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E.A.Tagirov*

QUANTUM MECHANICS IN GENERAL RELATIVITY AND ITS SPECIAL–RELATIVISTIC LIMIT

*E-mail: tagirov@thsun1.jinr.dubna.su

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1. Introduction

The present communication is devoted to construction of quantum mechanics of a particle in the general external gravitational field which is treated general-relativistically as the metric of *the Riemannian space-time* $V_{1,3}$. Quantum mechanics in $V_{1,3}$ has its own domain of application, at least speculative, but here I pursue the aim to look at the well-known but still rather mysterious theory from the viewpoint of the changed geometrical background. One may hope that thus some new knowledge can be achieved on the still rather mysterious quantum theory. To my opinion, the results of the present paper justify such an expectation and they concern not only quantum mechanics in $V_{1,3}$, but also in *the Minkowskian space-time* $E_{1,3}$.

There are two basically different approaches to the problem that has been set. The first, more traditional one is quantization of the classical mechanics. In the simplest case of a neutral and spinless point particle one should quantize the mechanics of the geodesic motion. The second, on which the present paper is concentrated, can be characterized as a restriction to the one-particle configurational subspace of the Fock space in the quantum theory of the (linear) field which corresponds to the particle. Simply speaking, I consider a particle as a spatially localized configuration of the quantum scalar field (QFT) and suppose that creation and annihilation of particles by the external gravitation are negligible. Then, the one-to-one particle matrix elements of naturally and, in a sense, uniquely determined in the operators of momentum and spatial coordinate can be represented as matrix elements of Hermitean differential operators on a space Ψ of solutions $\psi(x)$ of a Schrödinger equation with the hamiltonian which is Hermitean with respect to the inner product induced by an $L_2(\Sigma; C)$ norm, where Σ is *a Cauchy hypersurface* in $V_{1,3}$. This $L_2(\Sigma; C)$ norm provides $\psi(x)$ by the Born probabilistic interpretation in the configurational space, and the structure formed by Ψ and the Hermitean operators induced from the QFT is similar to the standard nonrelativistic quantum mechanics (NRQM), but the velocity of light c is finite in it. I shall refer further to this structure as *the quasinonrelativistic representation* of the relativistic quantum mechanics. In the general $V_{1,3}$ where the metric depends on time in any system of reference (see definition in Sec.4) this representation can be constructed only in the form of asymptotic expansions in c^{-2}, which are valid just for the case where creation and annihilation of particles can be neglected. In the globally static $V_{1,3}$, the number of particles does not change, the representation has a closed form and is valid formally for any value of c^{-2} (see Sec.6.).

A question of fundamental importance is: do these two approaches, quantization of the classical mechanics and the field-theoretical approach, lead in any sense to the same quantum mechanics? In spite of that in $V_{1,3}$ many other questions remain open in both the approaches, the answer to the question posed is definitely negative because there are distinctions between the resulting structures of quantum mechanics even in the limiting case of free motion in $E_{1,3}$.

The main distinction is in that the quasinonrelativistic operators of momenta and (curvilinear) spatial coordinates in Ψ, which follow from the QFT, are different, in general, from the canonical ones which are postulated for the immediate quantization of mechanics , see Sec.2; exceptions are the cases of $c^{-1} = 0$ for any $V_{1,3}$ and of Cartesian coordinates and momenta conjugate to them for free motion in $E_{1,3}$. In particular, with the indicated exceptions , any field-theoretically defined quasinonrelativistic operators of coordinates are noncommutative whereas commutativity of them is a postulate of of canonical quantization of mechanics.

In the case of free motion in $E_{1,3}$ the representation space Ψ is the space of the negative-frequency relativistic wave functions in the Feshbach – Villars representation [1]. However, Feshbach and Villars took the canonical expressions for operators of the momentum and Cartesian coordinates in fact postulatively whereas I deduce them from naturally and, in a sense, uniquely defined QFT–operators. Besides, remaining on the level of primary quantized theory Feshbach and Villars considered the complex scalar field as describing electrically charged particles. However, a complex structure of representation space is a general property of quantization (this is well explained, for example, in [2]) and second quantization of a linear field theory consists essentially in specification of a space of complex solutions of the field equation if even the field were real in the initial classical theory and the corresponding particles ("quanta" of the field) are not charged. I consider the neutral scalar field because only chargeless point particles move in $V_{1,3}$ along geodesics and the classical dynamics of electrically charged particles in $V_{1,3}$ is essentially different from the geodesical one even locally as it was shown by Hobbs [3] who corrected results of Brehme and DeWitt [4].

Thus, the change of geometrical background of quantum theory to the Riemannian one lead to the mentioned unusual conclusions should not seem strange if one recalls that the canonical quantization is only a postulate for restoration of a quantum theory from its classical counterpart up to $O(\hbar^2)$. The deformation quantizations, popular now, and related noncommutative geometries are in fact attempts to go outside the limits of this postulate.

Few words on other attempts of field–theoretical approach to quantum mechanics in the general external gravitational field. (There is also a great activity in study of the cases of particular space–time simmetries, but I concentrate here on the generally nonsymmetric space–time specifically with the hope that it may reveal more distinctly the role of the symmetry in the quantum theory.) A systematic study of the problem on the level of the "primary quantized" theory is done by Gorbatzevich [6] who applied in $V_{1,3}$ the operator method by Stephani [7] of transformation of the Dirac equation to the form of Pauli equation originally done for the case of the external electromagnetic field in $E_{1,3}$. The Stephani representation differs essentially from the more known Foldy–Wouthuysen one in that it, contrary to the latter, leads to the $L_2(E_3; C) \oplus L_2(E_3; C)$ norm of the representation space. The present paper emphasizes in particular the necessity of $L_2(\Sigma; C)$ structure (Σ is a space–like hypersurface in $V_{1,3}$, the configurational space of a particle) for the particle representation of QFT, the fact seeming obvious but often not recognized.

However, in [6], as well as in [7], the quantum–mechanical operators of observables are introduced "manually" as the canonical ones. Contrary to this, in papers [8–10] of the present author the mean values of quantum–mechanical observables were defined as natural quadratic functionals of the relativistic field corresponding to scalar and Dirac particles and these functionals are expressed as the diagonal matrix elements of Hermitean operators acting in the Feshbach – Villars and Stephani representation spaces respectively. In the present paper this construction for the real scalar field is consecutively justified on the basis of the quantum theory of the real scalar field and its consequences are studied.

The paper is organized as follows. In Sec.2 the postulates of quantization of a finite–dimensional Hamiltonian system are recalled and results of formal general-relativistic application of these postulates to the mechanics of a spinless particle in $V_{1,3}$ are presented according to Sniatycki [11]. However, this approach, being mathematically impeccable, does not provide with the standard probabilistic interpretation of the vectors of representation space which is essentially based on an 1+3-foliation of $V_{1,3}$ on time and space. Quantization of the geodesic motion in $V_{1,3}$ in the 1+3-formalism is not developed yet but the results in [11] suggest the general form of operators of basic mechanical observables in the 1+3-foliation formalism too.

Further, in Sec.3, the general set of Fock representations of the canonically quantized real scalar field in the general $V_{1,3}$ is considered and QFT-operators of basic observables are introduced. The main problem here is specification of the Fock space, the vectors of which have a particle interpretation. In Sec.5 its asymptotic solution with respect c^{-2} is proposed, which is based on the idea on a quantum particle as a stable field configuration localized on the normal geodesic translations $S(x) = const$ of a given initial *Cauchy hypersurface* Σ.

In Sec.5 the matrix elements of the introduced QFT-operators of spatial position and momentum between the asymptotic one-particle states are represented as matrix elements of Hermitean (self-adjoint) differential operators in the space of solutions of a Schrödinger equation in the configurational representation determined by the introduced Σ. The obtained structure looks as a generally covariant generalization of the standard NRQM in the Schrödinger representation, but the differential operators of position and momenta acquire relativistic (asymptotic in c^{-2}) corrections of any given order N.

In Sec.6 the cases of globally static $V_{1,3}$ and, in particular, of $E_{1,3}$ are considered. In these cases a particular normal geodesic congruences (the frames of reference) exist in which the external gravitation do not change the number of particles. Just in these cases, the asymptotic expansions can be converted for $N \to \infty$ to formally closed exactly relativistic expressions. Nonlocality of relation between relativistic and quasinonrelativistic wave functions is discussed in Sec.7 in connection with the so called Hegerfeldt theorem.

A short discussion of results and prospects of refinement and development of the obtained structure is given in the concluding Sec.8.

It should be noted at once that an heuristic (or naive) level of mathematical rigor is adopted and a majority of assertions of are of general situation, that is the necessary mathematical conditions are supposed to be fulfilled. For example, "asymptotic" means actually "formal asymptotic" throughout the paper. I hope that it is plausible because my first aim is to reveal possible changes in quantum mechanics related to or suggested by introduction of the Riemannian geometry of space–time. A necessary mathematical refinement can be made if the primary results and further development of them prove to be interesting.

Notation is standard for general relativity and, as a rule, in the simple index form, though, when it cannot cause a confusion, indexless notation, like, e.g., $\tau\nabla \equiv \tau^\alpha \nabla_\alpha$, will also be used for brevity. *The dot between differential operators denotes operator product* of them, i.e. $\hat{A} \cdot \hat{B}$ means that $\hat{A} \cdot \hat{B}\, \psi(x) \equiv \hat{A}(\hat{B}\psi(x))$.

