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DOUBLE COMPLEXES AND COHOMOLOGICAL HIERARCHY IN A SPACE OF WEAKLY INVARIANT LAGRANGIANS OF MECHANICS

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1 Introduction

The cohomology of the symmetries algebra has important consequences for properties of the corresponding theory [1,2] and cohomological methods play essential role in many problems of modern field theory. For example, their application made an understanding of algebraic origin of gauge anomalies more clear. As it was shown in [1], one can consider axial anomalies of four-dimensional gauge theory in terms of infinitesimal cocycles in a representation of gauge group.

Another example is the BRST formalism which initially was formulated in terms of symplectic geometry of the phase space expanded by the ghosts and antighosts. Later it was understood [3,4,5,6,7] that the language of homological algebra is more deeply related to physical meaning of this formalism: Inclusion of ghosts and antighosts corresponds to the construction of the chain of free modules (free resolvent) on the phase space of the constrained system, where the constraints cannot be resolved in a direct way. The operator corresponding to the BRST charge becomes the differential of the complex of these resolvents. Further the investigation of local BRST cohomology was performed with use of developed homological methods (see [8,9,10] and the citations there).

In this paper, we consider a more modest problem. We study relations between the Noether identities and related phenomena for global symmetries of Lagrangians and cohomological properties of the algebra of these symmetries.

Our considerations will be carried out for mechanics but the scheme has the straightforward generalization to the case of field theory Lagrangians.

The standard statement (the Noether 1-st Theorem) is: if the Lagrangian L is invariant under the action of the Lie algebra $\mathcal{G}$ of rigid symmetries $\{\delta_k\}$, then to every symmetry δ_k there corresponds the charge $N_k(L)$ which is preserved on the equations of motion [11].

If the Lie algebra of vector fields $\{\mathbf{X}_k = X_k^\mu \frac{\partial}{\partial q^\mu}\}$ (infinitesimal transformations of the configuration space) corresponds to $\{\delta_k\}$, then

$$\begin{aligned} \delta_k L = 0 = \tfrac{d}{dt}\left(N_k(L)\right) + X_k^\mu \mathcal{F}_\mu(L)\,, \\ \text{where}\quad N_k(L) = X_k{}^\mu \frac{\partial L}{\partial \dot q^\mu} \quad\text{and}\quad \mathcal{F}_\mu(L) = \frac{\partial L}{\partial q^\mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot q^\mu} \end{aligned} \tag{1.1}$$

is the left hand side (l.h.s.) of the equations of motion $\mathcal{F}_\mu = 0$ of the Lagrangian L.

The statement of the Noether theorem is valid also in the case when the Lagrangian is preserved up to a total derivative of some functions $\{\alpha_k(q)\}$ under the actions of transformations $\{\delta_k\}$,

$$\delta_k L = 0 \rightarrow \delta_k L = d\alpha_k, \text{ then } N_k(L) \rightarrow N_k(L) - \alpha_k\,. \tag{1.2}$$

To what extent is this total derivative essential? Redefinition of L by adding a total derivative $L \rightarrow L + df$ changes α_k to $\alpha_k + \delta_k f$. The algebra of symmetries of the Lagrangian can be considered generalized, if $d\alpha_k$ is not equal to 0 in (1.2), and it is *essentially generalized*, if it cannot be canceled by redefinition of Lagrangian with a total derivative, i.e., $\delta_k L = d\alpha_k$ but the equations

$$d(\alpha_k + \delta_k f) = 0 \tag{1.3}$$

have no solution.

Using the basic properties of the operators δ and d: $\delta^2 = d^2 = 0$, $d\delta = \delta d$ (see the Section 2) we obtain from (1.2) that

$$0 = \delta^2 L = \delta d\alpha_k = d\delta\alpha_k, \text{ so } (\delta\alpha)_{km} = w_{km} = \text{constant}, \tag{1.4}$$

where $(\delta\alpha)_{km} = \mathcal{L}_k\alpha_m - \mathcal{L}_m\alpha_k - c^i_{km}\alpha_i$ and c^i_{km} are structure constants of the symmetry Lie algebra ($\mathcal{L}_k\alpha_m = \delta_k\alpha_m$ is the Lie derivative of α_m along the symmetry vector field $\mathbf{X}_k$).

