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ON INTERBASIS EXPANSION FOR ISOTROPIC OSCILLATOR ON TWO-DIMENSIONAL SPHERE

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1 Introduction

The present article is devoted to the oscillator system on the two-dimensional sphere $s_1^2+s_2^2+s_3^2=R^2$, which is also known as a Higgs oscillator [1]

$$V=\frac{\alpha^2R^2}{2}\frac{s_1^2+s_2^2}{s_3^2}, \tag{1}$$

where s_i are the Cartesian coordinates in the ambient Euclidean space and R is a radius of sphere. As a "flat" space partner [2], this is a superintegrable system and has the same properties as accidental degeneracy of the energy spectrum [1], separation of variables in more than one coordinate systems [3, 4] and nontrivial realization of hidden symmetries [5] (see also [6]).

The aim of this paper is to describe of solutions to the Schrödinger equation for the potential (1) in three spherical systems of coordinates and to calculate the coefficients of interbasis expansion between the corresponding wave functions.

2 Quantum motion on two-dimensional sphere

The Schrödinger equation on the two-dimensional sphere has the following form:

$$H\Psi=\left[-\frac{1}{2}\Delta_{LB}+V\right]\Psi=E\Psi \tag{2}$$

where Δ_{LB} is the Laplace–Beltrami operator

$$\Delta_{LB}=\frac{1}{R^2}(L_1^2+L_2^2+L_3^2) \tag{3}$$

and L_i are the generators of the Lie algebra $o(3)$

$$L_i=-\epsilon_{ikj}s_k\frac{\partial}{\partial s_j}, \quad [L_i,L_k]=\epsilon_{ikj}, \quad i,k=1,2,3. \tag{4}$$

For $V=0$ the separated eigenfunctions of the Laplace-Beltrami operator satisfy

$$\Delta_{LB}\Psi=-\frac{l(l+1)}{R^2}, \quad I\Psi=k\Psi, \quad \Psi_{lk}(\alpha,\beta)=\psi_{lk}(\alpha)\psi_{lk}(\beta) \tag{5}$$

where I is a second order operator in the enveloping algebra of $o(3)$

$$I=a_{ik}L_iL_k, \qquad a_{ik}=a_{ki}\,. \tag{6}$$

The matrix a_{ik} can be diagonalized to give [7]

$$I(a_1,a_2,a_3)=a_1L_1^2+a_2L_2^2+a_3L_3^2\,. \tag{7}$$

When all three eigenvalues a_i are different, the separable coordinates in (5) are elliptic [8]. If the two eigenvalues of a_i are equal, e.g. $a_1=a_2\neq a_3$ or $a_1\neq a_2=a_3$, or $a_1=a_3\neq a_2$ we can transform the operator I into the operators: $I(0,0,1)=L_3^2$, $I(0,1,0)=L_2^2$, or $I(1,0,0)=L_1^2$. Thus, the corresponding separable coordinates on S_2 are the three type of spherical coordinates

$$\begin{array}{llll} s_1 &= R\sin\theta\cos\varphi &= R\cos\theta' &= R\sin\theta''\sin\varphi'', \\ s_2 &= R\sin\theta\sin\varphi &= R\sin\theta'\cos\varphi' &= R\cos\theta'', \\ s_3 &= R\cos\theta &= R\sin\theta'\sin\varphi' &= R\sin\theta''\cos\varphi'' \end{array} \tag{8}$$

where $\varphi\in[0,2\pi)$, $\theta\in(0,\pi)$. The eigenfunctions of the three sets of operators $\{\Delta_{LB},L_i\}$ are the usual spherical functions on S_2:

$$\Delta_{LB}Y_{lm_i}(\theta,\varphi)=-\frac{l(l+1)}{R^2}Y_{lm_i}(\theta,\varphi) \quad L_i^2Y_{lm_i}(\theta,\varphi)=m_i^2Y_{lm_i}(\theta,\varphi). \tag{9}$$

