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THE SOLUTION OF THE $N=(0|2)$
SUPERCONFORMAL f -TODA LATTICE

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1 Introduction

In paper [1], the one-dimensional (1D) f-Toda mapping (chain) responsible for the existence of the $N = 2$ supersymmetric Nonlinear Schrödinger (NLS) hierarchy [2] was introduced. Using this mapping as a starting point, the Hamiltonian structure and recursion operator connecting all evolution systems of the $N = 2$ NLS hierarchy were explicitly constructed there. On the other hand, it has been proved in [3] that the 1D f-Toda chain with fixed ends is exactly integrable system, and the method of constructing its general solutions in terms of a perturbative series with a finite number of terms was proposed. However, these calculations are sufficiently boring, and hence the explicit solution was presented only for a very particular case [3]. Interest in the general solution of the 1D f-Toda chain with fixed ends is mainly motivated by the fact that they, in turn, allow one to construct explicitly multi-soliton solutions of equations belonging the $N = 2$ NLS hierarchy.

After this, the $N = 2$ supersymmetric Toda lattice hierarchy whose first bosonic flow is equivalent to the 1D f-Toda chain was constructed, and its Hamiltonian and Lax-pair descriptions were developed in [4].

The aim of the present letter is to generalize the 1D f-Toda chain to the two-dimensional case and to construct its general solutions. It turns out that the integrable generalization — 2D f-Toda lattice — actually exists and its general solutions in the case of fixed ends can be expressed in terms of matrix elements of different fundamental representations of the $SL(n|n-1)$ supergroup. Our construction is mainly based on results of Ref. [5], where the method of constructing a wide class of integrable systems related to the $sl(n)$ algebras has been proposed. We observe that in a framework of this method there exists some hidden, omitted in [5] non-trivial possibility for deriving a new class of integrable systems. The developed here method is applied to both the case of algebras $sl(n)$ and superalgebras $sl(n|n-1)$. The 2D f-Toda lattice just belongs to a new class and gives the first, simplest representative that possesses the $N = (0|2)$ superconformal symmetry.

2 Structure of $sl(n|n-1)$ and $sl(2n-1)$ (super)algebras

In this section we shortly summarize the main facts concerning the (super)algebras $sl(n|n-1)$ and $sl(2n-1)$ which we use in what follows (for more detail, see [6, 7, 8] and references therein). We consider them simultaneously keeping the explicit dependence of all relevant formulae on their Grassmann nature via the factor \mathcal{P} which is $\mathcal{P} = 0$ or $\mathcal{P} = 1$, respectively.

The (super)algebra $sl(2n-1)$ ($sl(n|n-1)$) can be generated by a set of $4n(n-1)$ graded $(2n-1) \times (2n-1)$ matrices with zero (super)trace. In the Serre-Chevalley basis its defining commutation relations are¹

$$\begin{aligned} [H_i, H_j] &= 0, & [H_i, X_j^\pm] &= \pm K_{ij} X_j^\pm, & [X_i^+, X_j^-] &= \delta_{ij} H_j, \\ & & & & & (1 \leq i, j \leq 2(n-1)), \end{aligned} \quad (1)$$

where H_j and X_j^\pm are the generators of the Cartan subalgebra and raising/lowering operators, respectively, K_{ij} is the symmetric Cartan matrix,

$$K_{ij} = K_{ji} = (-1)^{(i\mathcal{P}+1)}(\delta_{i+1,j} - (1 + (-1)^\mathcal{P})\delta_{i,j} + (-1)^\mathcal{P}\delta_{i,j+1}), \quad (2)$$

and the brackets $[\]$ denote the graded commutator. All its other generators can be derived via the formula

$$\begin{aligned} Y_j^{\pm(k+1)} &= [X_j^\pm, [X_{j+1}^\pm, \dots [X_{j+k-1}^\pm, X_{j+k}^\pm] \dots]], \\ & (1 \leq k < r, \quad 1 \leq j \leq (r-k)), \end{aligned} \quad (3)$$

where r is the rank of the (super)algebra, $r = 2(n-1)$.

As opposed to the algebra $sl(2n-1)$, the superalgebra $sl(n|n-1)$ possesses several inequivalent systems of simple roots. The Cartan matrix (2) for $\mathcal{P} = 1$ corresponds to a purely fermionic simple-root system with the fermionic raising/lowering operators X_j^\pm which together with the Cartan generators can be represented by the graded matrices

$$\begin{aligned} H_j &= (-1)^{j+1}(E_{j,j} + E_{j+1,j+1}), & X_j^+ &= (-1)^{j+1}E_{j,j+1}, & X_j^- &= E_{j+1,j}, \\ (E_{i,j})_{pq} &\equiv \delta_{i,p}\delta_{j,q} \end{aligned} \quad (4)$$

with the zero (super)trace,

$$str(M) \equiv \sum_{j=1}^{2n-1} (-1)^{j+1} M_{jj}, \quad (5)$$

¹Hereafter, we understand that there is no summation over repeated indices.

and the Grassmann parity d_{ij} of their entries defined as $d_{ij} = (-1)^{i+j}$, where the value $d_{ij} = 1$ ($d_{ij} = -1$) corresponds to bosonic (fermionic) statistics of the entries M_{ij} . At such Z_2 -grading the generators (3) $Y_j^{\pm(2k)}$ are bosonic while $Y_j^{\pm(2k+1)}$ are fermionic ones. The Cartan generator of principal $osp(1|2)$ subalgebra of $sl(n|n-1)$

$$H = \frac{1}{2} \sum_{i=1}^{2(n-1)} \sum_{j=1}^{2(n-1)} (K^{-1})_{ji} H_i \quad (6)$$

defines a half-integer grading of superalgebra $sl(n|n-1) \equiv \mathcal{G}_- \mathcal{G}_0 \mathcal{G}_+ = (\oplus_{k=1}^{n-1} \mathcal{G}_{-\frac{k}{2}}) \mathcal{G}_0 (\oplus_{k=1}^{n-1} \mathcal{G}_{\frac{k}{2}})$, and $H_j \in \mathcal{G}_0$, $X_j^{\pm} \in \mathcal{G}_{\pm\frac{1}{2}}$ and $Y_j^{\pm(k)} \in \mathcal{G}_{\pm\frac{k}{2}}$. Thus, bosonic generators have integer grading, while fermionic ones have half-integer grading, and positive (negative) grading corresponds to upper (lower) triangular matrices. Here, $(K^{-1})_{ji}$ is the inverse Cartan matrix, $K^{-1}K = KK^{-1} = I$.