It is easy to see that w_{km} is a cocycle of algebra $\mathcal{G}$ with coefficients in constants. In the case when w_{km} is not a coboundary, one can see that the symmetries are essentially generalized. Indeed, if, according to (1.3), $\alpha_k = -\delta_k f + t_k$, where t_k are constants, then w_{km} in (1.4) is a coboundary in constants: $w_{km} = (\delta t)_{km} = -c^i_{km}t_i$.

Let us consider for example the algebra of space translations. This algebra has 2-cohomologies in constants represented by antisymmetric tensors B_{km}. (This algebra is abelian, so $\delta B = 0$ and the equation $B = \delta t$ has no solutions in constants.) To obtain Lagrangian which possesses generalized translation symmetries corresponding to these cocycles, we note that for this Lagrangian $\alpha_k = A_{km}q^m$. Redefining Lagrangian by a total derivative, one can reduce A_{km} to an antisymmetric tensor, and we come to the Lagrangian

$$L = f(\dot q) + q^k\, B_{km}\dot q^m\,. \tag{1.5}$$

If $f(\dot q) = \frac{m\dot q^2}{2}$, it is the well-known Lagrangian of a particle in constant magnetic field.

In Section 5 we consider a similar statement for the Galilean group: one comes to the Lagrangian of free particle as to a unique Lagrangian corresponding to the Bargmann cocycle of the Lie algebra of the Galilean group.

We see that one of the reasons generalized symmetries arise is the existence of 2-cohomology of the corresponding Lie algebra [1]. Of course a situation is more complicated. For example, by evident reasons for this phenomenon the de Rham cohomology of configuration space is responsible. If L_{inv} is a $\mathcal{G}$-invariant Lagrangian and $L = L_{inv} + A_\mu(q)\dot q^\mu$, where $A_\mu(q)dq^\mu$ is a closed differential 1-form which is not exact ($A_\mu(q)dq^\mu \neq df$), then in general L is not $\mathcal{G}$-invariant. It has the same equations of motion, but it differs from L_{inv} by Aharonov-Bohm like effects [15].

[1]The role of 2-cohomologies of symmetry group on the level of classical Lagrangians maybe at first was recognized in [12]. Many examples where physical properties of weakly invariant Lagrangians interplay with cohomology of configuration space and 2-cohomology of symmetry group and corresponding Lie algebra were actively investigated in physics. It is of great importance for clarifying geometry of quantization, for revealing the structure of Wess-Zumino terms. (See [1], [2], [13], [14] and references there.)

Even in the case when the de Rham cohomology is trivial and the cocycle w_{km} in (1.4) is a coboundary, the symmetries of Lagrangian can be essentially generalized. The coboundary condition $w_{ij} = -c^k_{ij}t_k$ is necessary but not sufficient for (1.3) to have a solution. It is other cohomologies of symmetry algebra which prevent a Lagrangian to be reduced to a $\mathcal{G}$-invariant by redefinition with a total derivative.

The purpose of our paper is to investigate systematically these phenomena.

For the algebra $\mathcal{G}$ of vector fields on the configuration space M and a Lagrangian $L(q,\dot q)$ on M, we considered the following possible cases of generalized symmetries arising

$$\begin{array}{c}
\text{1) The action of } \mathcal{G} \text{ on a Lagrangian } L \text{ produces a 2-cocycle on } \mathcal{G}\text{:}\\
\delta_k L(q,\dot q) = \frac{d}{dt}\alpha_k(q),\ w_{km} = \mathcal{L}_k\alpha_m - \mathcal{L}_m\alpha_k - c^i_{km}\alpha_i\,,\\
\text{2) The action of } \mathcal{G} \text{ on a Lagrangian } L \text{ produces a 2-cocycle,}\\
\text{but it is trivial: } w_{km} = -c^i_{km}t_i\,,\\
\text{3) The Lagrangian } L \text{ differs from the invariant one by a closed form:}\\
L = L_{inv} + A_\mu(q)\dot q^\mu, (\partial_\mu A_\nu - \partial_\nu A_\mu = 0)\\
\text{hence } \delta_k L = \frac{d}{dt}(A_\mu X^\mu_k) \text{ and } w_{km} = 0\,,\\
\text{4) The Lagrangian } L \text{ differs from the } \mathcal{G}\text{-invariant one}\\
\text{by an exact form (total derivative):}\\
L = L_{inv} + \partial_\mu f(q)\dot q^\mu = L_{inv} + \frac{d}{dt}f(q),\ \delta_k L_{inv} = 0\,.
\end{array} \tag{1.6}$$

One can see that

$$\text{“4”} \Rightarrow \text{“3”} \Rightarrow \text{“2”} \Rightarrow \text{“1”}\,. \tag{1.7}$$

We briefly discuss how generalized symmetries reveal themself in Hamiltonian mechanics and in a quasiclassical approximation of quantum mechanics [13,14].