2. Quantization of Classical Mechanics in Riemannian Space–Time

Quantization, according to Dirac [12], is a linear map $Q : f \to \hat{f}$ of the Poisson algebra of functions $f \in C^\infty(M)$ on a symplectic manifold (M_{2n}, ω), ω being a symplectic form, to a set of operators acting in a pre-Hilbert space $\mathcal{H}$ (*the representation space*), provided the following conditions are fulfilled:

1) $1 \to \hat{1}$;

2) $\{f,g\}_{\text{Poisson}} \quad \to i\hbar^{-1}[\hat{f},\hat{g}] \stackrel{def}{=} i\hbar^{-1}(\hat{f}\hat{g}-\hat{g}\hat{f})$;

3) $\hat{\bar{f}} = (\hat{f})^{\dagger}$, where *the dagger denotes the Hermitean conjugation with respect the scalar product of* $\mathcal{H}$;

4) a complete set of operators $\hat{f}_1, ..., \hat{f}_n$ exists, such that, if $\left[\hat{f},\ \hat{f}_i\right] = 0$ for any i, then $\hat{f} = \hat{f}(\hat{f}_1, ..., \hat{f}_n)$.

The map Q cannot be found for an arbitrary M_{2n} but for the dynamics of a point particle in $V_{1,3}$ a solution of the problem in the framework of the geometric quantization is presented in the monograph by Śniatycki [11]. I have no possibility (and a capacity, too) to enter into details of the geometrical quantization in the present paper. Instead, as a primitive user, I describe very briefly the initial M_{2n} and resulting map Q related to the geodesic dynamics in $V_{1,3}$ following to [11].

For a point-like particle moving along geodesic lines in $V_{1,3}$, the manifold M_{2n} is $T^*V_{1,3}$, a cotangent bundle over $V_{1,3}$ with a projection $\pi : T^*V_{1,3} \to V_{1,3}$. Any appropriate set $\{q^{(\alpha)}(x)\}$, $x \in V_{1,3}$ of four functions which satisfy the condition $\det \|\partial_\alpha q^{(\beta)}\| \neq 0$ defines on $T^*V_{1,3}$ a set of functions $q^{(\alpha)} = q^{(\alpha)}(x) \circ \pi$ which are constant on fibers of $T^*V_{1,3}$ and will be referred following [11], as *position type functions*. It is important to keep in mind that in the present Sec.2 $q^{(\alpha)}$ and $q^{(\alpha)}(x)$ are different functions: their domains are $T^*V_{1,3}$, and $V_{1,3}$ respectively. Then, a given chart $\{U;\ x^0,\ x^1,\ x^2,\ x^3\}$ in $V_{1,3}$ defines on a canonical chart

$$\left\{\pi^{-1}(U),\ q^{(0)}, ..., q^{(3)}, p_{(0)}, ..., p_{(3)}\right\}$$

on $T^*V_{1,3}$ where functions $p_{(\alpha)}$ are determined so that $\omega = dp_{(\alpha)} \wedge dq^{(\alpha)}$ on $\pi^{-1}(U)$.

An important point for us in this construction is that, on the background of the initial arbitrary curvilinear coordinates $\{x^\alpha\}$ which provide $U_A \subset V_{1,3}$ with an abstract arithmetization, we have introduced a set $q^{(\alpha)}(x)$ of four scalar functions which is related to the phase space of the particle, may be quantized and will be considered further as classical observable of space–time position since the values of the functions also define a point on $V_{1,3}$.

In the introduced notation the general-relativistic dynamics of a point-like particle of the rest mass m on U is determined by the constraint

$$m^2c^2 = p_{(\alpha)}p_{(\beta)}\left(g^{(\alpha)(\beta)}(x)\circ\pi\right), \tag{1}$$

where

$$g^{(\alpha)(\beta)}(x)\big|_U = \frac{\partial q^{(\alpha)}(x)}{\partial x^\gamma}\frac{\partial q^{(\beta)}(x)}{\partial x^\delta}\, g^{\gamma\delta}(x).$$

Thus, on the classical level, the primary observables of the basic physical interest and a constraint on them are introduced. The resulting map Q of quantization for these observables can be exposed according to [11], Sections 1.8, 10.1, as follows.

The representation space $\mathcal{H}$ is $L_2(V_{1,3}, C)$, a space of the complex valued square-integrable over $V_{1,3}$ and sufficiently smooth functions $\varphi(x)$. The $L_2(V_{1,3}, C)$ norm generates an inner product in $\mathcal{H}$ determined as

$$<\varphi_1,\varphi_2> = \int_{V_{1,3}} \overline{\varphi}_1\,\varphi_2\, dv_4, \qquad \varphi_1,\varphi_2 \in \mathcal{H}. \tag{2}$$

dv being the invariant volume element of $V_{1,3}$, i.e.

$$dv\big|_U = (-g)^{1/2}(x)dx^0dx^1dx^2dx^3, \quad g(x) \stackrel{def}{=} \det\ g_{\alpha\beta}(x)\big|_U$$

The operators $\hat{q}^{(\alpha)}$ associated to the position type in $T^*V_{1,3}$ variables $q^{(\alpha)}$ which may play the role of a complete set of functions in the condition 4) of quantization act on $\mathcal{H}$ as

$$\hat{q}^{(\alpha)}\big|_{\pi^{-1}(U)}\,\varphi(x) = q^{(\alpha)}(x)\,\varphi(x), \quad x \in U. \tag{3}$$

Thus $\hat{q}^{(\alpha)}$ form a complete commutative set of operators and the condition 4) of quantization is satisfied.

Instead of operators $\hat{p}_{(\alpha)}$, it is convenient to introduce first an operator of projection of the momentum on a given smooth vector field $K^\alpha(x)$, $x \in V_{1,3}$

$$\hat{p}_K(x)\big|_{\pi^{-1}(U)} = i\hbar\ \left(K^\alpha(x)\nabla_\alpha + \frac{1}{2}\nabla_\alpha K^\alpha(x)\right). \tag{4}$$

where ∇_α is *the covariant derivative* in $V_{1,3}$. The operators canonically conjugate to $\hat{q}^{(\alpha)}\big|_{\pi^{-1}(U)}$ are given by the fields $K^\gamma_{(\beta)}$ which are defined so that $K^\alpha_{(\beta)}\partial_\alpha q^{(\gamma)}(x) = \delta^{(\gamma)}{}_{(\beta)}$.

The operators $\hat{q}^{(\alpha)}$, $\hat{p}_K$ are obviously Hermitean with respect to the inner product $<.,.>$. Also, one can easily see that

$$[\hat{p}_K, \hat{p}_L] = i\hbar\ \hat{p}_{[K,L]_{\text{Lie}}}, \tag{5}$$

where $[K, L]_{\text{Lie}}$ is the Lie derivative of the vector field L along K. Hence, there is a commutative set of four operators $\hat{p}_{K_{(\alpha)}}$, since the vector fields $K^\beta_{(\alpha)}$ commute.

The constraint Eq (1) is mapped by Q to the condition specifying in $\mathcal{H}$ a subspace of functions satisfying the equation

$$\Box\varphi + \zeta\,R(x)\,\varphi + \left(\frac{mc}{\hbar}\right)^2\varphi \ = \ 0, \quad x \in V_{1,3} \tag{6}$$
$$\Box \stackrel{def}{=} g^{\alpha\beta}\nabla_\alpha\nabla_\beta,$$

with $\zeta = 1/6$ and not with $\zeta = 0$ as one might expect from the viewpoint of minimality of the *coupling to gravitation*. This is just consistent with the result of [13, 14] where it had been shown that $\zeta = 1/6$ is necessary for correct particle interpretation of the quantum theory of the scalar field $\varphi(x)$ in $V_{1,3}$. (Despite of that in [13, 14] only QFT in de Sitter space–time was considered, nevertheless the conclusion on necessity of $\zeta = 1/6$ is quite general, as it was indicated in [14].)

However, the presented construction meets a serious difficulty in physical interpretation. It manifests in that those solutions of Eq.(6) which can be considered in QFT in particular space–times $V_{1,3}$ as one-particle wave functions have a diverging $L_2(V_{1,3}, C)$ norm because it demands on $\varphi(x)$ to decrease in time-like directions. The simplest examples are any superposition of the negative-frequency solutions of in $E_{1,3}$ and of the analogous solutions in the De Sitter space-time obtained in [13, 14]. Such property is not compatible with the physical idea of a particle as a stable object in $E_{1,3}$.

It is clear that the roots of the divergence are in the choice of $T^*V_{1,3}$ as initial M_{2n} and in the symmetrical treatment of space and time coordinates. However, a moment of time, contrary to the space position, is not a property of a particle. Therefore the considered scheme of quantization does not lead to any analog of the standard quantum mechanics where the time represents an evolution parameter. Quantization of the geodesical dynamics, after some sort of the 1+3–foliation of $V_{1,3}$ by space-like hypersufaces serving as configuration spaces enumerated by a time-like parameter, would correspond better to this purpose, but it is not apparently done. However, I am not ready here to develop this version of quantization. To my mind, its result is covered by an alternative field–theoretical approach to the construction of quantum mechanics which starts with basic setting of quantum field theory in $V_{1,3}$. Nevertheless, the exposed results of general-relativistic quantization of the particle mechanics suggests how to treat curvilinear coordinates as observables covariantly. It is clear also that, in the 1+3–foliation formalism and immediate quantization of geodesical dynamics, the canonical operators of a spatial coordinate and (5) of pojection of momentum will have the same form of Eqs. (3) and (5) with the modification of q and K to the analogous objects on the spatial sections sections of $V_{1,3}$, at least, in the case when the sections are formed by the normal geodesic translation of a given Cauchy hypersurface Σ, see Sec.4.