The highest weight vector $|j\rangle$ and its dual vector $\langle j|$ ($1 \leq j \leq 2(n-1)$) of the j -th fundamental representation possess the following properties:

$$\begin{aligned} X_i^+ |j\rangle = 0, \quad H_i |j\rangle = \delta_{i,j} |j\rangle, \quad \langle j| X_i^- = 0, \quad \langle j| H_i = \delta_{i,j} \langle j|, \\ \langle j||j\rangle = 1. \end{aligned} \quad (7)$$

The representation is exhibited by repeated applications of the lowering operators X_i^- to $|j\rangle$ and extraction of all linear-independent vectors with non-zero norm. Its first few basis vectors are

$$|j\rangle, \quad X_j^- |j\rangle, \quad X_{j\pm 1}^- X_j^- |j\rangle. \quad (8)$$

In the fundamental representations, matrix elements of the group $G \in SL(n|n-1)$ ($G \in SL(2n-1)$) satisfy the following important identity² [7]:

$$sdet \begin{pmatrix} \langle j| X_j^+ G X_j^- |j\rangle, & \langle j| X_j^+ G |j\rangle \\ \langle j| G X_j^- |j\rangle, & \langle j| G |j\rangle \end{pmatrix} = \prod_{i=1}^{2(n-1)} \langle i| G |i\rangle^{-K_{ji}}, \quad (9)$$

where K_{ji} is the Cartan matrix (2). It can be used as a generating relation for a number of other useful identities connecting different matrix element

²Let us remind the definition of the superdeterminant, $sdet \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv det(A - BD^{-1}C)(detD)^{-1}$.

of the group G . Indeed, new identities can be derived by replacing G by $\exp(t_+ l_+) G \exp(t_- l_-)$ on both sides of eqs. (9), where l_{\pm} are arbitrary linear functionals of the generators $X_i^{\pm}, Y_i^{\pm(k)}$ (3) and differentiating the resulting expression over the parameters³ t_{\pm} at $t_{\pm} = 0$. For example, let us present the following identity:

$$\begin{aligned} sdet \begin{pmatrix} \langle j| X_j^+ l_+^p G l_-^q X_j^- |j\rangle, & \langle j| X_j^+ l_+^p G |j\rangle \\ \langle j| G l_-^q X_j^- |j\rangle, & \langle j| G |j\rangle \end{pmatrix} \\ = (\partial_{t_+}^p \partial_{t_-}^q \prod_{i=1}^{2(n-1)} \langle i| e^{t_+ l_+} G e^{t_- l_-} |i\rangle^{-K_{ji}}) |_{t_{\pm}=0} \\ \equiv l_+^p \circ l_-^q \circ \prod_{i=1}^{2(n-1)} \langle i| G |i\rangle^{-K_{ji}}, \end{aligned} \quad (10)$$

which is only valid for the operators l_+ (l_-) annihilating the highest weight vector $|j\rangle$ ($\langle j|$), $l_+ |j\rangle = 0$ ($\langle j| l_- = 0$) and $p, q = 0, 1$. The operation $l_+ \circ$ ($l_- \circ$) is defined by eq. (10), and it represents the left (right) infinitesimal shift of the group G by the generator l_+ (l_-).

The identity (9) represents a generalization of the famous Jacobi relation connecting determinants of $(n+1)$, n and $(n-1)$ orders of some special matrices to the case of arbitrary semisimple Lie (super)groups. As we will see in the next section, this identity is so important in deriving integrable mappings and lattices that one can even say that it is responsible for their existence. We call it the first Jacobi identity. Besides eqs. (9), there is another independent identity [5]

$$\begin{aligned} (-1)^p \frac{\langle j| X_j^+ X_{j-1}^+ G |j\rangle}{\langle j| G |j\rangle} + \frac{\langle j-1| X_{j-1}^+ X_j^+ G |j-1\rangle}{\langle j-1| G |j-1\rangle} = \\ (-1)^{j^p} \frac{\langle j| X_j^+ G |j\rangle \langle j-1| X_{j-1}^+ G |j-1\rangle}{\langle j| G |j\rangle \langle j-1| G |j-1\rangle} \end{aligned} \quad (11)$$

which we use also in what follows and call the second Jacobi identity. It is responsible for the existence of hierarchies of integrable equations which are invariant with respect to integrable mappings. From this identity one can generate other useful identities in the same way as it has been explained after formula (9).

³The Grassmann parity of l_{\pm} coincides with the parity of the operator l_{\pm} .

3 The $SUToda(2, 2; \{s_j^+, s_j^-\})$ mappings and lattices

In this section we derive new integrable mappings together with the corresponding interrupted lattices on the basis of the representation theory for the algebras $sl(n|n-1)$ ($\mathcal{P} = 1$) and $sl(2n-1)$ ($\mathcal{P} = 0$) summarized in the previous section.

Our starting point is the following representation for the group element $G(z_+, z_-) \in SL(n|n-1)$ ($SL(2n-1)$) [6, 5]:

$$G \equiv M_+^{-1} M_- \quad (12)$$

in terms of the product of upper and lower triangular (including a diagonal) matrices $M_+(z_+)$ and $M_-(z_-)$, respectively, which are defined as solutions of the following equations:

$$\begin{aligned} \mathcal{A}_\pm \equiv M_\pm^{-1} \partial_\pm M_\pm &= (\mp 1)^{\mathcal{P}} \left(\sum_{j=1}^{2(n-1)} (\partial_\pm \phi_j^\pm(z_\pm) H_j + \nu_j^\pm(z_\pm) X_j^\pm) \right. \\ &\quad \left. + \sum_{l=1}^{2(n-1)-1} s_l^\pm(z_\pm) Y_l^{\pm(2)} \right), \end{aligned} \quad (13)$$

where z_\pm are bosonic coordinates ($\partial_\pm \equiv \frac{\partial}{\partial z_\pm}$), $\phi_j^\pm(z_\pm)$ and $s_l^\pm(z_\pm)$ are arbitrary bosonic functions, while $\nu_j^\pm(z_\pm)$ are arbitrary fermionic (bosonic) ones. It is well known that for any finite-dimensional representations of a (super)algebra, the equations for M_\pm can be integrated in quadratures, and their solutions contain only a finite number of terms involving products of X_j^\pm and $Y_l^{\pm(2)}$. The fields \mathcal{A}_\pm belonging to a (super)algebra can be treated as components of two different pure gauge connections in the two-dimensional space with coordinates (z_+, z_-) (see appendix).