If the Lagrangian is $\mathcal{G}$-invariant, then to the Noether charges $N_k(L)$ in (1.1) in the Hamiltonian framework the charges $N^{ham}_k = X^\mu_k p_\mu$ correspond. They generate a $\mathcal{G}$-algebra structure via Poisson brackets

$$\{N^{ham}_k, N^{ham}_m\} = c^i_{km}N^{ham}_i\,. \tag{1.8}$$

In quasiclassical approximation of quantum mechanics the operators $X^\mu_k\hat p_\mu$ correspond to these charges. Their action on quasiclassical wave function in the configuration representation is reduced to a infinitesimal transformation of wave functions argument:

$$\imath\hat\delta_k\Psi = \Psi(q^\mu + \delta_k q^\mu) - \Psi(q^\mu)\,.$$

In the case when the symmetry algebra is generalized, one can see that due to (1.2)

$$N^{ham}_k = X^\mu_k p_\mu - \alpha_k\,.$$

The corresponding operators act not only on quasiclassical wave functions argument but also on its phase:

$$\hat\delta_k\Psi = -\imath X^\mu_k\frac{\partial\Psi(q)}{\partial q^\mu} - \alpha_k(q)\Psi(q)\,. \tag{1.9}$$

In the case when the Lagrangian does not possess the property "2" in (1.6), i.e., generalized symmetries lead to a non-trivial cocycle, the Lie algebra of Hamiltonian Noether charges N_i^{ham} is the central extension of the Lie algebra $\mathcal{G}$ which corresponds to the cohomology class of the cocycle w_{km}:

$$\{N_i^{ham}, N_j^{ham}\} = c_{ij}^k N_k^{ham} + w_{ij}\,. \tag{1.10}$$

Respectively in this case in (1.9) an essentially projective representation of the Lie algebra $\mathcal{G}$ is realized.

In the case when the Lagrangian possesses the property "2" in (1.6), one can choose α_k such that (1.8) is satisfied and the quantum representation (1.9) of $\mathcal{G}$ becomes linear. But if this Lagrangian does not possess the property "4" in (1.6), then the action of quantum transformation on the phase factor cannot be removed by redefinition $\Psi \to e^{if}\Psi$ of the wave function corresponding to redefinition of Lagrangian with a total derivative. In this case one can say that the linear transformation (1.9) is splitted into a space-like transformation + intrinsic spin-like transformation. Nevertheless if the Lagrangian possesses the property "3", i.e., it differs with an invariant Lagrangian by Aharonov-Bohm like effects, then the action on a phase in (1.9) can be removed locally [15].

We call a time-independent Lagrangian $L(q,\dot q)$ *weakly $\mathcal{G}$-invariant* if l.h.s. of its motion equations (1.1) is $\mathcal{G}$-invariant. For example, the Lagrangian L in (1.2) is weakly $\mathcal{G}$-invariant. One can show that if L is weakly $\mathcal{G}$-invariant Lagrangian, then

$$\delta_k L = c_k + w_k\,, \tag{1.11}$$

where c_k are constants and w_k correspond to closed forms: $w_k = w_{k\mu}(q)\dot q^\mu$, where differential 1-forms $w_{k\mu}(q)dq^\mu$ are closed (see in details below).

If $\{w_k\}$ correspond to exact forms: $w_{k\mu}(q)dq^\mu = d\alpha_k(q)$, $w_{k\mu}(q)\dot q^\mu = \partial_\mu\alpha_i(q)\dot q^\mu = d\alpha_k(q)/dt$ and

$$c_k = 0\,, \tag{1.12}$$

then we come to (1.2). In the case if (1.12) does not obey the corresponding Noether charges,

$$N_k = X_k{}^\mu \frac{\partial L}{\partial \dot q^\mu} - \alpha_k - c_k t \tag{1.13}$$

depend on time.

