(\alpha+\beta-1)+O(z^{-2})\right).$$

we finally obtain

$$\lim_{R\to\infty}\frac{1}{\sqrt{R}}\Psi^{EQ}_{\rho\lambda}(\alpha,\tau_2) = \sqrt{\frac{k}{2}}H^{(1)}_{i\lambda}(kr)e^{i\lambda(\tau_2+i\frac{\pi}{2})},$$

where $H^{(1)}_\nu(z)$ is the Hankel function of the first kind.

6.2 Pseudo-spherical basis on L_2 to Cartesian basis on $E_{1,1}$

The normalized pseudo-spherical eigenfunctions $\Psi^S_{\rho m}(\tau,\varphi)$ have the form

$$\Psi^S_{\rho m}(\tau,\varphi) = \sqrt{\frac{2\rho\sinh\pi\rho}{\pi}}|\Gamma(\frac{1}{2}-m+i\rho)|P^m_{i\rho-1/2}(\cosh\tau)e^{im\varphi}.$$

Let us write the Legendre function in terms of hypergeometric functions accordingly [8]

$$P^m_{i\rho-1/2}(\cosh\tau) = \frac{\sqrt{\pi}2^m(\sinh\tau)^{-m}}{\Gamma(\frac{3}{4}-\frac{m+i\rho}{2})\Gamma(\frac{3}{4}-\frac{m-i\rho}{2})}\cdot$$

$$\Bigg\{{}_2F_1\left(\frac{1}{4}-\frac{m+i\rho}{2},\frac{1}{4}-\frac{m-i\rho}{2};\frac{1}{2};\cosh^2\tau\right)+2\cosh\tau\frac{\Gamma(\frac{3}{4}-\frac{m+i\rho}{2})\Gamma(\frac{3}{4}-\frac{m-i\rho}{2})}{\Gamma(\frac{1}{4}-\frac{m+i\rho}{2})\Gamma(\frac{1}{4}-\frac{m-i\rho}{2})}$$

$${}_2F_1\left(\frac{3}{4}-\frac{m+i\rho}{2},\frac{3}{4}-\frac{m-i\rho}{2};\frac{3}{2};\cosh^2\tau\right)\Bigg\},$$

in the contraction limit $R\to\infty$ we put

$$\rho\sim kR,\ m\sim k_1R,\quad \coth\tau\sim\frac{t}{R},\quad \cot\varphi\sim\frac{x}{R};\quad k^2+k_1^2=k_0^2.$$

Using two asymptotic formulae

$$\lim_{R\to\infty}{}_2F_1\left(\frac{1}{4}-\frac{m+i\rho}{2},\frac{1}{4}-\frac{m-i\rho}{2};\frac{1}{2};\cosh^2\tau\right)={}_0F_1\left(\frac{1}{2};-\frac{k_0^2t^2}{4}\right)=\cos(k_0t).$$

$$\lim_{R\to\infty}{}_2F_1\left(\frac{3}{4}-\frac{m+i\rho}{2},\frac{3}{4}-\frac{m-i\rho}{2};\frac{3}{2};\cosh^2\tau\right)={}_0F_1\left(\frac{3}{2};-\frac{k_0^2t^2}{4}\right)=\frac{\sin(k_0t)}{k_0t}$$

and formula (59) we finally obtain

$$\lim_{R\to\infty}\sqrt{R}|\Gamma(i\rho)|\Psi^S_{\rho m}(\tau,\varphi)=\Phi_{k_0k_1}(t,x)=\sqrt{\frac{2}{k_0}}e^{ik_0t-ik_1x}.$$