At this point it is necessary to make an important, for further consideration, remark concerning equations (13). Similar equations have been used in [5]. They have only but crucial difference as compared to eqs. (13): all higher grade functions s_j^\pm , which belong to \mathcal{A}_\pm , were chosen equal to unity. Of course, this can always be done by suitable gauge transformation⁴, but

⁴Let us recall that the form of equations (13) for the matrices M_\pm is invariant with respect to gauge transformations $M_\pm \rightarrow g_\pm^{-1} M_\pm g_\pm$ generated by the Cartan subgroup $g_\pm = \exp(\sum_{j=1}^{2(n-1)} f_j^\pm(z_\pm) H_j)$, where $f_j^\pm(z_\pm)$ are parameter-functions.

only in the case if $s_j^\pm \neq 0$ for any j . Obviously, in the opposite case, i.e. if $s_l^\pm = 0$ for some set of indices l , it is impossible to reach the value $s_l^\pm = 1$ for this kind of indices l by any gauge transformations. Thus, we are led to the conclusion that the space of gauge inequivalent solutions of eqs. (13) is parameterised by the numbers $\{s_j^+, s_j^-, j = 1, \dots, 2(n-1)\}$, which take only two possible values, 0 or 1. In general, different sets of $\{s_j^+, s_j^-\}$ correspond to inequivalent integrable systems which can be derived from various fundamental representations of the group element G (12). Due to this reason as well as to the fact that $UToda(2, 2)$ mapping [5] was derived at the particular choice $s_j^\pm = 1$ in eqs. (13), it is natural to call them the $SUToda(2, 2; \{s_j^+, s_j^-\})$ mappings (lattices). In the following section, we analyze their simplest representatives and demonstrate that they indeed possess quite different properties and solutions.

Besides the independent functions $\phi_j^\pm(z_\pm)$, $s_l^\pm(z_\pm)$ and $\nu_j^\pm(z_\pm)$ entering into equations (13), we introduce also a set of dependent fermionic ($\mathcal{P} = 1$) or bosonic ($\mathcal{P} = 0$) functions expressed in terms of matrix elements of fundamental representations of the group G (12):

$$\begin{aligned} \mu_j^+(z_+, z_-) &\equiv \frac{\langle j | X_j^+ G | j \rangle}{\langle j | G | j \rangle}, & \mu_j^-(z_+, z_-) &\equiv \frac{\langle j | G X_j^- | j \rangle}{\langle j | G | j \rangle}, \\ \gamma_j^\mp(z_+, z_-) &\equiv \frac{\partial_\mp \mu_j^\pm}{g_j} \end{aligned} \quad (14)$$

and the bosonic functions⁵

$$\begin{aligned} g_j &\equiv (-1)^{j\mathcal{P}} \prod_{i=1}^{2(n-1)} \langle i | G | i \rangle^{-K_{ji}} \\ &= (-1)^{j\mathcal{P}} \frac{\langle j-1 | G | j-1 \rangle^{(-1)^{(j+1)\mathcal{P}}} \langle j+1 | G | j+1 \rangle^{(-1)^{j\mathcal{P}}}}{\langle j | G | j \rangle^{2(1-\mathcal{P})}}, \\ &1 \leq j \leq 2(n-1). \end{aligned} \quad (15)$$

Our nearest goal is to demonstrate that the functions $\{g_j, \gamma_j^+, \gamma_j^-\}$ satisfy the following closed system of equations:

$$\begin{aligned} \partial_+ \partial_- \ln g_j &= s_{j+1}^+ s_{j-1}^- g_{j+1} g_{j+2} - g_j (s_j^+ s_j^- g_{j+1} + s_{j-1}^+ s_{j-1}^- g_{j-1}) \\ &+ s_{j-2}^+ s_{j-2}^- g_{j-1} g_{j-2} + g_{j+1} \gamma_{j+1}^- \gamma_{j+1}^+ - (1 + (-1)^{\mathcal{P}}) g_j \gamma_j^- \gamma_j^+ \\ &+ (-1)^{\mathcal{P}} g_{j-1} \gamma_{j-1}^- \gamma_{j-1}^+, \quad \partial_\pm \gamma_j^\mp = s_j^\mp g_{j+1} \gamma_{j+1}^\pm - s_{j-1}^\mp g_{j-1} \gamma_{j-1}^\pm \end{aligned} \quad (16)$$

⁵Hereafter, in order to simplify formulae we use the following definitions: $X_{-k}^\pm = X_{2(n-1)+k+1}^\pm \equiv 0$ ($k \geq 0$), and $\langle 0 | \Gamma | 0 \rangle = \langle 2(n-1) + 1 | \Gamma | 2(n-1) + 1 \rangle \equiv 1$ for any group element $\Gamma \in SL(n|n-1)$ ($SL(2n-1)$).

with the boundary conditions $g_0 = g_{2(n-1)+1} = 0$. They has been called above the

SUToda(2, 2; $\{s_j^+, s_j^-\}$) lattice. The main steps of deriving eqs. (16) repeat the corresponding calculations of ref. [5] concerning the $sl(n)$ -case, but for a reader's convenience we briefly present them here.

At first, using eqs. (7), identities (10) with a group G (12), and definitions (14)–(15), we obtain other, equivalent expressions⁵ for γ_j^\pm ,

$$\gamma_j^\mp = (-1)^{j\mathcal{P}} \nu_j^\mp + s_j^\mp \mu_{j+1}^\mp - s_{j-1}^\mp \mu_{j-1}^\mp, \quad (0 \leq j \leq 2(n-1) + 1), \quad (17)$$

which being differentiated with respect to z_\pm give equations (16) for γ_j^\pm .