We denote by $\mathcal{V}_{0.0}$ the space of weakly $\mathcal{G}$-invariant Lagrangians on M and by $\mathcal{V}_{0.1}$ the subspace of $\mathcal{V}_{0.0}$ for which the condition (1.12) is satisfied. We denote by $\mathcal{V}_{s.1}$ ($s = 1,2,3,4,$) the space of Lagrangians for which the property "s" in (1.6) is satisfied. According to (1.7),

$$\mathcal{V}_{4.1} \subseteq \mathcal{V}_{3.1} \subseteq \mathcal{V}_{2.1} \subseteq \mathcal{V}_{1.1} \subseteq \mathcal{V}_{0.1} \subseteq \mathcal{V}_{0.0}\,. \tag{1.14}$$

One can also consider subspaces $\{\mathcal{V}_{s.0}\}$ of the space $\mathcal{V}_{0.0}$

$$\mathcal{V}_{4.0} \subseteq \mathcal{V}_{3.0} \subseteq \mathcal{V}_{2.0} \subseteq \mathcal{V}_{1.0} \subseteq \mathcal{V}_{0.0}\,, \quad \mathcal{V}_{s.1} \subseteq \mathcal{V}_{s.0}\,, \tag{1.15}$$

which correspond to $\{\mathcal{V}_{s.1}\}$ if we ignore the condition (1.12): a weakly $\mathcal{G}$-invariant Lagrangian L belongs to $\mathcal{V}_{1.0}$, if $\delta_i L = d\alpha_i + c_i$. It is easy to see that $\delta\alpha$ is also a 2-cocycle in this case as in (1.4). Moreover $L \in \mathcal{V}_{2.0}$, if this cocycle is trivial, $L \in \mathcal{V}_{4.0}$, if $\alpha_i = \delta_i f$, and $L \in \mathcal{V}_{3.0}$, if it differs from $\mathcal{V}_{4.0}$ by a closed form. Lagrangians in $\mathcal{V}_{s.0}$ have time-dependent Noether currents (1.13) in general.

What else can we say about embeddings (1.14, 1.15)? Does weakly $\mathcal{G}$-invariant Lagrangian possess generalized symmetries (1.2)? Can it be reduced to a $\mathcal{G}$-invariant one? Does there exist Lagrangian which belongs to the space $\mathcal{V}_{s.0}$ and which does not belong to the space $\mathcal{V}_{s+1.0}$ or $\mathcal{V}_{s.1}$? If an answer is "no" what are the reasons?

To answer these questions, we establish a hierarchy in the space of weakly $\mathcal{G}$-invariant Lagrangians. This hierarchy relates the phenomena discussed above with cohomology groups of the Lie algebra $\mathcal{G}$ and of the configuration space M.

The scheme of our considerations is the following. We fix a configuration space M and a finite-dimensional algebra $\mathcal{G}$ of its transformations. Then we establish relations between weakly $\mathcal{G}$-invariant Lagrangians on M and the cohomologies of the algebra $\mathcal{G}$ and of M. From the considerations above we see that in the phenomena we are investigating two differentials are interplaying, δ and $d_{E.L}$, where the differential δ corresponds to the symmetries and $d_{E.L}$ is the prolongation of the exterior differential which acts on Lagrangians. It is the variational derivative, whose action leads to the Euler-Lagrange equation. (See in details Section 2.) These differentials, as well as differentials d and δ, satisfy the conditions: $\delta^2 = d_{E.L}^2 = d_{E.L}\delta - \delta d_{E.L} = 0$. We naturally come to the differential $Q = d_{E.L} \pm \delta$ which is strictly related to our problem. For example the condition $Q(L,\alpha_i) = (d_{E.L}L, 0, w = \delta\alpha)$ corresponds to the condition $\delta L = d\alpha$ in (1.2). The changing of the cochain (L,α_i) on a coboundary: $(L,\alpha_i) \mapsto (L,\alpha_i) + Qf = (L+df, \alpha_i + \delta_i f)$ corresponds to redefinition of Lagrangian by a total derivative $L \mapsto L + df$.