Geometrically, the spherical coordinates (8) are connected with each other by rotation which may be expressed through the Euler angles (α,β,γ) in accordance with the relations [10]

$$\begin{aligned} \cos\theta' &= \cos\theta\cos\beta+\sin\theta\sin\beta\cos(\varphi-\alpha) \\ \cot(\varphi'+\gamma) &= \cot(\varphi-\alpha)\cos\beta-\frac{\cot\theta\sin\beta}{\sin(\varphi-\alpha)}\,. \end{aligned}$$

Correspondingly, the spherical functions $Y_{lm}(\theta,\varphi)$ are transformed by the formulae [10]

$$\begin{aligned} Y_{l,m'}(\theta',\varphi') &= \sum_{m=-l}^{l}D^l_{mm'}(0,\frac{\pi}{2},\frac{\pi}{2})Y_{l,m}(\theta,\varphi), \\ Y_{l,m''}(\theta'',\varphi'') &= \sum_{m=-l}^{l}D^l_{mm''}(\frac{\pi}{2},\frac{\pi}{2},0)Y_{l,m}(\theta,\varphi), \\ Y_{l,m''}(\theta'',\varphi'') &= \sum_{m'=-l}^{l}D^l_{m'm''}(0,\frac{\pi}{2},\frac{\pi}{2})Y_{l,m'}(\theta',\varphi'), \end{aligned}$$

where $D^l_{m_1,m_2}(\alpha,\beta,\gamma)$ - are the Wigner D-functions.

3 Higgs oscillator on the two-dimensional sphere

3.1 Solution to the Schrödinger equation

3.1 The oscillator potential (1) in the spherical coordinate (θ,φ) is

$$V=\frac{\alpha^2R^2}{2}\frac{s_1^2+s_2^2}{s_3^2}=\frac{\alpha^2R^2}{2}\tan^2\theta \tag{10}$$

and the Schrödinger equation (2) has the following form:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Psi}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2}+2R^2\left[E-\frac{\alpha^2R^2}{2}\tan^2\theta\right]\Psi=0. \tag{11}$$

Choosing the wave function according to

$$\Psi(\theta,\varphi)=\frac{Z(\theta)}{\sqrt{\sin\theta}}\frac{e^{im\varphi}}{\sqrt{2\pi}}, \qquad m\in\mathbf{Z}, \tag{12}$$

after the separation of variables in equation (11) we come to the Pöschl-Teller - type equation:

$$\frac{d^2Z}{d\theta^2}+\left[\varepsilon-\frac{m^2-\frac{1}{4}}{\sin^2\theta}-\frac{\nu^2-\frac{1}{4}}{\cos^2\theta}\right]Z=0 \tag{13}$$

where $\nu=\sqrt{\alpha^2R^4+\frac{1}{4}}$ and $\varepsilon=2R^2E+\alpha^2R^4+\frac{1}{4}$. The solution of the above equation orthonormalised in the interval $\theta\in[0,\pi/2]$ is

$$Z(\theta)\equiv Z_{n_rm}(\theta) = \sqrt{\frac{2(2n_r+|m|+\nu+1)(n_r)!\Gamma(n_r+|m|+\nu+1)}{(n_r+|m|)!\Gamma(n_r+\nu+1)}} \cdot (\sin\theta)^{|m|+\frac{1}{2}}(\cos\theta)^{\nu+\frac{1}{2}}P_{n_r}^{(|m|,\nu)}(\cos2\theta) \tag{14}$$

where $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials [11] and the energy E takes the values

$$E_n=\frac{1}{2R^2}\left[(n+1)(n+2)+(2\nu-1)(n+1)\right] \tag{15}$$

where n_r is a "radial" quantum number and $n=2n_r+|m|$ is the principal quantum number. The degree of degeneracy of the energy spectrum, like the flat two-dimensional oscillator system, is equal to $2n+1$. Note also that in contraction limit when $R\to\infty$, we have $\nu\sim\alpha R^2$ and from formula (15) the energy spectrum for two-dimensional circular oscillator is restored [9].