At second, taking into account eqs. (7)–(8) and (12) one can derive the following relation

$$\partial_+ \partial_- \ln \langle j | G | j \rangle = s \det \left(\begin{array}{cc} \langle j | X_j^+ l_{j+} G l_{j-} X_j^- | j \rangle, & \langle j | X_j^+ l_{j+} G | j \rangle \\ \langle j | G l_{j-} X_j^- | j \rangle, & \langle j | G | j \rangle \end{array} \right) \quad (18)$$

with $l_{j\pm} = (\mp)^{\mathcal{P}+1} \nu_j^\pm + s_{j-1}^\pm X_{j-1}^\pm - (-1)^{\mathcal{P}} s_j^\pm X_{j+1}^\pm$. Using identities (10), definitions (14)–(15) and eqs. (17), equation (18) becomes

$$\begin{aligned} \partial_+ \partial_- \ln \langle j | G | j \rangle \\ = (-1)^{(j+1)\mathcal{P}} g_j (s_j^+ s_j^- g_{j+1} + (-1)^{\mathcal{P}} s_{j-1}^+ s_{j-1}^- g_{j-1} + \gamma_j^- \gamma_j^+), \end{aligned} \quad (19)$$

and it can easily be transformed into equation (16) for g_j .

Thus, we conclude that the SUToda(2, 2; $\{s_j^+, s_j^-\}$) lattice (16) is integrable, and its general solutions are given by formulae (12)–(15) and (17) in terms of matrix elements of fundamental representations of the (super)groups $SL(n|n-1)$ and $SL(2n-1)$.

To close this section, we would like to stress that the presented here formalism, underlying the identities between matrix elements of various fundamental representations of a group, guarantees the existence of a zero-curvature representation with a pure gauge connection defined by a single group element G (see appendix). An advantage of the developed scheme is that starting with a graded algebra and properties of fundamental representations of the corresponding group we simultaneously obtain both integrable mappings and general solutions of their interrupted versions which represent finite-dimensional exactly integrable lattices.

4 Examples: supersymmetric mappings

The consideration of the previous section was based on a pure superalgebraic level. However, in physical applications integrable systems possessing supersymmetry cause more interest. In this connection the important question arises which concerns the existence and classification of supersymmetric mappings encoded in the above-constructed superalgebraic integrable systems. We do not know the complete answer to this question and analyze here only two important representatives of the SUToda(2, 2; $\{s_j^+, s_j^-\}$) lattices (16), characterized by

$$\begin{aligned} I) \quad & s_j^\pm = 1, \quad (1 \leq j \leq 2(n-1)), \\ II) \quad & s_{2j}^- = s_{2j+1}^- = 1, \quad s_{2j}^+ = 1, \quad s_{2j+1}^+ = 0, \quad (0 \leq j \leq (n-1)). \end{aligned} \quad (20)$$

We show that for the $sl(n|n-1)$ superalgebra, i.e. when γ_j^+ and γ_j^- are fermionic fields, they represent two inequivalent integrable systems which indeed possess higher supersymmetries, the $N = (2|2)$ and $N = (0|2)$ superconformal symmetries, respectively. The first system is the $N = (2|2)$ superconformal Toda lattice equation (see, [9] and references therein), while the second one represents two-dimensional generalization of the 1D f-Toda chain [1, 3].

1. The $N = (2|2)$ superconformal Toda lattice. In the case of I) (20) equations (16) become

$$\begin{aligned} \partial_+ \partial_- \ln g_j &= g_{j+1} g_{j+2} - g_j (g_{j+1} + g_{j-1}) + g_{j-1} g_{j-2} \\ &+ g_{j+1} \gamma_{j+1}^- \gamma_{j+1}^+ - (1 + (-1)^{\mathcal{P}}) g_j \gamma_j^- \gamma_j^+ + (-1)^{\mathcal{P}} g_{j-1} \gamma_{j-1}^- \gamma_{j-1}^+, \\ \partial_\pm \gamma_j^\mp &= g_{j+1} \gamma_{j+1}^\pm - g_{j-1} \gamma_{j-1}^\pm, \quad (1 \leq j \leq 2(n-1)). \end{aligned} \quad (21)$$

For the $sl(n|n-1)$ superalgebra, i.e. when $\mathcal{P} = 1$ and γ_j^\pm are fermionic fields, they coincide with the component form of the $N = (1|1)$ superconformal Toda lattice equation

$$\mathcal{D}_- \mathcal{D}_+ \ln B_j = (-1)^{j+1} (B_{j+1} - B_{j-1}) \equiv \sum_i K_{ji} B_i. \quad (22)$$

Here, K_{ji} is the symmetric Cartan matrix (2) of the $sl(n|n-1)$ superalgebra, $B_j(z_+, \vartheta_+; z_-, \vartheta_-)$ is the bosonic $N = 1$ superfield with the components

$$g_j \equiv (-1)^j B_j|, \quad \gamma_j^\pm \equiv \mathcal{D}_\pm \ln B_j|, \quad (23)$$

where $|$ means the $\vartheta_{\pm} \rightarrow 0$ limit, and \mathcal{D}_{\pm} are the $N = 1$ supersymmetric fermionic covariant derivatives

$$\mathcal{D}_{\pm} = \frac{\partial}{\partial \vartheta_{\pm}} \mp \vartheta_{\pm} \partial_{\pm}, \quad \mathcal{D}_{\pm}^2 = \mp \partial_{\pm}, \quad \{\mathcal{D}_{+}, \mathcal{D}_{-}\} = 0 \quad (24)$$

in the $N = (1|1)$ superspace $(z_{+}, \vartheta_{+}; z_{-}, \vartheta_{-})$.

Let us remark that after rescaling $B_j \rightarrow (-1)^j B_j$ the factor $(-1)^{j+1}$ completely disappears from eq. (22) which in this case corresponds to anti-symmetric Cartan matrix $K_{ij} = \delta_{i+1,j} - \delta_{i,j+1}$. This form of equation (22) has been discussed in [10] where an infinite family of solutions of its symmetry equation were constructed. These solutions describe integrable evolution equations belonging the $N = (1|1)$ superconformal Toda lattice hierarchy.