It is the cohomology of the differential Q, which allows us to reveal the relations between generalized symmetries of Lagrangians and cohomologies of the configuration space and the symmetry Lie algebra. We do it in the following way. Using the technique of spectral sequences, we calculate the cohomology of Q via cohomology of $d_{E.L}$ by modulo δ, then vice versa via cohomology of δ by modulo $d_{E.L}$. Calculating cohomology of the operator Q in the first way, we come to the spaces $\{K_s\}$ which are expressed in terms of cohomologies of the Lie algebra and the configuration space. On the other hand, calculating the same cohomology in the second way, we come naturally to the space $\mathcal{V}_{0.0}$ of weakly $\mathcal{G}$-invariant Lagrangians and to its subspaces $\{\mathcal{V}_{s.\sigma}\}$ (1.14,1.15). Natural relations, which arise between the results of calculations in the first and in the second way, lead to the sequence of homomorphisms between the spaces $\{\mathcal{V}_{s.\sigma}\}$ and $\{K_s\}$ defining these spaces in a recurrent way via the kernels of the corresponding homomorphisms.

This construction establishes hierarchy in the space of weakly $\mathcal{G}$-invariant Lagrangians making links between the physical properties of Lagrangians and pure mathematical objects: the condition that Lagrangian belongs to some space $\mathcal{V}_{s.\sigma}$ and does not belong to the space $\mathcal{V}_{s+1.\sigma}$ or $\mathcal{V}_{s.\sigma+1}$ in terms of this hierarchy is reformulated to the condition that the corresponding homomorphism does not vanish on it. The problem of analyzing the content of the spaces $\{\mathcal{V}_{s.\sigma}\}$ and differences between them is reduced to the problem of calculating the corresponding homomorphisms. For example in the case, when the space K_s is trivial one has $\mathcal{V}_{s-1.\sigma} = \mathcal{V}_{s.\sigma}$. In particular, if all the spaces K_s are trivial then all weakly invariant Lagrangians are invariant (up to a total derivative).

The plan of the paper is as follows.

In Section 2 we consider the complex of Lagrangians and clarify its relations with corresponding complex of differential forms.

In Section 3 we calculate cohomology of the differential Q of the double complex of cochains on the Lie algebra $\mathcal{G}$ and taking values in the functions on M and in Lagrangians of classical mechanics. Using the results of these calculations, in the Section 4, we establish hierarchy in the space of weakly invariant Lagrangians and consider some general properties of this hierarchy. It is the main result of the paper. In this Section we consider also from our point of view the hierarchy for Lagrangians polynomial in velocities.

In Section 5, using this hierarchy, we calculate the content of the subspaces $\mathcal{V}_{s.\sigma}$ in (1.14, 1.15) for some special cases of configuration spaces and symmetry algebras. In particular, we perform this analysis for $so(3)$, Poincaré and Galilean algebras.

In Section 6, we give some motivations for the technique we used in this paper.

In Appendixes we give a brief sketch on the notion of Lie algebra cohomology and calculation of double complexes cohomology via corresponding spectral sequences.

2 The complexes of Lagrangians and Differential Forms

Let M be an n-dimensional manifold (configuration space) and $\mathcal{G}$ be a Lie algebra acting on it. It means that a homomorphism Φ from $\mathcal{G}$ in the Lie algebra of vector fields on M is defined:

$$\mathcal{G} \ni x \xrightarrow{\Phi} \tilde{x} \text{ (fundamental vector field)}: \quad [\tilde{x}, \tilde{y}] = \widetilde{[x, y]}. \tag{2.1}$$

We denote this pair by $[\mathcal{G}, M]$.

Let $\Omega^j(M)$ be the space of differential j-forms on M. The linear spaces $\Omega^j(M)$ for any given j can be considered as $\mathcal{G}$-modules if we define the action of the algebra on forms via Lie derivatives along corresponding fundamental vector fields: $h \circ w = \mathcal{L}_{\tilde{h}} w$. One can consider the $\mathcal{G}$-differential corresponding to this module structure and cohomology spaces $H^i(\mathcal{G}, \Omega^j(M))$, which are $\mathcal{G}$-cohomologies with coefficients in $\Omega^j(M)$. (See e.g.[16] or Appendix 1).

Consider the de Rham complex $\{\Omega^j(M), d\}$ and extend the exterior differential d onto i-cochains $C^i(\mathcal{G}, \Omega^j(M)) = C^i(\mathcal{G}) \otimes \Omega^j(M)$ by setting $d(c \otimes w) = c \otimes dw$. The differentials d and δ commute with each other, $d\delta = \delta d$, and one can consider the corresponding double complex $\{C^i(\mathcal{G}, \Omega^j(M)), d, \delta\}$.