3. Quantum Field Theory in Riemannian Space–Time

3.1 The Fock Representation Spaces

Now let us pass to the idea that a structureless neutral particle is, in a sense, a quantum of the canonically quantized *real* scalar field $\hat\varphi(x)$, $x \in V_{1,3}$ satisfying Eq.(6). The general structure of a Fock representation space can be described as follows, see, e.g., [15, 16]

Consider in $\Phi_c = \Phi \otimes \mathbf{C}$, the complexification of the vector space Φ of real solutions to Eq.(6), and a subspace $\Phi'_c \subset \Phi_c$ such that

$$\Phi'_c = \Phi^- \oplus \Phi^+ \tag{7}$$

where $\Phi^\pm$ are supposed to be mutually complex conjugate vectors spaces. They selected so that the sesquilinear (i.e. linear for the second argument and antilinear for the first one) functional

$$\{\varphi_1,\ \varphi_2\}_\Sigma \stackrel{def}{=} i \int_\Sigma d\sigma^\alpha(x)\,(\overline{\varphi_1}(x)\,\partial_\alpha\varphi_2(x) - \partial_\alpha\overline{\varphi_1}(x)\,\varphi_2(x)), \tag{8}$$

where $\Sigma = \{x \in V_{1,3};\ \Sigma(x) = const,\ \partial_\alpha\Sigma\partial^\alpha\Sigma > 0\}$ is a Cauchy hypersurface $V_{1,3}$ and $d\sigma^\alpha(x)$ is its *normal volume element*, is positive (negative) semidefinite on Φ^- (Φ^+). The value of the form does not depend on the choice of the Cauchy hypersurface Σ so far as φ_1 and φ_2 both are solutions of the field equation (6). Therefore the form can be considered as a scalar product in Φ^- providing the latter with a pre-Hilbert structure.

Suppose further that there is a basis $\{\varphi(x;\,A)\}$ in Φ^- enumerated by a multi-index A having values on a set $\{A\}$ with a measure $\mu(A)$ and orthonormalized with respect to the inner product Eq.(8), i.e.

$$\int_{\{A\}} d\mu(A)\ f(A)\ \{\varphi(.\,;A),\ \varphi(.\,;B)\}_\Sigma = f(B) \tag{9}$$

for any function $f(A)$ on $\{A\}$. (The assumption on existence of a basis can be considered as an auxiliary one to come to basic functionals defined below by Eqs.(23), (25) and (26)). Then, the quantum field operator is represented as

$$\hat\varphi(x) = \int_{\{A\}} d\mu(A)\,(c^+(A)\,\overline{\varphi}(x;\,A) + c^-(A)\,\varphi(x;\,A)) \equiv \hat\varphi^+(x) + \hat\varphi^-(x), \tag{10}$$

with the operators $c^+(A)$ and $c^-(A)$ of creation and annihilation of the field modes $\varphi^-(x;\,A) \in \Phi^-$ (of*quasiparticles*), which satisfy the canonical commutation relations

$$[c^+(A),\,c^+(A')] = [c^-(A),\,c^-(A')] = 0,\ \int_{\{A\}} d\mu(A)\,f(A)\,[c^-(A),\,c^+(A')] = f(A')$$

for any smooth function $f(A)$. They act in the Fock space $\mathcal{F}$ with the cyclic vector $|0>$ (*the quasivacuum*) defined by equations

$$c^-(A)\,|0>= 0. \tag{11}$$

Since the decomposition, Eq.(10), in general, can be done by an infinite set of ways and still there is no reason to single out one of them, a question arises of physical interpretation, at least, of a particular choice of the decomposition. It seems natural to look for a decomposition in which the modes might be interpreted as relativistic wave functions of a particle. However, then a question arises on the meaning of the notion "a particle". I shall return to these questions in Sec.4 and now, for time being, continue with the introduced arbitrary Fock spaces.

3.2. Basic Operators of Observables in Fock Spaces

Now operators of observables acting in $\mathcal{F}$ should be introduced. Having in mind reconstruction of quantum mechanics in the configurational space, it is natural to consider as basic operators of observables in QFT the following ones.

The operator of the number of modes (or of quasiparticles) is defined by straightforward generalization to $V_{1,3}$ of the operator of the number of spinless neutral particles of the standard QFT in $E_{1,3}$, see, e.g., [1], Chapter 7, Sec.3:

$$\hat{\mathcal{N}}(\hat\varphi;\,\Sigma) \stackrel{def}{=} \{\hat\varphi^+,\,\hat\varphi^-\}_\Sigma. \tag{12}$$

The operator of projection of the momentum of the field $\hat\varphi(x)$ on a given vector field $K^\alpha(x)$ is also a standard expression:

$$\hat{\mathcal{P}}_K(\hat\varphi;\,\Sigma) = :\int_\Sigma d\sigma^\alpha\ K^\beta T_{\alpha\beta}(\hat\varphi):, \tag{13}$$

where the colons mean the normal product in expression between them and $T_{\alpha\beta}$ is the metrical energy–momentum tensor for $\hat\varphi$, see [13]:

$$\begin{aligned} 2h^{-1}\,T_{\alpha\beta}(\hat\varphi) \;=\; & \partial_\alpha\hat\varphi\,\partial_\beta\hat\varphi + \partial_\alpha\hat\varphi\,\partial_\beta\hat\varphi - g_{\alpha\beta}\left(\partial^\gamma\hat\varphi\,\partial_\gamma\hat\varphi + \left(\frac{mc}{h}\right)^2\hat\varphi^2 + \zeta R\hat\varphi^2\right) \\ & -2\zeta\,(R_{\alpha\beta} + \nabla_\alpha\nabla_\beta - g_{\alpha\beta}\Box)\,\hat\varphi^2. \end{aligned} \tag{14}$$

(The factor h is introduced in Eq.(14) from considerations of the dimensions assuming the dimension of φ is that of the inverse length). It is well known that $\hat{\mathcal{P}}_K(\hat\varphi;\,\Sigma)$ does not depend on the choice of Σ if K^α satisfies the Killing equation

$$\nabla_\alpha K_\beta + \nabla_\beta K_\alpha = 0 \tag{15}$$

and thus defines an isometry of $V_{1,3}$. Then the condition of invariance of the quasivacuum with respect to this symmetry is

$$\hat{\mathcal{P}}_K(\hat\varphi;\,\Sigma)\,|0>= 0 \tag{16}$$

and it distinguishes a particular class of decompositions (7) reduced in $E_{1,3}$ to the unique decomposition for the linear envelops of negative- and positive-frequency exponentials.

A less obvious point is introduction of QFT-operators which in appropriate cases will give observables describing the space localization of a quantum particle. Contrary to the considered case of the momentum, the Lagrangian formalism of the field theory does not provide with classical prototypes of these observables, what is natural. However, in QFT such an observable can make sense and three corresponding operators can be almost uniquely and covariantly introduced if one recalls the position type functions $q^{(\alpha)}(x)$ introduced in Sec.2 and adopts the general structure of the operators $\hat{\mathcal{N}}$ and $\hat{\mathcal{P}}_K$ introduced above. Then, if one accepts the point of view that, for the observables in the NRQM, there exist some prototypes in the relativistic QFT, then the following line of reasoning seems to be satisfactory to define these prototypes.

For a given Cauchy hypersurface Σ, three functions $q_\Sigma^{(i)}(x)$, $i, j, ... = 1, 2, 3$, satisfying the conditions

$$\partial^\alpha\Sigma\, \partial_\alpha q_\Sigma^{(i)}\Big|_\Sigma = 0, \qquad \mathrm{rank}\left\|\partial_\alpha q_\Sigma^{(i)}\right\| = 3. \tag{17}$$

define a point on Σ. Their restrictions on Σ can serve as internal coordinates on it. They may be called spatial position type functions with respect to Σ. It is natural to impose on the corresponding three QFT–operators the following conditions:

1. They should be real local quadratic functionals, like $\hat{\mathcal{P}}_K(\hat{\varphi};\Sigma)$, in the operators $\hat{\varphi}^\pm(x)$ and linear functionals in $q_\Sigma^{(i)}(x)$ expressed as invariant integrals over Σ.
2. They should not contain derivatives of $q_\Sigma^{(i)}(x)$.
3. They should lead to the operator of multiplication by the corresponding argument of the wave function in the limit of the standard NRQM (i.e. $c^{-1} = 0$) in the inertial frame of reference.

These conditions lead apparently to the unique set of three operators on $\mathcal{F}$ which will be called further *(spatial) position type operators* (with respect to Σ):

$$\hat{\mathcal{Q}}^{(i)}\{\hat{\varphi};\Sigma\} \stackrel{def}{=} i\int_\Sigma d\sigma^\alpha\, q_\Sigma^{(i)}(x)\; \left(\hat{\varphi}^+(x)\,\partial_\alpha\hat{\varphi}^-(x) \;-\; \partial_\alpha\hat{\varphi}^+(x)\,\hat{\varphi}^-(x)\right) \tag{18}$$

If $V_{1,3} \sim E_{1,3}$, $\Sigma \sim E_3$ and $q_{E_3}^{(i)}(x) \equiv x^i$, x^i being Cartesian coordinates on E_3, this operator coincides with one of two versions of the position operators that had been considered by Polubarinov [17]. Actually, for reasons of causality, which are not correct from the point of view adopted in the present paper, Polubarinov had preferred another definition of the Cartesian version of position operator. However, along with some other unsatisfactory properties, the latter of his definitions does not satisfy the third of the conditions formulated above. Therefore I proceed with the definition Eq.(18) which, in a certain sense, leads to a generalization for $V_{1,3}$ of the known Newton–Wigner operator.