The general solutions of the $N = (1|1)$ superconformal Toda in terms of matrix elements of fundamental representations of the $SL(n|n-1)$ supergroup with the group element G (12)-(13) can easily be derived from formulae (14)–(15) and (17) and look like

$$\begin{aligned} \gamma_j^- &= (-1)^j \nu_j^- + \frac{\langle j+1 | GX_{j+1}^- | j+1 \rangle}{\langle j+1 | G | j+1 \rangle} - \frac{\langle j-1 | GX_{j-1}^- | j-1 \rangle}{\langle j-1 | G | j-1 \rangle}, \\ \gamma_j^+ &= (-1)^j \nu_j^+ + \frac{\langle j+1 | X_{j+1}^+ G | j+1 \rangle}{\langle j+1 | G | j+1 \rangle} - \frac{\langle j-1 | X_{j-1}^+ G | j-1 \rangle}{\langle j-1 | G | j-1 \rangle}, \\ g_j &= (-1)^j \left(\frac{\langle j+1 | G | j+1 \rangle}{\langle j-1 | G | j-1 \rangle} \right)^{(-1)^j}. \end{aligned} \quad (25)$$

These expressions can be promoted to a compact superfield form in terms of the $N = 1$ superfield $B_j(z_{+}, \vartheta_{+}; z_{-}, \vartheta_{-})$ (22)–(23),

$$\begin{aligned} B_j &= \left(\frac{\langle j+1 | \tilde{G} | j+1 \rangle}{\langle j-1 | \tilde{G} | j-1 \rangle} \right)^{(-1)^j}, \quad \nu_j^{\pm} \equiv \tilde{\nu}_{j+1}^{\pm} - \tilde{\nu}_{j-1}^{\pm}, \\ \tilde{G} &\equiv e^{\vartheta_{+} \sum_{j=1}^{2(n-1)} (\tilde{\nu}_j^{+} H_j - (-1)^j X_j^{+})} G e^{\vartheta_{-} \sum_{j=1}^{2(n-1)} (\tilde{\nu}_j^{-} H_j - (-1)^j X_j^{-})}. \end{aligned} \quad (26)$$

Actually, equation (22) possesses the $N = (2|2)$ superconformal symmetry and can be rewritten in a manifestly $N = (2|2)$ supersymmetric form [9]

$$D_{+} D_{-} \ln \bar{\Phi}_j = \Phi_{j-1} - \Phi_j, \quad \bar{D}_{+} \bar{D}_{-} \ln \Phi_j = \bar{\Phi}_j - \bar{\Phi}_{j+1} \quad (27)$$

in terms of two bosonic chiral and antichiral $N = 2$ superfields $\Phi_j(z_{+}, \theta_{+}, \bar{\theta}_{+}; z_{-}, \theta_{-}, \bar{\theta}_{-})$ and $\bar{\Phi}_j(z_{+}, \theta_{+}, \bar{\theta}_{+}; z_{-}, \theta_{-}, \bar{\theta}_{-})$, $D_{\pm} \Phi_j = \bar{D}_{\pm} \bar{\Phi}_j = 0$, respectively, with the components

$$g_{2j} \equiv \bar{\Phi}_j|, \quad g_{2j+1} \equiv \Phi_j|, \quad \gamma_{2j}^{\pm} \equiv D_{\pm} \ln \bar{\Phi}_j|, \quad \gamma_{2j+1}^{\pm} \equiv \bar{D}_{\pm} \ln \Phi_j|. \quad (28)$$

Here, D_{\pm} and \bar{D}_{\pm} are the $N = 2$ supersymmetric fermionic covariant derivatives

$$\begin{aligned} D_{\pm} &= \frac{\partial}{\partial \theta_{\pm}} \mp \frac{1}{2} \bar{\theta}_{\pm} \partial_{\pm}, \quad \bar{D}_{\pm} = \frac{\partial}{\partial \bar{\theta}_{\pm}} \mp \frac{1}{2} \theta_{\pm} \partial_{\pm}, \\ D_{\pm}^2 &= \bar{D}_{\pm}^2 = \{D_{\pm}, \bar{D}_{\mp}\} = 0, \quad \{D_{\pm}, D_{\pm}\} = \mp \partial_{\pm} \end{aligned} \quad (29)$$

in the $N = 2$ superspace $(z_{\pm}, \theta_{\pm}, \bar{\theta}_{\pm})$. The $N = 2$ superfield solutions can be restored from eqs. (25)–(27) and read as

$$\begin{aligned} \Phi_j &= \frac{\langle 2j | \Omega | 2j \rangle}{\langle 2(j+1) | \Omega | 2(j+1) \rangle}, \quad \bar{\Phi}_j = \frac{\langle 2j+1 | \bar{\Omega} | 2j+1 \rangle}{\langle 2j-1 | \bar{\Omega} | 2j-1 \rangle}, \\ \Omega &\equiv e^{\bar{\theta}_{+} \sum_{j=1}^{2(n-1)} (\tilde{\nu}_j^{+} H_j - (-1)^j X_j^{+})} G(z_{+} + \frac{1}{2} \theta_{+} \bar{\theta}_{+}, z_{-} - \frac{1}{2} \theta_{-} \bar{\theta}_{-}) e^{\bar{\theta}_{-} \sum_{j=1}^{2(n-1)} (\tilde{\nu}_j^{-} H_j - (-1)^j X_j^{-})}, \\ \bar{\Omega} &\equiv e^{\theta_{+} \sum_{j=1}^{2(n-1)} (\tilde{\nu}_j^{+} H_j - (-1)^j X_j^{+})} G(z_{+} - \frac{1}{2} \theta_{+} \bar{\theta}_{+}, z_{-} + \frac{1}{2} \theta_{-} \bar{\theta}_{-}) e^{\theta_{-} \sum_{j=1}^{2(n-1)} (\tilde{\nu}_j^{-} H_j - (-1)^j X_j^{-})}, \end{aligned} \quad (30)$$

where the functions $\tilde{\nu}_j^{\pm}$ are defined in eqs. (26). The chiral and antichiral group elements Ω and $\bar{\Omega}$, $D_{\pm} \Omega = \bar{D}_{\pm} \bar{\Omega} = 0$, are related by the involution of the algebra of $N = 2$ fermionic derivatives (29),

$$\theta_{\pm}^* = \bar{\theta}_{\pm}, \quad \bar{\theta}_{\pm}^* = \theta_{\pm}, \quad z_{\pm}^* = z_{\pm} \Rightarrow \Omega^* = \bar{\Omega}, \quad \bar{\Omega}^* = \Omega. \quad (31)$$