3.2 In the second spherical coordinate (θ',φ') the potential (1) has the form

$$V=\frac{\alpha^2R^2}{2}\left(\frac{1}{\sin^2\theta'\sin^2\varphi'}-1\right). \tag{16}$$

After the substitution

$$\Psi(\theta',\varphi')=\frac{1}{\sqrt{\sin\theta'}}S(\theta')S(\varphi') \tag{17}$$

we come to the system of differential equations

$$\frac{d^2S}{d\theta'^2}+\left[\varepsilon-\frac{A^2-\frac{1}{4}}{\sin^2\theta'}\right]S=0 \qquad \frac{d^2S}{d\varphi'^2}+\left[A^2-\frac{\nu^2-\frac{1}{4}}{\sin^2\varphi'}\right]S=0 \tag{18}$$

where A is the separation constant. Solving equations (18) we obtain

$$A=n_1+\nu+\frac{1}{2}, \quad \varepsilon=\left(n_2+A+\frac{1}{2}\right)^2=(n_1+n_2+\nu+1)^2=(n+\nu+1)^2 \tag{19}$$

where $n_1,n_2\in\mathbf{N}$ and the principal quantum number $n=n_1+n_2$, so that the energy spectrum is given by equation (15). The orthonormalized eigenfunctions $\Psi(\theta',\varphi')$ could be written as

$$\Psi_{n_1n_2}(\theta',\varphi')=\frac{1}{\sqrt{\sin\theta'}}S_{n_2}^{A}(\theta')S_{n_1}^{\nu}(\varphi') \tag{20}$$

where

$$S_n^a(\varphi) = \frac{\Gamma(a+1)\Gamma(n+a+\frac{1}{2})}{\Gamma(n+a+1)}\sqrt{\frac{(n+a+1/2)n!}{\pi\Gamma(n+2a+1)}}(\sin\varphi)^{\frac{1}{2}+a}C_n^{a+\frac{1}{2}}(\cos\varphi) \tag{21}$$

and C_n^λ are the Gegenbauer polynomials [11]. Finally, note that the operator characterizing the separation solutions in this coordinate system is

$$J_1\Psi_{n_1n_2}=\left(\frac{\partial^2}{\partial\varphi'^2}-\frac{\nu^2-\frac{1}{4}}{\sin^2\varphi'}\right)\Psi_{n_1n_2}=\left[L_1^2-(s_2^2+s_3^2)\frac{\nu^2-\frac{1}{4}}{s_3^2}\right]\Psi_{n_1n_2}=-A^2\Psi_{n_1n_2} \tag{22}$$

3.3 For the potential (1) in the coordinate system (θ'',φ'') we have

$$V=\frac{\alpha^2R^2}{2}\left(\frac{1}{\sin^2\theta''\cos^2\varphi''}-1\right). \tag{23}$$

The orthonormalized solution to the Schrödinger equation (2) have the following form:

$$\Psi_{l_1l_2}(\theta'',\varphi'')=\frac{1}{\sqrt{\sin\theta''}}S_{l_1}^{\nu}(\varphi''+\frac{\pi}{2})\,S_{l_2}^{B}(\theta'') \tag{24}$$

where $l_1,l_2\in\mathbf{N}$, the principal quantum number $n=l_1+l_2$ and the constant $B=l_1+\nu+\frac{1}{2}$. For the energy spectrum we come to expression (15) and the wave function $S_n^a(\theta)$ is given by formula (21).