3.3. Restriction to the One–Quasiparticle Subspace of a Fock Space

Let us consider a one–quasiparticle state vector in $\mathcal{F}$

$$|\varphi> \stackrel{def}{=} \{\varphi,\varphi\}_\Sigma^{-1/2}\int_{\{A\}} d\mu(A)\;\tilde{\varphi}(A)\,c^+(A)\,|0>, \tag{19}$$

determined by a complexified field configuration

$$\Phi^- \;\ni\; \varphi(x) = \int_{\{A\}} d\mu(A)\;\tilde{\varphi}(A)\,\varphi(x;A) \tag{20}$$

It is normalized, i.e.

$$<\varphi|\varphi> = 1 \tag{21}$$

because according to Eq.(9)

$$\{\varphi,\varphi\}_\Sigma = \int_{\{A\}} d\mu(A)\;|\tilde{\varphi}(A)|^2 \tag{22}$$

Consider now the matrix elements of operators $\hat{\mathcal{N}}(\hat{\varphi};\Sigma)$, $\hat{\mathcal{P}}_K(\hat{\varphi};\Sigma)$ and $\hat{\mathcal{Q}}^{(i)}\{\hat{\varphi};\Sigma\}$ between two such states $|\varphi_1>$ and $|\varphi_2>$. Simple calculations with the use of Eqs.(23), (14), (18) and (19) give

$$<\varphi_1|\hat{\mathcal{N}}(\hat{\varphi};\Sigma)\,|\varphi_2> = \frac{\{\varphi_1,\varphi_2\}_\Sigma}{\{\varphi_1,\varphi_1\}_\Sigma^{1/2}\{\varphi_2,\varphi_2\}_\Sigma^{1/2}}, \tag{23}$$

$$<\varphi_1|\hat{\mathcal{P}}_K(\hat{\varphi};\Sigma)|\varphi_2> = \frac{P_K(\varphi_1,\varphi_2;\Sigma)}{\{\varphi_1,\varphi_1\}_\Sigma^{1/2}\{\varphi_2,\varphi_2\}_\Sigma^{1/2}} \tag{24}$$

where

$$P_K(\varphi_1,\varphi_2;\Sigma) = \hbar\int_\Sigma d\sigma^\alpha\left(\partial_\alpha\overline{\varphi}_1\,K^\beta\partial_\beta\varphi_2 + K^\beta\partial_\beta\overline{\varphi}_1\,\partial_\alpha\varphi_2\right.$$
$$\left. -\,K_\alpha\left(\partial_\beta\overline{\varphi}_1\,\partial^\beta\varphi_2 - \left(\left(\frac{mc}{\hbar}\right)^2 + \zeta R\right)\overline{\varphi}_1\varphi_2\right) - \zeta\,K^\beta\left(R_{\alpha\beta} + \nabla_\alpha\nabla_\beta - g_{\alpha\beta}\Box\right)\left(\overline{\varphi}_1\varphi_2\right)\right), \tag{25}$$

and

$$<\varphi_1|\,\hat{\mathcal{Q}}^{(i)}\{\hat{\varphi};\Sigma\}\,|\varphi_2> = \frac{\{\varphi_1, q_\Sigma^{(i)}\varphi_2\}_\Sigma}{\{\varphi_1,\varphi_1\}_\Sigma^{1/2}\{\varphi_2,\varphi_2\}_\Sigma^{1/2}} \tag{26}$$

The right–hand side of Eq.(25) can be simplified by subtraction of the divergence $\nabla^\alpha S_{\alpha\beta}$ of an antisymmetric tensor $S_{\alpha\beta}$ from the integrand, which does not contribute to the integral according to the Gauss theorem. Taking

$$S_{\alpha\beta} \stackrel{def}{=} \zeta\left(K_\alpha\partial_\beta - K_\beta\partial_\alpha + \frac{1}{2}(\nabla_\alpha K_\beta - \nabla_\beta K_\alpha)\right)(\overline{\varphi}_1\varphi_2),$$

one obtains for the last three terms in Eq.(25)

$$\int_\Sigma d\sigma^\alpha\,\zeta K^\beta\left(R_{\alpha\beta} + \nabla_\alpha\nabla_\beta - g_{\alpha\beta}\Box\right)(\overline{\varphi}_1\varphi_2) = \int_\Sigma d\sigma^\alpha\,\zeta(\tilde{K}_{\alpha\beta}\partial^\beta - \nabla^\beta\tilde{K}_{\alpha\beta})(\overline{\varphi}_1\varphi_2) \tag{27}$$

where

$$\tilde{K}_{\alpha\beta} \stackrel{def}{=} \nabla_\alpha K_\alpha + \nabla_\beta K_\alpha - \nabla K\, g_{\alpha\beta} \tag{28}$$

and $\nabla K \stackrel{def}{=} \nabla_\gamma K^\gamma$. The tensor $\tilde{K}_{\alpha\beta}$ evidently vanishes when K_α is a Killing vector and $\mathcal{P}_K(\varphi;\Sigma)$ is a conserved quantity (that is it does not depend on the choice of Σ).

Eqs.(23)–(26) for the matrix elements of the basic observables in the one–quasiparticle subspace of a given $\mathcal{F}$ reveal a projective structure in the space Φ^- : $\varphi(x)$ and $const\cdot\varphi(x)$ are equivalent one–quasiparticle wave functions for calculation of matrix elements $<\varphi_1|\hat{\mathcal{N}}\{\hat{\varphi};\Sigma\}|\varphi_2>$, $<\varphi_1|\hat{\mathcal{P}}_K\{\hat{\varphi};\Sigma\}|\varphi_2>$ and $<\varphi_1|\hat{\mathcal{Q}}^{(i)}\{\hat{\varphi};\Sigma\}|\varphi_2>$ though they are different as complex superpositions of real classical scalar fields.

The sesquilinear functionals $\{\varphi_1,\varphi_2\}_\Sigma$, $P_K(\varphi_1,\varphi_2;\Sigma)$ and $\{\varphi_1, q_\Sigma^{(i)}\varphi_2\}_\Sigma$ of $\varphi_1(x)$, $\varphi_2(x)\in\Phi^-$ are obviously Hermitean in the sense that, given a functional $\mathcal{Z}(\varphi_1,\varphi_2;\Sigma)$, the following equality takes place:

$$\mathcal{Z}(\varphi_1,\varphi_2;\Sigma) = \overline{\mathcal{Z}(\varphi_2,\varphi_1;\Sigma)}. \tag{29}$$

They determine quantum mechanics of a quasiparticle specified by decomposition (7) provided that the processes of creation and annihilation of the quasiparticles by the gravitational field can be neglected.

In principle, one could proceed further with $\{\varphi_1, \varphi_2\}_\Sigma$, $P_K(\varphi_1, \varphi_2; \Sigma)$ and $\{\varphi_1, q^{(i)}_\Sigma \varphi_2\}_\Sigma$ as quantum-mechanical amplitudes of transition under measurement of the corresponding observable. S.Weinberg [18] formulated a version of NRQM with a nonlinear Schrödinger equation in terms of an open algebra of Hermitean functionals of observables. An extension of Weinberg's formalism to our case seems possible in principle and even useful for consideration of quantum mechanics in essentially different frames of reference but this question needs a special study. Here I shall develop the traditional operator formalism.

4. Asymptotic Quasinonrelativistic One–Particle Functions

Now the main problem is to distinguish that space Φ^- which could be interpreted on sufficient physical basis as the space of wave functions of particles instead of the ambiguous notion of quasiparticles. In $E_{1,3}$ and globally static space-times (see definition in Sec.5) there exists a unique decomposition Eq.(7) such that on Φ^- an irreducible representation of the space-time symmetry is realized. Such distinguished Fock spaces are singled out also in the de Sitter and Friedman–Robertson–Walker nonstationary cosmological models but only by combination of the symmetry arguments with additional physical arguments such as a correct quasiclassical behavior of $\varphi(x) \in \Phi^-$ [13, 19], minimality of the rate of cosmological particle creation [20], diagonalization of the field hamiltonian [21].

In the general $V_{1,3}$ one has no symmetry arguments and can appeal only to an intuitive idea of a quantum particle as a localized object, which is firmly formulated only in the Schrödinger representation of the standard nonrelativistic quantum mechanics where $c^{-1} = 0$. A choice of the Fock space realizing this idea can be done only by an immediate construction of a space Φ^-, which, in turn, can be done in the general $V_{1,3}$ only by approximate methods. Having in mind the standard NRQM as a guideline, I shall consider the space $\Phi^-(\Sigma; N)$ of formal asymptotic solutions *of Eq.(6) of an order* N *in* c^{-2} *of the following WKB-type form:*

$$\varphi(x) \;=\; \sqrt{\hbar/2mc}\; exp\left(-i\frac{mc}{\hbar}\, S_\Sigma(x)\right)\, \phi(x). \tag{30}$$

The function $S_\Sigma(x)$ is assumed to be a solution of the Hamilton-Jacobi equation

$$\partial_\alpha S_\Sigma\, \partial^\alpha S_\Sigma \;=\; 1, \tag{31}$$

with the initial value $S_\Sigma(x)\,|_\Sigma = S_0 \equiv const.$ on an initially fixed Cauchy hypersurface Σ. Thus, any hypersurface $S_\Sigma(x) = const$ forms a level surface of a geodesic flow normal to Σ. These hypersurfaces which will be denoted further as S_Σ or simply as S may be called *a normal geodesic translation* S_Σ *of a given* Σ. Thus an 1+3-foliation of $V_{1,3}$ is introduced and the value of S_Σ at given $x \in V_{1,3}$ can be considered as an evolution parameter. Of course, this is only a covariantization introduction of the semigeodesic, or Gaussian, coordinates in $V_{1,3}$.