For a particular case of the $sl(2|1)$ superalgebra eq. (27) amounts to the $N = 2$ supersymmetric Liouville equation of Ref. [11], where its general solution has been presented. For the general $sl(n|n-1)$ case the solution of the superconformal Toda lattice in another parameterisation of the group element and in another basis in the space of the functions $\{g_j, \gamma_j^+, \gamma_j^-\}$ has been found in [12].

For the $sl(n)$ algebra, i.e. when $\mathcal{P} = 0$ and γ_j^{\pm} are bosonic fields, eqs. (21) reproduce the $UToda(2, 2)$ lattice of ref. [5].

2. The $N = (0|2)$ superconformal Toda lattice. Now, we consider the second system characterized by relation II) (20). In the new basis $\{U_j, V_j, \Psi_j, \bar{\Psi}_j, \alpha_j, \bar{\alpha}_j, (0 \leq j \leq n-1)\}$ in the space of the fields $\{g_j, \gamma_j^+, \gamma_j^-\}$,

$$\begin{aligned} U_j &\equiv g_{2j}, & \bar{\Psi}_j &\equiv \gamma_{2j}^+, & \alpha_j &\equiv \gamma_{2j}^-, \\ V_j &\equiv -g_{2j+1}, & \Psi_j &\equiv -\gamma_{2j+1}^+, & \bar{\alpha}_j &\equiv -\gamma_{2j+1}^-, \end{aligned} \quad (32)$$

equations (16) become

$$\begin{aligned} \partial_- \partial_+ \ln(U_j V_{j-1}) &= \partial_- (\Psi_j \bar{\Psi}_j + (-1)^P \Psi_{j-1} \bar{\Psi}_{j-1}) \\ &\quad - (1 + (-1)^P) ((\partial_- \Psi_j) \bar{\Psi}_j + \Psi_{j-1} \partial_- \bar{\Psi}_{j-1}), \\ \partial_+ \left(\frac{1}{U_j} \partial_- \Psi_j \right) &= V_j \Psi_j - V_{j-1} \Psi_{j-1}, \quad \partial_+ \left(\frac{1}{V_j} \partial_- \bar{\Psi}_j \right) = U_j \bar{\Psi}_j - U_{j+1} \bar{\Psi}_{j+1}, \\ \partial_+ \partial_- \ln V_j &= U_j V_j - U_{j+1} V_{j+1} + (\partial_- \Psi_{j+1}) \bar{\Psi}_{j+1} \\ &\quad - (1 + (-1)^P) \Psi_j \partial_- \bar{\Psi}_j + (-1)^P (\partial_- \Psi_j) \bar{\Psi}_j, \end{aligned} \quad (33)$$

$$\alpha_j = \frac{1}{U_j} \partial_- \Psi_j, \quad \bar{\alpha}_j = \frac{1}{V_j} \partial_- \bar{\Psi}_j \quad (34)$$

with the boundary conditions: $U_0 = V_{n-1} = 0, \Psi_0 = \nu_1^+, \bar{\Psi}_{n-1} = \nu_{2(n-1)}^+$.

Let us discuss equations (33) in more detail in the case corresponding to the superalgebra $sl(n|n-1)$, i.e. when $P = 1$ and fields ψ_j and $\bar{\psi}_j$ are fermionic.

In this case, the first equation of system (33) has the form of a conservation law with respect to the coordinate z_+ ,

$$\partial_- \mathcal{I} = 0, \quad \mathcal{I} \equiv \partial_+ \ln(U_j V_{j-1}) - \Psi_j \bar{\Psi}_j + \Psi_{j-1} \bar{\Psi}_{j-1}, \quad (35)$$

and as a result, the quantity \mathcal{I} depends only on the coordinate z_+ , i.e. $\mathcal{I} = \mathcal{I}(z_+)$. It can easily be expressed in terms of the functions $\phi_j^+(z_+)$ and $\nu_j^+(z_+)$, introduced by eqs. (13),

$$\mathcal{I}(z_+) = \partial_+ \ln(\eta_{j-1}(z_+)/\eta_j(z_+)), \quad \eta_j(z_+) \equiv e^{(\phi_{2j}^+ - \phi_{2j+1}^+ - \int \nu_{2j}^+ \nu_{2j+1}^+ dz_+)}, \quad (36)$$

where we have used the definitions (32), (14)–(15), (17) and the second Jacobi identity (11). Keeping in mind relations (35)–(36), after rescaling

$$U_j \rightarrow u_j \equiv \eta_j U_j, \quad V_j \rightarrow v_j \equiv \frac{V_j}{\eta_j}, \quad \Psi_j \rightarrow \psi_j \equiv \eta_j \Psi_j, \quad \bar{\Psi}_j \rightarrow \bar{\psi}_j \equiv \frac{\bar{\Psi}_j}{\eta_j} \quad (37)$$

we rewrite equations (33) in an equivalent form

$$\begin{aligned} \partial_+ \ln(u_j v_{j-1}) &= (\psi_j \bar{\psi}_j - \psi_{j-1} \bar{\psi}_{j-1}), \\ \partial_+ \left(\frac{1}{u_j} \partial_- \psi_j \right) &= v_j \psi_j - v_{j-1} \psi_{j-1}, \quad \partial_+ \left(\frac{1}{v_j} \partial_- \bar{\psi}_j \right) = u_j \bar{\psi}_j - u_{j+1} \bar{\psi}_{j+1}, \\ \partial_+ \partial_- \ln v_j &= u_j v_j - u_{j+1} v_{j+1} + (\partial_- \psi_{j+1}) \bar{\psi}_{j+1} - (\partial_- \psi_j) \bar{\psi}_j, \end{aligned} \quad (38)$$