The additional operator describing this solution and separation is

$$J_2\Psi_{l_1l_2}=\left(\frac{\partial^2}{\partial\varphi''^2}-\frac{\nu^2-\frac{1}{4}}{\cos^2\varphi''}\right)\Psi_{l_1l_2}=\left[L_2^2-(s_1^2+s_3^2)\frac{\nu^2-\frac{1}{4}}{s_3^2}\right]\Psi_{l_1l_2}=-B^2\Psi_{l_1l_2}. \tag{25}$$

3.2 Algebra

If we take the constant of motion in the form

$$\tilde{J}_3 = L_3, \qquad \tilde{J}_1 = L_1^2 - \alpha^2 R^4 \frac{s_2^2}{s_3^2}, \quad \tilde{J}_2 = L_2^2 - \alpha^2 R^4 \frac{s_1^2}{s_3^2},$$

we have the Hamiltonian

$$H = -\frac{1}{2R^2}\left[\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3\right],$$

and the commutator relations

$$\begin{aligned}
[\tilde{J}_1, \tilde{J}_2] &= \{L_1, \{L_2, L_3\}\} + 2\alpha^2 R^4 \left(\frac{s_2^2 - s_1^2}{s_3^2} + \frac{2 s_1 s_2}{s_3^2} L_3\right) \\
[\tilde{J}_1, \tilde{J}_3] &= -\{L_1, L_2\} - 2\alpha^2 R^4 \frac{s_1 s_2}{s_3^2} \\
[\tilde{J}_2, \tilde{J}_3] &= \{L_1, L_2\} + 2\alpha^2 R^4 \frac{s_1 s_2}{s_3^2} \qquad (26)
\end{aligned}$$

where $\{,\}$ is the anticommutator. To close this algebra, we use the redefined operators

$$S_1 = \tilde{J}_3, \qquad S_2 = \tilde{J}_1 - \tilde{J}_2, \qquad S_3 = [S_1, S_2] \qquad (27)$$

and derive the following relations:

$$S_3 = 2\{L_1, L_2\} + 4\alpha^2 R^4 \frac{s_1 s_2}{s_3^2}, \qquad (28)$$

$$[S_3, S_1] = 4S_2, \quad [S_3, S_2] = \frac{4HS_1}{R^2} + 8S_1^3 + 4\left(4\alpha^2 R^4 - 1\right) S_1. \qquad (29)$$

Thus, the operators S_1, S_2, S_3 a generate nonlinear algebra, the so-called cubic.or Higgs algebra.

4 Interbasis expansions

Let us now consider interbasis expansion between two spherical wave functions

$$\Psi_{n_1,n_2}(\theta', \varphi') = \sum_{m=-n}^{n} W_{n_1 n_2}^m \Psi_{n,m}(\theta, \varphi). \qquad (30)$$

To calculate an explicit form of the expansion coefficients $W_{n_1 n_2}^m$ it is sufficient to use the orthogonality for the wave function on one of the variables in the right-hand

side of (30) and to fix, at the most appropriate point, the second variable that does not participate in integration. Rewrite the left-hand side of (30) in the spherical coordinates (θ, φ) according to the formulae

$$\cos\theta' = \sin\theta\cos\varphi, \quad \cos\varphi' = \frac{\sin\theta\sin\varphi}{\sqrt{1 - \sin^2\theta\cos^2\varphi}}.$$

Then, by substituting $\theta = \frac{\pi}{2}$ and taking into account that

$$C_n^\lambda(1) = \frac{\Gamma(2\lambda + n)}{n!\Gamma(2\lambda)}$$

we obtain an equation depending only on the variable φ. Thus, using the orthogonality relation for the function $e^{im\varphi}$ upon the quantum number m, we arrive at the following integral representation for the coefficients W_{n_1,n_2}^m:

$$W_{n_1 n_2}^m(\nu) = \frac{(-1)^{\frac{n-|m|}{2}}}{2^{\nu+1}\pi} \sqrt{\frac{(n_2)!\left(n_1 + \nu + \frac{1}{2}\right)\Gamma(n_1 + 2\nu + 1)(\frac{n+m}{2})!(\frac{n-m}{2})!}{(n_1)!\Gamma(n + n_1 + 2\nu + 2)\Gamma(\frac{n-m}{2} + \nu + 1)\Gamma(\frac{n+m}{2} + \nu + 1)}}$$

$$\cdot \frac{\Gamma(n_1 + \nu + \frac{3}{2})\Gamma(n + \nu + 1)}{\Gamma(n + \nu + \frac{3}{2})} I_{n_1 n_2 m}^\nu \qquad (31)$$

where

$$I_{n_1 n_2 m}^\nu = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} (\sin\varphi)^{n_1} C_{n_2}^{n_1+\nu+1}(\cos\varphi) e^{-im\varphi} d\varphi. \qquad (32)$$