Table 1: Separable Coordinate Systems on the Two-Dimensional Hyperboloid

Coordinate System Integral of Motion I	Coordinates
I. Equidistant	$u_0 = R\cosh\tau_1\cosh\tau_2$
$\tau_{1,2} \in \mathbb{R}$	$u_1 = R\cosh\tau_1\sinh\tau_2$
$I_{EQ} = K_2^2$	$u_2 = R\sinh\tau_1$
II. Pseudo-spherical	$u_0 = R\cosh\tau$
$\tau > 0,\ \varphi \in [0, 2\pi)$	$u_1 = R\sinh\tau\cos\varphi$
$I_S = M_3^2$	$u_2 = R\sinh\tau\sin\varphi$
III. Horicyclic	$u_0 = R(\tilde{x}^2 + \tilde{y}^2 + 1)/2\tilde{y}$
$\tilde{y} > 0, \tilde{x} \in \mathbb{R}$	$u_1 = R(\tilde{x}^2 + \tilde{y}^2 - 1)/2\tilde{y}$
$I_{HO} = (K_1 + M_3)^2$	$u_2 = R\tilde{x}/\tilde{y}$
IV. Elliptic[a]	$u_0^2 = R^2(\rho_1 - a_3)(\rho_2 - a_3)/(a_1 - a_3)(a_2 - a_3)$
$a_3 < a_2 < \rho_2 < a_1 < \rho_1$	$u_1^2 = R^2(\rho_1 - a_2)(\rho_2 - a_2)/(a_1 - a_2)(a_2 - a_3)$
$I_E = M_3^2 + \sinh^2 f K_2^2$	$u_2^2 = R^2(\rho_1 - a_1)(a_1 - \rho_2)/(a_1 - a_2)(a_1 - a_3)$
V. Hyperbolic[b]	$u_0^2 = R^2(\rho_1 - a_2)(a_2 - \rho_2)/(a_1 - a_2)(a_2 - a_3)$
$\rho_2 < a_3 < a_2 < a_1 < \rho_1$	$u_1^2 = R^2(\rho_1 - a_3)(a_3 - \rho_2)/(a_1 - a_3)(a_2 - a_3)$
$I_H = K_2^2 - \sin^2\gamma M_3^2$	$u_2^2 = R^2(\rho_1 - a_1)(a_1 - \rho_2)/(a_1 - a_2)(a_1 - a_3)$
VI. Semi-Hyperbolic	$u_0 - iu_1 = R\sqrt{(1 - i\mu_1)(1 + i\mu_2)}$
$\mu_{1,2} > 0$	$u_0 + iu_1 = R\sqrt{(1 + i\mu_1)(1 - i\mu_2)}$
$I_{SH} = -\{K_1, M_3\}$	$u_2 = R\sqrt{\mu_1\mu_2}$
VII. Elliptic-Parabolic	$u_0 + u_1 = R(a^2 - \rho_1\rho_2)/\sqrt{a^3(\rho_1 + a)(\rho_2 + a)}$
$-a < \rho_2 < 0 < \rho_1$	$u_0 - u_1 = R\sqrt{(\rho_1 + a)(\rho_2 + a)}/\sqrt{a}$
$I_{EP} = (K_1 + M_3)^2 + K_2^2$	$u_2 = iR\sqrt{\rho_1\rho_2}/a$
VIII. Hyperbolic-Parabolic	$u_0 + u_1 = -iR(a^2 - \rho_1\rho_2)/\sqrt{a^3(\rho_1 + a)(\rho_2 + a)}$
$\rho_2 < -a < 0 < \rho_1$	$u_0 - u_1 = iR\sqrt{-(\rho_1 + a)(\rho_2 + a)}/\sqrt{a}$
$I_{HP} = (K_1 + M_3)^2 - K_2^2$	$u_2 = iR\sqrt{\rho_1\rho_2}/a$
IX. Semi-Circular-Parabolic	$u_0 = R\frac{(\xi^2 + \eta^2)^2 + 4}{8\xi\eta}$
$\xi, \eta > 0$	$u_1 = R\frac{(\xi^2 + \eta^2)^2 - 4}{8\xi\eta}$
$I_{SCP} = \{K_1, K_2\} + \{K_2, M_3\}$	$u_2 = R\frac{\eta^2 - \xi^2}{2\xi\eta}$

[a] Parameter f is determined by relation: $\sinh^2 f = (a_1 - a_2)/(a_2 - a_3) = k'^2/k^2$ $(k'^2 + k^2 = 1)$.

[b] Angle γ is determined by the formula: $\sin^2\gamma = (a_2 - a_3)/(a_1 - a_3)$ where 2γ is the angle between two focal lines.

Table 2: Orthogonal Separable Coordinate Systems on the pseudo-Euclidean plane $E_{1,1}$

Coordinate System	Integral of Motion X	Coordinates
I. Cartesian, type I	$X_C = p_1 p_2$	t x
II. Pseudo-polar $r \geq 0, -\infty < \tau_2 < \infty$	$X_S = M^2$	$t = r\cosh\tau_2$ $x = r\sinh\tau_2$
III. Parabolic type I. $v \geq 0, -\infty < u < \infty$	$X_P^I = \{p_2, M\}$	$t = \frac{1}{2}(u^2 + v^2)$ $x = uv$
IV. Parabolic type II. $-\infty < \eta, \zeta < \infty$	$X_P^{II} = \{p_1, M\}$ $+\{p_2, M\} - (p_1 - p_2)^2$	$t = (\eta - \zeta)^2 - (\eta + \zeta)$ $x = (\eta - \zeta)^2 + (\eta + \zeta)$
V. Hyperbolic type I. $-\infty < \eta, \zeta < \infty$	$X_H^I = M^2 - l^2 p_1 p_2$	$t = \frac{l}{2}\left(\cosh\frac{\eta-\zeta}{2} + \sinh\frac{\eta+\zeta}{2}\right)$ $x = \frac{l}{2}\left(\cosh\frac{\eta-\zeta}{2} - \sinh\frac{\eta+\zeta}{2}\right)$
VI. Hyperbolic type II. $-\infty < \eta, \zeta < \infty$	$X_H^{II} = M^2$ $+l^2(p_1 + p_2)^2$	$t = l\left(\sinh(\eta - \zeta) + e^{\eta+\zeta}\right)$ $x = l\left(\sinh(\eta - \zeta) - e^{\eta+\zeta}\right)$
VII. Hyperbolic type III. $-\infty < \eta, \zeta < \infty$	$X_H^{III} = M^2$ $-l^2(p_1 + p_2)^2$	$t = l\left(\cosh(\eta - \zeta) + e^{\eta+\zeta}\right)$ $x = l\left(\cosh(\eta - \zeta) - e^{\eta+\zeta}\right)$
VIII. Elliptic, type I. $-\infty < \eta, \zeta < \infty$	$X_E^I = M^2 + D^2 p_2^2$	$t = D\sinh\eta\cosh\zeta$ $x = D\cosh\eta\sinh\zeta$
IX. Elliptic, type II,III. $(i)\ -\infty < \eta < \infty, \zeta \geq 0$ $(ii)\ 0 < \eta < 2\pi, 0 \leq \zeta < \pi$	$X_E^{II} = M^2 - d^2 p_2^2$	(i) $t = d\cosh\eta\cosh\zeta$ $x = d\sinh\eta\sinh\zeta$, (ii) $t = d\cos\eta\cos\zeta$ $x = d\sin\eta\sin\zeta$