where the function $\eta_j(z_+)$ completely disappears. These equations reproduce the 1D f-Toda equations [1, 3] at the reduction $\partial_+ = \partial_-$ to the one-dimensional bosonic subspace. Therefore, they represent its two-dimensional integrable generalization. We call them the 2D f-Toda lattice. Summarizing all the above-given formulae we present their general solutions

$$\begin{aligned} u_j &= \eta_j \frac{\langle 2j+1 | G | 2j+1 \rangle}{\langle 2j-1 | G | 2j-1 \rangle}, \quad v_j = \frac{1}{\eta_j} \frac{\langle 2j | G | 2j \rangle}{\langle 2(j+1) | G | 2(j+1) \rangle}, \\ \bar{\psi}_j &\equiv \frac{1}{\eta_j} \frac{\langle 2j+1 | (\nu_{2j}^+ + X_{2j+1}^+) G | 2j+1 \rangle}{\langle 2j+1 | G | 2j+1 \rangle}, \\ \psi_j &= \eta_j \frac{\langle 2j | (\nu_{2j+1}^+ + X_{2j}^+) G | 2j \rangle}{\langle 2j | G | 2j \rangle} \end{aligned} \quad (39)$$

in an explicit form in terms of matrix elements of fundamental representations of the $SL(n|n-1)$ supergroup with the group element G (12)–(13).

Similar to its one-dimensional counterpart, the 2D f-Toda equations (38) admit the $N = 2$ supersymmetry and can be rewritten in the superfield form

$$F_{j+1} \bar{F}_{j+1} - F_j \bar{F}_j = \partial_+ \ln((\bar{D}_- F_{j+1})(D_- \bar{F}_j)) \quad (40)$$

in terms of chiral and antichiral fermionic $N = 2$ superfields $F_j(z_+; z_-, \theta_-, \bar{\theta}_-)$ and $\bar{F}_j(z_+; z_-, \theta_-, \bar{\theta}_-)$, $D_- F_j = \bar{D}_- \bar{F}_j = 0$, respectively, with the components

$$v_j \equiv -D_- \bar{F}_j|, \quad \bar{\psi}_j \equiv \bar{F}_j|, \quad u_j \equiv \bar{D}_- F_j|, \quad \psi_j \equiv F_j|, \quad (41)$$

where the $N = 2$ fermionic derivatives D_- and \bar{D}_- are defined in eqs. (29). The general solutions of eq. (40) have the following nice representation⁵:

$$F_j = \eta_j \frac{\langle 2j | (\nu_{2j+1}^+ + X_{2j}^+) \mathcal{G} | 2j \rangle}{\langle 2j | \mathcal{G} | 2j \rangle},$$

$$\begin{aligned}
\bar{F}_j &\equiv \frac{1}{\eta_j} \frac{(2j+1 | (\nu_{2j}^+ + X_{2j+1}^+) \bar{\mathcal{G}} | 2j+1)}{(2j+1 | \bar{\mathcal{G}} | 2j+1)}, \\
\mathcal{G} &\equiv G(z_+, z_- - \frac{1}{2} \theta_- \bar{\theta}_-) e^{-\bar{\theta}_- \sum_{j=1}^{2(n-1)} (-1)^j X_j^-}, \\
\bar{\mathcal{G}} &\equiv G(z_+, z_- + \frac{1}{2} \theta_- \bar{\theta}_-) e^{-\theta_- \sum_{j=1}^{2(n-1)} (-1)^j X_j^-}.
\end{aligned} \tag{42}$$

Here, $0 \leq j \leq n-1$, \mathcal{G} and $\bar{\mathcal{G}}$ are chiral and antichiral group elements, $D_- \mathcal{G} = \bar{D}_- \bar{\mathcal{G}} = 0$, which are related by involution (31), $\mathcal{G}^* = \bar{\mathcal{G}}$, $\bar{\mathcal{G}}^* = \mathcal{G}$.

In the superfield form one can easily recognize that the 2D f-Toda equations are actually invariant with respect to higher supersymmetry — $N = (0, 2)$ superconformal symmetry, which generates the transformations $(z_+; z_-, \theta_-, \bar{\theta}_-) \rightarrow (\tilde{z}_+; \tilde{z}_-, \tilde{\theta}_-, \tilde{\bar{\theta}}_-)$,

$$\tilde{z}_+ = \tilde{z}_+(z_+), \quad \Rightarrow \quad \partial_+ = (\partial_+ \tilde{z}_+) \tilde{\partial}_+, \tag{43}$$

$$\begin{aligned}
\tilde{z}_- &= \tilde{z}_-(z_-, \theta_-, \bar{\theta}_-), \quad \tilde{\theta}_- = \tilde{\theta}_-(z_-, \theta_-, \bar{\theta}_-), \quad \tilde{\bar{\theta}}_- = \tilde{\bar{\theta}}_-(z_-, \theta_-, \bar{\theta}_-), \\
D_- \tilde{\bar{\theta}}_- &= \bar{D}_- \tilde{\theta}_- = 0, \quad D_- \tilde{z}_- = -\frac{1}{2} \tilde{\bar{\theta}}_- D_- \tilde{\theta}_-, \quad \bar{D}_- \tilde{z}_- = -\frac{1}{2} \tilde{\theta}_- \bar{D}_- \tilde{\bar{\theta}}_-, \\
\Rightarrow D_- &= (D_- \tilde{\theta}_-) \tilde{D}_-, \quad \bar{D}_- = (\bar{D}_- \tilde{\bar{\theta}}_-) \tilde{\bar{D}}_-.
\end{aligned} \tag{44}$$

Under these transformations the superfields F_j and \bar{F}_j are transforming according to the rule

$$\begin{aligned}
F_j(z_+; z_-, \theta_-, \bar{\theta}_-) &= \varphi(z_+) (\partial_+ \tilde{z}_+)^{1-j} \tilde{F}_j(\tilde{z}_+; \tilde{z}_-, \tilde{\theta}_-, \tilde{\bar{\theta}}_-), \\
\bar{F}_j(z_+; z_-, \theta_-, \bar{\theta}_-) &= \varphi^{-1}(z_+) (\partial_+ \tilde{z}_+)^j \tilde{\bar{F}}_j(\tilde{z}_-; \tilde{z}_-, \tilde{\theta}_-, \tilde{\bar{\theta}}_-),
\end{aligned} \tag{45}$$

where $\varphi(z_+)$ is an arbitrary invertible function corresponding to the local internal $GL(1)$ -transformation. Thus, we conclude that the 2D f-Toda lattice (40) possesses the $N = (0|2)$ superconformal symmetry and due to this important property it can also be called the $N = (0|2)$ superconformal Toda lattice.