Until now, the matter resembles the quasiclassical approximation for a second order differential equation with small parameters $\hbar^2$, c^{-2}, or m^{-2} at second derivatives, see, e.g. [22]. One goes to *the quasinonrelativistic approximation* instead of the quasiclassical one when one introduces *a*

normal geodesic frame of reference, i.e. a time-like vector field

$$\tau^\alpha \stackrel{def}{=} c\,\partial^\alpha S_\Sigma, \qquad \tau^\alpha \tau_\alpha \;=\; c^2. \tag{32}$$

This is equivalent to the introduction of a variable $t = c^{-1}x^0$ of the dimensionality of the macroscopic time, after which the right-hand side of Eq.(r) ceases to be a c^{-1}-differential operator in terms of [22].

Then, if Eq.(30) is an asymptotic solution of Eq.(6), i.e.

$$\Box\varphi + \zeta R(x)\varphi + \left(\frac{mc}{\hbar}\right)^2 \varphi = O\left(c^{-2(N+1)}\right), \tag{33}$$

one comes through obvious iterations to the following evolution equation for $\phi(x)$

$$i\hbar\, T\, \phi(x) \;=\; H_N\, \phi(x), \quad H_N \stackrel{def}{=} H_0 \;+\; \sum_{n=1}^{N} \frac{h_n}{2mc^2} + O\left(c^{-2(N+1)}\right), \tag{34}$$

where

$$T \stackrel{def}{=} \tau^\alpha \nabla_\alpha \;+\; \frac{1}{2}\nabla_\alpha \tau^\alpha, \tag{35}$$

cf. Eq.(4), and

$$H_0 \stackrel{def}{=} -\frac{\hbar^2}{2m}\left(\Delta_{S_\Sigma} - \zeta R + \left(\frac{1}{2}(\partial S_\Sigma\; \partial\Box S_\Sigma) \;+\; \frac{1}{4}(\Box S_\Sigma)^2\right)\right). \tag{36}$$

$\Delta_{S_\Sigma}(x)$ is the Laplace-Beltrami operator on the hypersurface $S_\Sigma(x) = const.$ The differential operators h_n are determined by recurrence relations:

$$h_{n+1} \;=\; [-i\hbar\, T,\, h_n] \;-\; \sum_{k=0}^{n} h_k \cdot h_{n-k}, \quad n > 0; \quad h_0 \equiv H_0. \tag{37}$$

It is easy to see that the differential operator $H_N \equiv H_N(x)$ which will be further the main element of construction contains only *covariant derivatives* D_α *along the hypersurface* S to which the point x belongs:

$$D_\alpha \stackrel{def}{=} h^\beta_\alpha \nabla_\beta, \quad h_{\alpha\beta} \stackrel{def}{=} c^{-2}\,\tau_\alpha \tau_\beta \;-\; g_{\alpha\beta}, \tag{38}$$

i.e. $h_{\alpha\beta}$ is *the tensor of projection on* S. For example, $\Delta_S = -D^\alpha D_\alpha$.

Now, turn to the scalar product $\{\varphi_1, \varphi_2\}_\Sigma$, Eq.(8), on $\Phi^-(\Sigma; N)$, which, according to Eq.(23) is a matrix element of $\hat{\mathcal{N}}(\hat\varphi; \Sigma)$, the QFT-operator of the number of quasiparticles. It is obvious that

$$\{\varphi_1, \varphi_2\}_{S_\Sigma} = \{\varphi_1, \varphi_2\}_\Sigma + O\left(c^{-2(N+1)}\right). \tag{39}$$

and thus the quasiparticle is asymptotically stable in the neighborhood of Σ, in which the accepted approximation is valid.

The inner product $\{\varphi_1, \varphi_2\}_\Sigma$ is asymptotically positive definite on $\Phi^-(\Sigma; N)$ in the sense that, for $g_{\alpha\beta} \in C_{2N}(S)$, a sufficiently small value of $c^{-2} > 0$ exists for which $\{\varphi_1, \varphi_2\}_\Sigma$ is positive definite. In a physical sense this is, of course, a condition on the metric, the function $\varphi(x)$ and on their derivatives. Thus, it induces an asymptotic positive definite norm $\{\varphi, \varphi\}^{1/2}_\Sigma$ which, however, is not an $L_2(\Sigma; C)$ norm which would be a natural generalization of the $L_2(E_3; C)$ norm of the standard NRQM in the Schrödinger representation. The latter norm is essential in the quantum mechanics for a precise definition of localization of a particle in terms of the projection-valued measure on E_3 in Cartesian coordinates, see [23], Sec.13-1.

More simply speaking, the integrand of $\{\varphi, \varphi\}_\Sigma$ is not nonnegative and therefore can not be interpreted as a probability density on Σ. (For $E_{1,3}$, an example of a superposition of positive-frequency exponentials for which the integrand oscillates between positive and negative values can be found in [24].) If it were positive everywhere on Σ then one could restrict the integration in expression for $\{\varphi, \varphi\}_\Sigma$ to any domain $\Delta\Sigma \subset \Sigma$ and consider this modified quadratic functional as the probability of detecting a particle in $\Delta\Sigma$, what would correspond to a Born probabilistic interpretation of φ. Then one might restrict to $\Delta\Sigma$ the integrals in expressions for $P_K(\varphi, \varphi; \Sigma)$ and $\{\varphi, q_\Sigma^{(i)}\varphi\}_\Sigma$ and, substituting $P_K(\varphi, \varphi; \Delta\Sigma)$ and $\{\varphi, q_\Sigma^{(i)}\varphi\}_{\Delta\Sigma}$ thus obtained into Eqs.(24) and (26) for $< \varphi|\hat{\mathcal{P}}_K(\hat{\varphi}; \Sigma)|\varphi >$ and $< \varphi|\hat{\mathcal{Q}}^{(i)}\{\hat{\varphi}; \Sigma\}|\varphi >$, come to average values of these observables on $\Delta\Sigma$.

For a free motion in $E_{1,3}$, a mapping of Φ^- to a space with the $L_2(E_3; C)$ is given by the Feshbach - Villars transformation originally set in the momentum representation [1]. The presence of an external field forces to look for a similar transformation in the configurational representation. Therefore, I consider $\phi(x)$ (and, consequently, $\varphi(x)$) as an asymptotic transformation of another function $\psi(x)$:

$$\phi(x) = V_N(x, D)\,\psi(x), \qquad \psi(x)\,|_S \in L_2(S; C) \text{ for any } S \tag{40}$$

and define the asymptotical differential operator $V_N(x, D)$ which acts along the hypersurface S containing the point $x \in V_{1,3}$ so that the following relation takes place:

$$(\psi_1,\, \psi_2)_S \stackrel{def}{=} \int_S d\sigma(x)\,\overline{\psi}_1\,\psi_2 = \{\varphi_1,\, \varphi_2\}_S + O\left(c^{-2(N+1)}\right), \tag{41}$$

$d\sigma(x)$ being the invariant volume element of S. Hence and from Eq.(34) it follows that V_N satisfies up to multiplication from the right by an arbitrary unitary differential operator the equation

$$V_N \cdot V_N^\dagger = \left(1 + \frac{H_N + H_N^\dagger}{2mc^2}\right)^{-1} + O\left(c^{-2(N+1)}\right). \tag{42}$$

Here and further the Hermitean conjugation denoted by the dagger is defined with respect to the scalar product $(\psi_1,\, \psi_2)_S$, that is, for example,

$$(H\psi_1,\, \psi_2)_S = (\psi_1,\, H^\dagger\psi_2)_S\,. \tag{43}$$

It is obvious that Eq.(40) defines V_N up to multiplication from the right by an arbitrary asymptotically unitary differential operator.

It is easily seen from Eq.(34) that $\psi(x)$ satisfies the following Schrödinger equation:

$$i\hbar\, T\,\psi = \hat{H}_N\,\psi, \tag{44}$$

where

$$\hat{H}_N \stackrel{def}{=} V_N^{-1} \cdot (H_N \cdot V_N + [i\hbar T,\, V_N])\,. \tag{45}$$

Further, any differential operator $\hat{z}(x, D)$ which is a polynomial of the order $2N$ of the "spatial" derivatives D_α, Eq.(38), the commutator $[i\hbar T,\, \hat{z}]$, being restricted to the space of solutions of Eq.(44), is again such a polynomial. Hence, taking into account the relation

$$[i\hbar T,\, \hat{z}]^\dagger = -\,[i\hbar T,\, \hat{z}^\dagger], \tag{46}$$

which is not so obvious because there is no Hermitean conjugation for operator T (see [8] for the proof of the relation), one can see that the hamiltonian operator $\hat{H}_N$, in contrast to H_N, is asymptotically Hermitean, i.e.

$$\hat{H}_N = \hat{H}_N^\dagger + O\left(c^{-2(N+1)}\right). \tag{47}$$

Therefore the sesquilinear form $(\psi_1,\, \psi_2)_S$ does not depend on the value of S (though depends on the choice of the initial Σ which generates the 1+3-foliation) and can be considered as a scalar product in *the space* $\Psi(\Sigma; N)$ *of solutions of the Schrödinger equation* Eq.(44). Then $|\psi(x_1)|^2\, d\sigma(x_1)$ can be considered as a density of probability to observe the asymptotically stable configuration described by *the quasinonrelativistic wave function* $\psi(x)$, or, equally, by the corresponding $\varphi(x)$, at the point x_1 of the hypersurface $S(x) = S(x_1)$.