7 Conclusion

In this paper we continue investigation of some aspects of the theory of the Lie groups and Lie algebra contractions: the relation between separable coordinate systems in curved and flat spaces, related by the contraction of their isometry groups. We have considered the simplest meaningful example, namely the original Inönü-Wigner contraction from a $O(2,1)$ to $E(1,1)$, as applied to the two-sheeted hyperboloid L_2 and pseudo-Euclidean plane $E_{1,1}$.

We have followed through the contraction $R \to \infty$ at four levels: the Lie algebras as realized by vector fields and the Laplace-Beltrami operators in the two spaces, the second order operators in the enveloping algebras, characterizing separable systems, the separable coordinate systems themselves and the two separated eigenfunctions of the invariant operators. In particular, we have shown how *different* limiting procedures lead from separable systems on L_2 to separable systems on the plane $E_{1,1}$. We considered only the contraction first eight separable coordinate systems on L_2 to six separable coordinate systems on $E_{1,1}$.

Acknowledgements

The authors thank Professors P. Winternitz and V.M.Ter-Antonyan for very interesting discussions. One of the authors (G. P.) is thankful to the Organizers of the Conference "Quantum Groups, Deformations and Contractions" for financial support and very kind hospitality.

References

[1] A.A.Izmest'ev, G.S.Pogosyan, A.N.Sissakian and P.Winternitz, *Journal of Physics A: Mathematical and General*, **A29**, 5940, 1996.

[2] A.A.Izmest'ev, G.S.Pogosyan, A.N.Sissakian and P.Winternitz, *Int. J. Mod. Phys.*, **A12**, 53, 1997.

[3] E.G.Kalnins and W.Miller Jr., *J. Math. Phys.* **15**, 1263, 1974.

[4] M.N.Olevskiĭ, *Math. Sb.* **27**, 379, 1950 (in Russian).

[5] P.Winternitz, I.Lukac, and Ya.A.Smorodinskii, *Sov. J. Nucl. Phys.* **7**, 139, 1968.

[6] C.Grosche, G.S.Pogosyan, and A.N.Sissakian, *Phys. Part. Nucl.* **27**, 244, 1996.

[7] E.G.Kalnins, *SIAM J. Math. Anal.*, **6**, 340, 1975.

[8] H.Bateman and A.Erdélyi, *Higher Transcendental Functions* V.1 (New York Toronto London Mc Graw-Hill Book Company, Inc., 1953).

Received by Publishing Department
on April 6, 1998.

Изместьев А.А., Погосян Г.С. E2-98-83

Контракции алгебр Ли и разделение переменных.
От двумерного гиперболоида к двумерной псевдоевклидовой плоскости

Для установления связи между процедурами разделения переменных в уравнениях Гельмгольца на двух соответствующих однородных пространствах — двумерном двуполостном гиперболоиде L_2 и двумерной псевдоевклидовой плоскости $E_{1,1}$ — используется контракция Иноню—Вигнера из группы Лоренца $O(2,1)$ в псевдоевклидову группу $E(1,1)$. Рассматриваются: контракция алгебры Ли и контракция множества коммутирующих операторов второго порядка (интегралов движения), лежащих в обертывающей алгебре $o(2,1)$; контракция (и связь) ортогональных координатных систем на пространствах L_2 и $E_{1,1}$, а также соответствующая контракция некоторых собственных функций операторов Гельмгольца.

Работа выполнена в Лаборатории теоретической физики им. Н.Н.Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1998

Izmest'ev A.A., Pogosyan G.S. E2-98-83

Contractions of Lie Algebras and Separation Variables.
From Two-Dimensional Hyperboloid to Pseudo-Euclidean Plane

The Inönü–Wigner contraction from the rotation group $O(2,1)$ to the pseudo-Euclidean group $E(1,1)$ is used to relate the separation of variables in the Laplace–Beltrami operators on two corresponding homogeneous spaces. We consider the contractions on four levels: the Lie algebra, the commuting sets of second order operators in the enveloping algebra of $o(2,1)$, the coordinate systems and some eigenfunctions of the Laplace–Beltrami operators.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1998