Keeping in mind that at $\mathcal{P} = 1$ (i.e. for the case of $sl(n)$ algebra with bosonic fields Ψ_j and $\bar{\Psi}_j$) equations (33) represent the integrable bosonic counterpart of the 2D f-Toda lattice, it is reasonable to call them the 2D bosonic Toda (b-Toda) lattice equations.

5 Conclusion

In the present letter we have demonstrated the dominant role of the representation theory of graded superalgebras in the context of the problem of constructing superintegrable mappings. We have shown that the following chain of relations: representations of a graded algebra \Rightarrow representations of the corresponding group \Rightarrow integrable mappings (lattices), used in [5] for constructing the bosonic integrable mappings, perfectly works also in the super-algebraic case and gives an effective algorithm for constructing new mappings. We have developed this approach and constructed the new integrable mappings. All other ingredients of the theory of integrable systems such as deriving evolution equations, which belong to an integrable hierarchy, and their multi-soliton solutions are also present in the framework of this construction. Thus, for example, equations of hierarchy arise in this approach as solutions of the symmetry equation which corresponds to an integrable mapping, while their multi-soliton solutions are related to general solutions of the corresponding integrable lattices with fixed ends.

We have applied this approach to the case of $sl(n|n-1)$ superalgebra, and the 2D f-Toda lattice possessing the $N = (0|2)$ superconformal symmetry has been derived for the first time together with its general solutions in terms of matrix elements of different fundamental representations of the $SL(n|n-1)$ supergroup.

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Appendix

In this appendix we derive zero-curvature representation for the $SUToda(2, 2; \{s_j^+, s_j^-\})$ lattice (16) (compare with [6, 7]).

Firstly, we rewrite the group element G defined by eq. (12) in the following equivalent form:

$$G \equiv M_+^{-1} M_- \equiv N_- G_0 N_+. \quad (\text{A.1})$$

This relation defines the group elements N_+ , N_- and G_0 representing the exponentiation of the graded subspaces \mathcal{G}_+ , \mathcal{G}_- and \mathcal{G}_0 of the $sl(n|n-1)$ ($sl(n)$) superalgebra, respectively, in terms of the group element G . It is a simple exercise to deduce the following two chains of identities:

$$\begin{aligned} \mathcal{A}_- &= G^{-1} \partial_- G = N_+^{-1} G_0^{-1} (N_-^{-1} \partial_- N_-) G_0 N_+ + N_+^{-1} (G_0^{-1} \partial_- G_0) N_+ \\ &\quad + N_+^{-1} \partial_- N_+, \\ \mathcal{A}_+ &= G \partial_- G^{-1} = N_- G_0 (N_+ \partial_- N_+^{-1}) G_0^{-1} N_-^{-1} + N_- (G_0 \partial_- G_0^{-1}) N_-^{-1} \\ &\quad + N_- \partial_- N_-^{-1} \end{aligned} \quad (\text{A.2})$$

from relations (A.1) and (13). Comparing the decompositions over graded subspaces of the right-hand and left-hand sides of these identities, one can obtain decomposition rules for the following algebra-valued functions:

$$\begin{aligned} G_0^{-1} (N_-^{-1} \partial_- N_-) G_0 &\in \mathcal{G}_{-1} \oplus \mathcal{G}_{-2}, \\ G_0^{-1} (N_-^{-1} \partial_- N_-) G_0|_{-2} &= \mathcal{A}_-|_{-2}, \\ G_0 (N_+ \partial_- N_+^{-1}) G_0^{-1} &\in \mathcal{G}_{+1} \oplus \mathcal{G}_{+2}, \\ G_0 (N_+ \partial_- N_+^{-1}) G_0^{-1}|_{+2} &= \mathcal{A}_+|_{+2}, \end{aligned} \quad (\text{A.3})$$

which we use in what follows. Here, $|_{\pm k}$ means the projection on the grade subspace $\mathcal{G}_{\pm k}$.

Secondly, we introduce a new group element \mathcal{Q} ,

$$\mathcal{Q} \equiv M_- N_+^{-1} = M_+ N_- G_0, \quad (\text{A.4})$$

and the components A_+ and A_- ,

$$\begin{aligned} A_- &\equiv \mathcal{Q}^{-1} \partial_- \mathcal{Q} = G_0^{-1} (N_-^{-1} \partial_- N_-) G_0 + G_0^{-1} \partial_- G_0, \\ A_+ &\equiv \mathcal{Q}^{-1} \partial_+ \mathcal{Q} = N_+ \partial_- N_+^{-1}, \end{aligned} \quad (\text{A.5})$$

of a pure gauge by construction connection,

$$[\partial_+ - A_+, \partial_- - A_-] = 0, \quad (\text{A.6})$$

with the properties

$$\begin{aligned} A_- &\in \mathcal{G}_0 \oplus \mathcal{G}_{-1} \oplus \mathcal{G}_{-2}, \quad A_-|_{-2} = \mathcal{A}_-|_{-2}, \\ A_+ &\in \mathcal{G}_{+1} \oplus \mathcal{G}_{+2}, \quad A_+|_{+2} = G_0^{-1} \mathcal{A}_+|_{+2} G_0, \end{aligned} \quad (\text{A.7})$$

where in deriving these equations, relations (A.1) and (A.3) have been exploited.

Relations (A.4)–(A.7) represent the resolved form of the zero-curvature representation for the $SUToda(2, 2; \{s_j^+, s_j^-\})$ lattice equations (16).

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