To include Lagrangians in the game, we extend the complex $\{\Omega^j(M), d\}$ of differential forms to the complex $\{\Lambda^j(M), d_{E.L}\}$ of Lagrangians, following [17].

We define the space $\Lambda^j(M)$ of j-Lagrangians ($j \geq 1$) as the space of functions (Lagrangians) which depend on points q^μ of the manifold M and on derivatives $\frac{\partial q^\mu}{\partial \xi^\alpha}, \ldots, \frac{\partial^k q^\mu}{\partial \xi^{\alpha_1} \ldots \partial \xi^{\alpha_k}}$ of an arbitrary but finite order k of parameters $(\xi^1, \ldots, \xi^j)$ which take values in the j-dimensional space $\mathbf{R}^j$. In the case $j = 0$ we put $\Lambda^0(M) = \Omega^0(M)$ to be the space of functions on M. We say that Lagrangian has the rank k, if the highest degree of derivatives on which it depends is equal to k and we denote by Λ^j_k the corresponding subspace of Λ^j. The Lagrangians of classical mechanics considered in the following Sections belong to Λ^1_1.

If L is a Lagrangian in $\Lambda^j(M)$ then to every map (j-dimensional path)

$$q^\mu(\xi^1, \ldots, \xi^j): \quad \mathbf{R}^j \to M \tag{2.2}$$

corresponds the integral

$$S_L([q(\xi)]) = \int L\left(q^\mu(\xi), \frac{\partial q^\mu(\xi)}{\partial \xi^\alpha}, \ldots, \frac{\partial^k q^\mu(\xi)}{\partial \xi^{\alpha_1} \ldots \partial \xi^{\alpha_k}}\right) d\xi^1 \ldots d\xi^j . \tag{2.3}$$

This defines the natural embedding of the space $\Omega^j(M)$ of differential j-forms in $\Lambda^j_1(M)$:

$$w = w_{\mu_1 \ldots \mu_j}(q) dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_j} \mapsto L_w = n! w_{[\mu_1 \ldots \mu_j]}(q) \frac{\partial q^{\mu_1}}{\partial \xi^1} \cdot \ldots \cdot \frac{\partial q^{\mu_j}}{\partial \xi^j} . \tag{2.4}$$

The integral $S_{L_w}([q(\xi)])$ is equal to the integral of the differential form w over the surface defined by the map (2.2). It does not depend on the choice of parametrization $q(\xi)$ of this surface. We say that Lagrangian L_w corresponds to the differential form w and later on we often will not distinguish between w and L_w.

Remark In general, for an arbitrary Lagrangian the l.h.s. of (2.3) is not correctly defined on images of maps (2.2). It can be considered as a functional on embedded surfaces which does not depend on its parametrization in a case if Lagrangian L is a *density*, i.e. under reparametrization $q(\xi) \to q(\xi(\tilde{\xi}))$, one has $L \to L \cdot \det(\partial\xi/\partial\tilde{\xi})$ and L defines the volume form on surface $q(\xi)$ (see for example [18,19]). The Lagrangians corresponding to differential forms are the special examples of densities.

To define the complex of Lagrangians which generalizes de Rham complex we consider, following [17], the differential $d_{E.L}$, using Euler-Lagrange equations of motion for the functional (2.3):

$$d_{E.L}\colon \Lambda^j \to \Lambda^{j+1}, \; d_{E.L} L\left(q, \frac{\partial q^\mu}{\partial \xi^{\tilde{a}}}, \ldots, \frac{\partial^k q^\mu}{\partial \xi^{\tilde{\alpha}^1} \ldots \partial \xi^{\tilde{\alpha}^k}}\right) = \mathcal{F}_\mu(L) \frac{\partial q^\mu}{\partial \xi^{j+1}} , \tag{2.5}$$

where $\tilde{\alpha} = (1, \ldots, j, j+1), \alpha = (1, \ldots, j)$ and $\mathcal{F}_\mu(L)$ are l.h.s. of Euler-Lagrange equations of the Lagrangian L, i.e. the variational derivatives of the corresponding functional (2.3): $\mathcal{F}_\mu(L) = \frac{\delta}{\delta q^\mu} S_L([q(\xi)])$