5. Quasinonrelativistic Operators of Observables in the Field-Theoretical Approach

Having accepted the point of view that $\Psi(\Sigma; N)$ is the projective space of states of a quantum spinless particle in $V_{1,3}$ with the Born's probabilistic interpretation of $\psi(x)$ one should introduce a way to evaluate mechanical observables of the particle in the state defined by $\psi(x)$. Now I shall do it on the same field-theoretical basis.

According to Eqs.(23), (39), (41)

$$< \varphi_1|\,\hat{\mathcal{N}}(\hat{\varphi}; S)\,|\varphi_2 >= \frac{(\psi_1,\, \psi_2)_S}{(\psi_1,\, \psi_1)_S^{1/2}\,(\psi_2,\, \psi_2)_S^{1/2}} + O\left(c^{-2(N+1)}\right), \tag{48}$$

that is the operator of number of particles $\hat{\mathcal{N}}(\hat{\varphi}; \Sigma)$ is represented in $\Psi(\Sigma; N)$ by the unity operator as it should be in the quantum mechanics of a single stable particle.

5.1. Quasinonrelativistic Operators of Momentum and Energy of a Particle

Like $\hat{\mathcal{N}}$, the one-particle matrix element (13) of the QFT-operator of the projection of momentum can be represented as a matrix element of an differential operator $\hat{p}_K(x; S; N)$ *acting along* S *on* $\Psi(\Sigma; N)$, *i.e. containing only "spatial" derivatives* D_α:

$$< \varphi_1|\hat{\mathcal{P}}_K(\hat{\varphi}; S)|\varphi_2 >= \frac{(\psi_1,\, \hat{p}_K(.\,; S; N)\psi_2)_S}{(\psi_1,\, \psi_1)_S^{1/2}\,(\psi_2,\, \psi_2)_S^{1/2}} + O\left(c^{-2(N+1)}\right), \tag{49}$$

Obviously,

$$p_K(x; S; N) = p_K^\dagger(x; S; N) + O\left(c^{-2(N+1)}\right) \tag{50}$$

owing to the property (29) of $< \varphi_1|\hat{\mathcal{P}}_K(\hat{\varphi}; S)|\varphi_2 >$. It is natural to consider the operator $\hat{p}_K(x\,; S_\Sigma; N)$ as *the quasinonrelativistic operator of the projection of momentum* on a given vector field K^α in $V_{1,3}$, that is as an analog of the momentum operator of the standard NRQM, but now $c^{-1} \neq 0$.

A straightforward calculation with the use of properties of τ^α, of the relation

$$D_\alpha^\dagger = -D_\alpha - c^{-2}\tau_\alpha\,\nabla_\beta\tau^\beta,$$

and of Eqs. (34), (40), (42) gives

$$\hat{p}_K(x;\,S;\,N) = \frac{1}{2}\,V_N^\dagger\cdot\Big\{m\tau K + \Big(1+\frac{\tilde{H}_N^\dagger}{mc^2}\Big)\cdot m\tau K\,\Big(1+\frac{\tilde{H}_N}{mc^2}\Big)$$
$$-\Big(1+\frac{\tilde{H}_N^\dagger}{mc^2}\Big)\cdot i\hbar KD - (i\hbar KD)^\dagger\cdot\Big(1+\frac{\tilde{H}_N}{mc^2}\Big) - \frac{1}{mc^2}\Big((i\hbar D_\alpha)^\dagger\cdot\tau K\cdot i\hbar D^\alpha\Big)$$
$$+\frac{\zeta\hbar}{2mc^2}\Big(\frac{i}{c^2}\Big((\tau\tilde{K}\tau)\cdot\tilde{H}_N - \tilde{H}_N^\dagger\cdot(\tau\tilde{K}\tau)\Big) - \hbar W(K)\Big)\Big\}\cdot V_N + O\left(c^{-2(N+1)}\right), \quad (51)$$

where

$$\tilde{H}_N \stackrel{def}{=} H_N - \frac{1}{2}i\hbar\nabla\tau$$

, an indexless notation like

$$\nabla\tau \stackrel{def}{=} \nabla_\beta\tau^\beta,\ (\tau\tilde{K}\tau) \stackrel{def}{=} \tau^\alpha\tilde{K}_{\alpha\beta}\tau^\beta \quad (52)$$

is used for simplicity and

$$W(K) \stackrel{def}{=} D^\alpha(\tau^\beta\tilde{K}_{\alpha\beta}) - \tau^\beta\nabla^\alpha\tilde{K}_{\alpha\beta}. \quad (53)$$

There are two explicitly distinctive samplings of K^α: $\tau_\alpha K^\alpha \equiv \tau K = 0$ and $K^\alpha = c^{-1}\tau^\alpha$. In the first case one obtains the spatial projection of momentum:

$$\hat{p}_K(x;\,S;N)\,|_{\tau K=0} = -\frac{1}{2}\,V_N^\dagger\cdot\Bigg(\Big(\Big(1+\frac{\tilde{H}_N^\dagger}{mc^2}\Big)\cdot i\hbar KD + (i\hbar KD)^\dagger\cdot\Big(1+\frac{\tilde{H}_N}{mc^2}\Big)$$
$$-\frac{\hbar\zeta}{2mc^2}(i\,\nabla K\cdot H_N - i\,H_N^\dagger\cdot\nabla K - \hbar W(K))\Bigg)\cdot V_N + O\left(c^{-2(N+1)}\right). \quad (54)$$

For $N = 1$ and $V_1^\dagger = V_1$ Eq.(54) takes the form

$$\hat{p}_K(x;\,S;\,1)\,|_{\tau K=0} \;=\; i\hbar\left(K\nabla + \frac{1}{2}\nabla K\right) + \frac{\hbar}{4mc^2}(D\cdot K)\nabla\tau$$
$$+\;\frac{1-2\zeta}{4mc^2}\,[i\hbar\nabla K,\,H_0] - \frac{\hbar^2\zeta}{2mc^2}W(K) + O\left(c^{-4}\right) \quad (55)$$

We see that for $N = 0$, i.e. for exact nonrelativistic limit, this operator coincides with $\hat{p}_K\,|_U\,(x)$, Eq.(4). It is remarkable, however, that if K^α and L^α are two Killing vectors fields along the level hypersurfaces S, then Eq.(5) takes place for any value of N owing to the relation $[\tau\nabla,\,KD] = [\tau\nabla,\,LD] = 0$. Thus, the operators $\hat{p}_{K_*}(x;\,S;N)$ as well as $\hat{p}_{K_*}\,|_U$ realize a representations of the Lie algebra of the group of isometry defined by the Killing fields along surfaces S. (They are also Killing vectors of the interior geometry of Σ induced by the metric of $V_{1,3}$.) In particular, this means that in the Friedman–Robertson–Walker space–times there is a complete set of commuting asymptotic operators defined by their spatial symmetries, namely, $SO(3)$ for the closed model, $SO(1,2)$ for the open model and $E(3)$ for the spatially flat model.

In general, we come here to a very interesting topic of representation of Lie algebras by asymptotic operators, but it needs a special study.

In the case of $K^\alpha = c^{-1}\tau^\alpha$ simple transformations give the operator of energy

$$c\,\hat{p}_{\tau/c}(x;\,S;\,N) = mc^2 + V_N^\dagger\cdot\Big(H_0 + \frac{\tilde{H}_N^\dagger\cdot\tilde{H}_N}{2mc^2}$$
$$-\frac{\hbar\zeta}{2mc^2}\left([i\hbar T - \tilde{H}_N,\,\nabla\tau] - \hbar(\nabla\tau)^2 - \hbar R_{\alpha\beta}\tau^\alpha\tau^\beta\right)\Big)\cdot V_N + O\left(c^{-2(N+1)}\right). \quad (56)$$

Again for $V_1^\dagger = V_1$ one has

$$c\,\hat{p}_{\tau/c}(x;\,S;\,1) \;=\; mc^2 + H_0 - \frac{H_0^2}{2mc^2} + \frac{i\hbar}{4mc^2}\,[(\nabla\tau),\,(1-2\zeta)H_0 - 2\zeta i\hbar\nabla\tau]$$
$$+\;\frac{1+4\zeta}{8mc^2}\,\hbar^2(\nabla\tau)^2 + O\left(c^{-4}\right). \quad (57)$$

A very important point is that the energy operator $c\,\hat{p}_{\tau/c}(x;\,S;\,N)$ is unitarily equivalent to the hamiltonian $\hat{H}_N$ in Schrödinger equation (44) in the sense that the operator V_N , having been defined by Eq.(42) up to multiplication by an asymptotically unitary operator from the left, can be chosen so that the following equality will take place on $\Psi(S;\,N)$:

$$c\,\hat{p}_{\tau/c}(x;\,S;\,N) \;=\; mc^2 \;+\; \hat{H}_N \;+\; O\left(c^{-2(N+1)}\right). \quad (58)$$

The proof of this fact which is an important indication of the self-consistency of the approach is given in Appendix.

5.2. Quasinonrelativistic Operator of Spatial Position of a Particle

A normal 1+3-foliation of $V_{1,3}$ by the one -parametric set of hypersurfaces S_Σ having been done, the position type functions $q_\Sigma^{(i)}(x)$ of Subsec 3.2. can be introduced so that the conditions (17) are satisfied on each S_Σ. Thus, they are constant on each geodesic which is normal to Σ and translate an interior coordinate system of Σ to each S_Σ.