To calculate the integral $I_{n_1 n_2 m}^\nu$ it is sufficient to write the Gegenbauer polynomial $C_{n_2}^{n_1+\nu}(\cos\varphi)$ and $(\sin\varphi)^{n_1}$ as a series in terms of the exponents. After integration we obtain

$$\int_0^{2\pi} (\sin\varphi)^k C_n^\lambda(\cos\varphi) e^{-im\varphi} d\varphi = \frac{(-1)^{\frac{n-m}{2}} 2^{\lambda-k+\frac{1}{2}} \pi \Gamma(\lambda + n + \frac{1}{2}) k!}{n!\Gamma(\lambda + \frac{1}{2})\Gamma\left(\frac{n+k-m}{2} + 1\right)\Gamma\left(\frac{k-n+m}{2} + 1\right)}$$

$$\cdot \; {}_3F_2\left\{\begin{matrix} -n, & -\frac{n+k-m}{2}, & \lambda \\ -\lambda - n + 1, & \frac{k-n+m}{2} + 1 & \end{matrix}\middle| 1\right\}.$$

The introduction of (32) into (31) gives us the interbasis coefficients in the closed form

$$W_{n_1 n_2}^m(\nu) = (-1)^{\frac{|m|-m-n_1}{2}} \sqrt{\frac{2(n_1 + \nu + \frac{1}{2})(n_1)!\Gamma(n_1 + 2\nu + 1)}{(n_2)!\Gamma(n + n_1 + 2\nu + 2)\Gamma(\frac{n-m}{2} + \nu + 1)\Gamma(\frac{n+m}{2} + \nu + 1)}}$$

$$\cdot \sqrt{\frac{(\frac{n+m}{2})!}{(\frac{n-m}{2})!}} \frac{\Gamma(n + \nu + 1)}{\Gamma(\frac{n_1-n_2+m}{2} + 1)} \; {}_3F_2\left\{\begin{matrix} -n_2, & -\frac{n-m}{2}, & n_1 + \nu + 1 \\ -n - \nu, & \frac{n_1-n_2+m}{2} + 1 & \end{matrix}\middle| 1\right\}. \qquad (33)$$

The interbasis coefficients $W_{n_1 n_2}^{m}(\nu)$ could also be expressed in term of the Clebsch–Gordan coefficients for $SU(2)$ group, analytically continued to the real values of their arguments. Using the formula for the Clebsch-Gordan coefficients $C_{a,\alpha;b,\beta}^{c,\gamma}$ [10]

$$C_{a,\alpha;b,\beta}^{c,\gamma} = \delta_{\gamma,\alpha+\beta}\sqrt{\frac{(a+\alpha)!(b-\beta)!(c+\gamma)!(c-\gamma)!(2c+1)}{(a+b-c)!(a+b+c+1)!(a-\alpha)!(b+\beta)!}}$$

$$\frac{\sqrt{(a-b+c)!(c-a+b)!}}{(-b+c+\alpha)!(-a+c-\beta)!}\ {}_3F_2\left\{\begin{matrix} -a-b+c, & -a+\alpha, & -b-\beta \\ -a+c-\beta+1, & -b+c+\alpha+1 & \end{matrix}\middle| 1\right\} \qquad (34)$$

and following the property of the polynomial hypergeometric function ${}_3F_2$

$${}_3F_2\left\{\begin{matrix} a, & b, & c \\ d, & e & \end{matrix}\middle| 1\right\} = \frac{\Gamma(d)\Gamma(d-a-b)}{\Gamma(d-a)\Gamma(d-b)}\ {}_3F_2\left\{\begin{matrix} a, & b, & e-c \\ a+b-d+1, & e & \end{matrix}\middle| 1\right\}. \qquad (35)$$