Then, similarly to the cases of operators $\hat{N}_\Sigma$ and $\hat{\mathcal{P}}_K(\hat{\varphi};\Sigma)$ a spatial operator of position which is Hermitean in Ψ_N is defined by the equality of matrix elements:

$$<\varphi_1|\,\hat{Q}^{(i)}\{\hat{\varphi};\,S\}\,|\varphi_2> = \frac{(\psi_1,\,\hat{q}^{(i)}(.\,;\,S;\,N)\,\psi_2)_S}{(\psi_1,\,\psi_1)_S^{1/2}\,(\psi_2,\,\psi_2)_S^{1/2}} + O\left(c^{-2(N+1)}\right) \quad (59)$$

from which it follows that

$$\hat{q}^{(i)}(x,\,S;N) \;=\; V_N^\dagger\cdot\left(q_S^{(i)}(x) \;+\; \frac{H_N^\dagger\cdot q_S^{(i)}(x) + q_S^{(i)}(x)H_N}{2mc^2}\right)\cdot V_N \;+\; O\left(c^{-2(N+1)}\right). \quad (60)$$

For $V_2^\dagger = V_2$

$$\hat{q}^{(i)}(x,\,S;\,2) \;=\; q_S^{(i)}(x) \;-\; \frac{1}{(2mc^2)^2}\left([i\hbar T - \frac{1}{2}H_0,\,[H_0,\,q_S^{(i)}(x)]]\right) + O\left(c^{-6}\right)$$
$$=\; q_S^{(i)}(x) + \frac{1}{(2mc^2)^2}\left(\frac{\hbar^2}{m}\hat{p}^0_{[\tau,\partial q^{(i)}]} + \frac{i\hbar}{2m}\,[H_0,\,\hat{p}^0_{\partial q^{(i)}}]\right) + O\left(c^{-6}\right). \quad (61)$$

where $\hat{p}^0_K \stackrel{def}{=} \hat{p}_K(x,\,S;\,0)$, the latter expression in Eq.(61) is a consequence of Eq.(5) and of the relation

$$[H_0,\,q_S^{(i)}(x)] = \frac{i\hbar}{m}\hat{p}^0_{\partial q^{(i)}}.$$

It is remarkable that the first relativistic correction vanishes and the operators of space position type functions commute up to $O\left(c^{-4}\right)$ and may be taken up to this accuracy as a complete set of operators of the observables, but for $N > 1$ they are noncommutative. Thus, the field-theoretically determined operators of the space position $\hat{q}^{(i)}(x;\,S;\,N)$ and of the space momentum

$\hat{p}_{\partial_{q^{(i)}}}(x;\, S;\, N)$ cannot coincide with the canonically conjugated primary operators of quantized mechanics unless $N = 0$, i.e. except the exact nonrelativistic limit.

6. Quasinonrelativistic Operators of Observables in Globally Static and Minkowskian Space–Times

Consider now a globally static $V_{1,3}$ where a normal frame of reference $\tau^{\alpha}(x)$ exists that satisfies the Killing equation $\nabla_{\alpha}\tau_{\beta} + \nabla_{\beta}\tau_{\alpha} = 0$. It means that τ^{α} is a covariantly constant vector field. Then, if $S(x)$ is chosen so that $\tau_{\alpha} = c\,\partial_{\alpha}S$, one has $[T,\ H_0] = 0$ and, having taken $V_N^{\dagger} = V_N$, comes to the following formal closed expressions for $N \to \infty$:

$$H_{\infty} = \hat{H}_{\infty} \;=\; mc^2\left(\left(1+\frac{2H_0}{mc^2}\right)^{1/2} - 1\right), \qquad H_0 = -\frac{\hbar^2}{2m}(\Delta_S - \zeta\, R), \tag{62}$$

$$V_{\infty} \;=\; \left(1+\frac{2H_0}{mc^2}\right)^{-1/4}, \tag{63}$$

$$\hat{p}_K(x;\,S;\,\infty)\,|_{(K\tau)=0} \;=\; -\frac{i\hbar}{2}\,V_{\infty}^{-1}\cdot(KD)\cdot V_{\infty} + \frac{i\hbar}{2}\,V_{\infty}\cdot(KD)^{\dagger}\cdot V_{\infty}^{-1} - \frac{\hbar\zeta}{2mc^2}\,V_{\infty}\cdot(\tau\nabla)(\nabla K)\cdot V_{\infty} \tag{64}$$

$$c\,\hat{p}_{\tau/c}(x;\,S;\,\infty) \;=\; mc^2\left(1+\frac{2H_0}{mc^2}\right)^{1/2}, \tag{65}$$

$$\hat{q}^{(i)}(x,\,S;\infty) \;=\; q_S^{(i)}(x) \;+\; \frac{1}{2}\left[[V_{\infty},\ q_S^{(i)}(x)],\ V_{\infty}^{-1}\right]. \tag{66}$$

It should be emphasized that these formulae are exact relativistic ones in the sense that they are not asymptotic and valid for any value of c^{-1}. However, one should keer in mind that they are take place in the particular frame of reference determined by the symmetry of the globally static $V_{1,3}$.

One can come from Eqs.(63) – (66) to the following conclusions.

1) Not only the hamiltonian $\hat{H}$, but also operators of spatial momentum, $hatp_K$, and position, $\hat{q}^{(i)}$, are generally non-local, except the case of $c^{-1} = 0$, i.e. in the exact nonrelativistic limit, when they coincide in form with Eqs.(5) and (3).

2) The operators of spatial projections of momentum become local and coincide in form with that of geometric quantization, Eq.(5), if

$$[KD, H_0] = 0, \tag{67}$$

what means that $K^{\alpha}(x)$ is a Killing vector of each level hypersurface S with respect to metric induced by $V_{1,3}$ and, as a consequence, of $V_{1,3}$ itself.

3) The operators $\hat{q}^{(i)}$ of position on S are noncommutative, again except the case $c^{-1} = 0$ or when functions $q^{(i)}(x)$ are a Cartesian coordinates x^i on a space-like hyperplane in $E_{1,3}$.

In the case of the globally static $V_{1,3}$ the distinction between the field-theoretically determined operators of observables and those that are postulated in immediate quantization of mechanics may not be related to the processes of particle creation and annihilation by the external field.

Now consider the simplest case of the inertial frame of reference in $E_{1,3}$, what means that Σ is a hyperplane E_3. It is easy to see that in this case $E_{1,3}$ the space $\Phi^-(E_3;\,\infty)$ determined by Ψ is a linear envelope of the negative–frequency exponentials. Then, if $q^{(i)} \stackrel{def}{=} x^i$, x^i being Cartesian coordinates on E_3 and $K^{\alpha}_{(i)} = \delta^{\alpha}_i$, it follows from from Eqs.(64) and (66) that

$$\hat{q}^{(i)}(x;\ E_3;\ \infty) \equiv \hat{x}^i = x^i\,\mathbf{1}, \qquad \hat{p}_i = -i\hbar\frac{\partial}{\partial x^i}, \tag{68}$$

i.e. the canonical expressions.

However, the relativistic corrections to the position operator are nonzero even in $E_{1,3}$ and th inertial frames of reference, if curvilinear coordinates on E_3 are taken as $q^{(i)}(x\,;E_3;\,\infty)$. Let, e.g., $q^{(1)} = r$, the ordinary radial coordinate. Then, for $< r > \gg \hbar/mc$, one has from Eq.(66)

$$\hat{r} = r + \left(\frac{\hbar}{2mc}\right)^4 \frac{1}{r^3}\,\Delta_{S_2} + O\left(\left(\frac{\hbar}{2mcr}\right)^6\right), \tag{69}$$

where Δ_{S_2} is the laplacian on the sphere. Apparently, Eqs.(68), (69) mean also that radial positions of a quantum particle determined by a direct measurement of the coordinate r and calculated after measuring of three Cartesian coordinates x^i will differ if the relativistic corrections are taken into account. Speculatively the matter looks as if the measurement of the distance between a spherical radar and a quantum particle would give a result which is different from the result of locating the particle by a huge three–dimensional wire chamber and subsequent calculation of the distance.

The operator $\hat{x}^i$, Eq.(68), can be transformed to the Newton–Wigner operator [25] for to the following reason which is valid for the general $V_{1,3}$. It is easy to see that $\varphi(x) \in \Phi^-(\Sigma;\ N)$ corresponding to $\psi(x) \in \Psi(\Sigma;\ N)$ satisfies the equation:

$$i\hbar\,T\,\varphi(x) \;=\; (mc^2 + H_N)\,\varphi(x), \tag{70}$$

so that owing to Eq.(42)

$$\{\varphi_1,\ \varphi_2\}_S \;= <\varphi_1,\ \varphi_2>_S \stackrel{def}{=} \frac{2mc}{\hbar}\int_S d\sigma\,\overline{\varphi}_1\left(V_N\cdot V_N^{\dagger}\right)^{-1}\,\varphi_2, \tag{71}$$

The operators of position $\check{q}^{(i)}$ with respect to the new scalar product $< .\,,.>$ in $\Phi^-(\Sigma;\ N)$ can be introduced by the relation

$$(\psi_1,\ \hat{q}^{(i)}\psi_2)_S \equiv \{\varphi_1,\ q^{(i)}\varphi_2\}_S \stackrel{def}{=} <\varphi_1,\ \check{q}^{(i)}\varphi_2>_S \tag{72}$$

Hence it follows that

$$\check{q}^{(i)} = q^{(i)} + \frac{1}{2mc^2}V_N^2\cdot[q^{(i)},\ H_N] + O\left(c^{-2(N+1)}\right). \tag{73}$$

In the case when $V_{1,3} \sim R_{1,3}$ and $\Sigma \sim E_3$ and $q^{(i)}(x) \equiv x^i$, x^i being Cartesian coordinates on E_3, Eq.(73) reads as

$$\check{x}^i = x^i + \frac{\hbar}{m(mc^2 + H_0)}\,\frac{\partial}{\partial x^a}, \tag{74}$$

and x^i thus defined is just the Newton–Wigner operator of position in the x–representation. Therefore one may consider Eq.(73) as a generalization of the Newton–Wigner operator to $V_{1,3}$.