We can rewrite the formula (34) in the form

$$C_{a,\alpha;b,\beta}^{c,\gamma} = \delta_{\gamma,\alpha+\beta}\sqrt{\frac{(2c+1)(b+c-a)!(b-\beta)!(c+\gamma)!(c-\gamma)!}{(a+b-c)!(a-b+c)!(a+b+c+1)!(a+\alpha)!(a-\alpha)!(b+\beta)!}}$$

$$\frac{(2a)!(c-b+\alpha)!}{(c-b+\alpha)!(c-a-\beta)!}\ {}_3F_2\left\{\begin{matrix} -a-b+c, & -a+\alpha, & b-a+c+1 \\ -2a, & c-a-\beta+1 & \end{matrix}\middle| 1\right\}. \qquad (36)$$

By comparing equations (36) and (33) we finally obtain

$$W_{n_1 n_2}^{m}(\nu) = (-1)^{\frac{|m|-m-n_1}{2}}\ C_{\frac{n+\nu}{2},\frac{\nu+m}{2};\frac{n+\nu}{2},\frac{\nu-m}{2}}^{n_1+\nu,\nu}. \qquad (37)$$

The inverse expansion of (30), namely

$$\Psi_{nm}(\theta,\varphi) = \sum_{n_1=0}^{n} \tilde{W}_{nm}^{n_1}(\nu)\Psi_{n_1 n_2}(\theta',\varphi') \qquad (38)$$

immediately follows from the orthogonality property of the $SU(2)$ Clebsch-Gordan coefficient. Thus, the interbasis coefficients in expansion (38) are given by

$$\tilde{W}_{nm}^{n_1}(\nu) = (-1)^{\frac{|m|-m+n_1}{2}}\ C_{\frac{n+\nu}{2},\frac{\nu+m}{2};\frac{n+\nu}{2},\frac{\nu-m}{2}}^{n_1+\nu,\nu} \qquad (39)$$

and may be expressed in terms of the ${}_3F_2$ function through (33).

Using the same method we could calculate the coefficients of the interbasis expansion for the wave functions (24) and (12). We have

$$\Psi_{l_1 l_2}(\theta'',\varphi'') = \sum_{m=0}^{n}(-1)^{n+\frac{m}{2}} W_{l_1 l_2}^{m}(\nu)\Psi_{nm}(\theta,\varphi) \qquad (40)$$

where the coefficients $W_{l_1 l_2}^{m}(\nu)$ are given by formulae (33) or (37) by replacing the quantum number $n_i \rightarrow l_i$.

The last interbasis expansion between two spherical wave functions (24) and (17) can be constructed by using equations (40) and (38)

$$\Psi_{l_1 l_2}(\theta'',\varphi'') = \sum_{n_1=0}^{n} U_{l_1 l_2}^{n_1}(\nu)\Psi_{n_1 n_2}(\theta',\varphi'), \qquad (41)$$

where

$$U_{l_1 l_2}^{n_1}(\nu) = (-1)^{l_2+\frac{l_1+n_1}{2}} \sum_{m=-n}^{n}(-1)^{\frac{m}{2}} C_{\frac{n+\nu}{2},\frac{\nu+m}{2};\frac{n+\nu}{2},\frac{\nu-m}{2}}^{l_1+\nu,\nu}\, C_{\frac{n+\nu}{2},\frac{\nu+m}{2};\frac{n+\nu}{2},\frac{\nu-m}{2}}^{n_1+\nu,\nu}. \qquad (42)$$

Finally, note that direct methods of calculation of the coefficients in expansion (41) give us the hypergeometrical function ${}_4F_3$ from the unit argument.

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