However, introduction of the Riemannian background becomes natural for covariant consideration of curvilinear coordinates and curved initial hypersurfaces Σ even in $E_{1,3}$.

7. Note on the Hegerfeldt Theorem

Of course, the representation space Ψ is in an one-to-one correspondence with the space Φ^- of solutions of Eq.(6) spanned by the negative-frequency exponentials in the sense that any $\varphi(x) \in \Phi^-$ can be represented in Cartesian coordinates in the form of Eq.(30)

$$\varphi(x) = \sqrt{\hbar/2mc}\, exp\left(-i\frac{mc}{\hbar}x^0\right)\; V_\infty \psi(x). \tag{75}$$

However, the correspondence is obviously nonlocal owing to the operator V_∞. This nonlocality is apparently a manifestation of a paradox in quantum theory which is referred sometimes to as the Hegerfeldt theorem and, in application to a single particle, consists in that its wave function having initially a compact support $X \subset E_3$, acquires nonzero values at space-like intervals from X in subsequent moments of time. *This looks as a nonzero probability of superluminal propagation of particles.* Hegerfeldt and Ruijsenaars [25] proposed a resolution of the paradox consisting in that a localization in a compact domain is not possible at all, but, in application to our case, they meant the localization in terms of the field $\varphi(x)$. However, the probability density of localization of a particle is determined by the field $\psi(x)$. The initial data for $\varphi(x)$ are related to the initial $\psi(x)$ nonlocally by Eqs.(62), (40), (70). Therefore, even if the particle is localized in the quasi-nonrelativistic sense which is apparently the unique correct sense, nevertheless, the corresponding initial data for the relativistic field are smeared out over the whole E_3.

8. Concluding Remarks

An essential feature of the approach exposed here is that the $L_2(\Sigma; C)$ structure of the representation space and operators of observables acting on it are traced to the corresponding field-theoretical notions of a number of quanta, the energy-momentum and "the position of a quantum". The latter which leads to the most inconvenient conclusions on noncommutativity of operators of coordinates may seem rather artificial but it is unique and necessary for an intrinsic congruence of the approach which certainly deserves to be considered. Indeed, why could not one choose such *sets from the limits of these operators for $c^{-1} = 0$ ($N = 0$),* i.e. *simply as multiplications by functions* $q^{(i)}$, which just suggest by the canonical and geometrical quantization? Of course, one could, but in QFT no physically sensible quantity would correspond to these operators. Particularly, one should then refuse the convenient definition of conserved quantities by Eq.(13) following from the Noether theorems and equivalence between the energy operator $\hat{p}_{\tau/c}$ and the hamiltonian $\hat{H}$.

The present approach and that of quantization of mechanics are together obviously a manifestation of the wave-corpuscular dualism in quantum theory. The former corresponds to the point of view that the Schrödinger wave function $\psi(x)$ is not only a mathematical object but related to the field $\varphi(x)$ carrying an energy-momentum. At the same time, it leads to the inconvenient conclusion that in an external field generally neither operators of momentum nor of coordinates generate a complete set of commuting observables.

Another point for doubts on the presented scheme might be the question of completeness of the spaces $\Phi^-(\Sigma; N)$ and $\Psi(\Sigma; N)$ since $\varphi(x)$ and $\psi(x)$ are subordinated to conditions of validity of the asymptotic expansions. Of course, this question needs an investigation as well as many other points where the adjective "asymptotic" is used (the asymptotic inner product, asymptotic Hermiticity, asymptotic unitarity, the range of validity the asymptotic expansions along the frame of reference etc.). However, the situation looks not worse than with the standard NRQM which, in fact, is also a limit of a more general relativistic theory, but nevertheless its mathematically refinement is developed as it were a closed self-consistent theory. Besides, in the case of the globally static $V_{1,3}$ our construction is not asymptotic and in this sense it is closed.

There are questions which are more specific to quantum mechanics in $V_{1,3}$. For example, generally the frame of reference $\tau^\alpha(x)$ has focal points in the future or past (or both) of Σ, in which the normal geodesics from different points of Σ intersect and the solution of the Cauchy problem for the Hamilton-Jacobi equation (31) has a singularity. The question arises: Can a method of extension of an asymptotic solution over the point be elaborated for the quasinonrelativistic asymptotics analogous to that by Maslov and Fedoriuk [22], for the quasiclassical one? A simple instance when this problem should be studied is quantum mechanics in $E_{1,3}$ determined by a curved Σ instead of a convenient E_3.

Another interesting direction of study is to consider nongeodesic normal frames of reference defined by a Hamilton-Jacobi equation different from Eq.(31) thus distributing an action of external nongravitational forces between the frame of reference and the quantum dynamics of a particle in it.

An important question is: what relation exists between quantum mechanics' determined by different Cauchy hypersurfaces Σ? The operators of observables are determined only in each $\Psi^-(\Sigma; N)$ and can be transformed to the corresponding $\Phi^-(\Sigma; N)$. Each quasinonrelativistic quantum mechanics thus defined forms a coherent, or irreducible lattice, see [23], Sec.8-2, and the quantum principle of superposition takes place in it. On the other hand, a closure of the set theoretical union $\bigcup \Phi^-(\Sigma_n; N)$ for two or even infinite number of essentially different hypersurfaces Σ_n has also the structure of vector space since any superposition of wave functions from $\bigcup \Phi^-$ is again a solution of the field equation (6). However, if one takes φ_1 and φ_2 from different spaces $\Phi^-(\Sigma_n; N)$, then there one may expect (and draft calculations support this expectation though a rigorous proof do not seem easy for the general case) the sesquilinear functionals $\{\varphi_1, \varphi_2\}_\Sigma$, $P_K(\varphi_1, \varphi_2; \Sigma)$ and $\{\varphi_1, q^{(i)}_\Sigma \varphi_2\}_\Sigma$ defined by Eqs.(23), (25), and (26) asymptotically vanish. If it is the case, then a superposition of φ_1 and φ_2 is corresponds to asymptotically mixed state. The situation is just as if the spaces $\Phi^-(\Sigma_n; N)$ form ***superselection sectors*** in $\bigcup \Phi^-(\Sigma_n; N)$ which is a reducible lattice. If this understanding is correct, one reveals a very interesting class of superselection rules associated to frames of reference.

Having recalled the quantum field-theoretical origin of the presented construction, one could also attempt to connect the sets of operators of creation and annihilation of particles, which are determined on two different Fock spaces $\mathcal{F}_a$, $a = 1, 2$, corresponding to given spaces $\Phi^-(\Sigma_a; N)$, by a Bogoliubov transformation. Apparently, this is possible for sufficiently smooth metrics of $V_{1,3}$, and then a one-particle state, say, in $\mathcal{F}_1$ will be represented by a superposition of an infinite set of different many-particles states in $\mathcal{F}_2$.

At last, it is interesting to apply the field-theoretical approach exposed here to fields of nonzero spin. Since the operators of spin projections are defined by the lagrangian of the field one may expect that their algebraic properties are different from the standard ones.

These problems may be criticized as academical ones. However, I think that without study of them our knowledge of quantum theory would be essentially incomplete.

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Appendix

The proof of unitary equivalence of the operator energy, Eq.(56), to the hamiltonian in the Schrödinger equation (44) is the same for any value of ζ. Therefore consider for brevity the case of $\zeta = 0$.

Using expressions Eq.(45) for $\hat{H}_N$ Eq.(56) and relation Eq.(42 for V_N, one can rewrite Eq.(58) as an equation for V_N:

$$[i\hbar T, V_N] = \left\{ H_N - \left(1 + \frac{H_N^\dagger + H_N}{2mc^2}\right)^{-1} \left(H_0 + \frac{\tilde{H}_N^\dagger \cdot \tilde{H}_N}{2mc^2} \right) \right\} \cdot V_N + O\left(c^{-2(N+1)}\right). \qquad (76)$$

Since the operator V_N is a polynomial of the space derivatives D_α with coefficients depending on x Eq.(76) is equivalent to a linear evolution system along the field τ^α on these coefficients in virtue of the relation

$$[T,\ D_\alpha] = \tau^\gamma[\nabla_\gamma,\ \nabla_\alpha] + \nabla_\alpha \tau^\gamma D_\gamma + \frac{1}{2} D_\alpha \nabla_\gamma \tau^\gamma \qquad (77)$$

The first term at the right-hand side will generate in Eq.(76) a term proportional to the Riemann-Christoffel curvature tensor and consequently the commutator is again a polynomial of D_α of the same order. A solution of the Cauchy problem for this evolution system always exists in some neighborhood of an initial $S = \Sigma$. However, the solution should satisfy to the condition (42). The latter, having been imposed the Cauchy data on Σ, is fulfilled τ^α because

$$\left[\tau^\alpha \partial_\alpha,\ V_N^\dagger \cdot \left(1 + \frac{H_N^\dagger + H_N}{2mc^2}\right) \cdot V_N\right] = O\left(c^{-2(N+1)}\right) \qquad (78)$$

in virtue of Eq.(76) and the condition itself. Performing the commutations, one may easily verify that Eq.(78) is equivalent to the condition of the asymptotical hermiticity of the hamiltonian $\hat{H}_N$ expressed in terms of H_N.

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