For example, if $L \in \Lambda_1^j(M)$, $L = L(q, \frac{\partial q^\mu}{\partial \xi^\alpha})$, then

$$d_{E.L} L\left(q, \frac{\partial q^\mu}{\partial \xi^a}, \frac{\partial^2 q^\mu}{\partial \xi^a \partial \xi^\beta}\right) = \sum_\mu \sum_{1\leq\alpha,\beta\leq j} \left(\frac{\partial L}{\partial q^\mu} - \frac{\partial^2 L}{\partial q^\nu \partial q^\mu_\alpha}\frac{\partial q^\nu}{\partial \xi^\alpha} - \frac{\partial^2 L}{\partial q^\nu_\beta \partial q^\mu_\alpha}\frac{\partial^2 q^\nu}{\partial \xi^\alpha \partial \xi^\beta}\right)\frac{\partial q^\mu}{\partial \xi^{j+1}}. \tag{2.6}$$

(In general $d_{E.L}\Lambda_k^j \subseteq \Lambda_{2k}^{j+1}$)

One can show that as well as for exterior differential d, $d_{E.L}^2 = 0$ [17] and consider the cohomology of the complex

$$\{\Lambda^j(M), d_{E.L}\}: \quad \Lambda^0(M) \xrightarrow{d_{E.L}} \Lambda^1(M) \xrightarrow{d_{E.L}} \Lambda^2(M) \xrightarrow{d_{E.L}} \dots \tag{2.7}$$

From the definition of $d_{E.L}$ and from (2.4) it follows that $L_{dw} = d_{E.L} L_w$. The complex $\{\Omega^j(M), d\}$ of differential forms is subcomplex of the complex (2.7).

The spaces $\Lambda^j(M)$ of Lagrangians for any given j (and their subspaces $\Lambda_k^j(M)$ for any given j and k) as well as $\Omega^j(M)$ can be naturally considered as $\mathcal{G}$-modules if we define the action of Lie algebra elements on Lagrangians, as follows: if $x \in \mathcal{G}$ and $\tilde{x} = \Phi x = X^\mu(q)\partial/\partial q^\mu$ (see (2.1)), then

$$(x \circ L) = \mathcal{L}_{\tilde{x}} L = X^\mu \frac{\partial L}{\partial q^\mu} + (D_\alpha X^\mu)\frac{\partial L}{\partial q^\mu_\alpha} + (D_\beta D_\alpha X^\mu)\frac{\partial L}{\partial q^\mu_{\alpha\beta}} + \dots \tag{2.8}$$

where $D_\alpha = \frac{d}{d\xi^\alpha} = q^\mu_\alpha \frac{\partial}{q^\mu} + q^\mu_{\alpha\beta}\frac{\partial}{q^\mu_\beta} + \dots$ is the total derivative. If a Lagrangian corresponds to a differential form, then (2.8) corresponds to usual Lie derivative: $\mathcal{L} L_w = L_{\mathcal{L}w}$. To the identity $\mathcal{L}_\eta w = dw\rfloor\eta + d(w\rfloor\eta)$ for Lie derivative on forms there corresponds the identity $\mathcal{L}_\eta L = \eta^\mu \mathcal{F}_\mu(L) + D_\alpha N^\alpha$, which leads to Noether currents N^α in the case $\mathcal{L}_\eta L = 0$.

Considering $\mathcal{G}$-differential δ corresponding to this module structure we come to the spaces $H^i(\mathcal{G}, \Lambda_k^j(M))$ of $\mathcal{G}$-cohomologies with coefficients in $\Lambda_k^j(M)$.

In the same way as for differential forms one can extend the action of $d_{E.L}$ on the spaces $C^i(\mathcal{G}, \Lambda^j)$ of i-cochains with values in Λ^j and consider the double complex $\{C^i(\mathcal{G}, \Lambda^j), d_{E.L}, \delta\}$ because $d_{E.L}$ and δ commute also for Lagrangians. The complex $\{C^i(\mathcal{G}, \Omega^j), d, \delta\}$ is embedded in this complex.

The cohomology of the complex (2.7) evidently differs from the de Rham cohomology, but on the other hand one has

Proposition 1[2]

1. If Lagrangian L is exact, $L = d_{E.L} L'$ and it is a density (see the remark above), then it corresponds to an exact differential form.

[2]The complex (2.7) differs from the standard variational complex (see for example [20,26]). It was introduced in [17] by Th. Voronov for the Lagrangians on superspace. This complex and Proposition are useful in supermathematics where the concept of usual differential form is ill-defined [18,19,21].

2. If Lagrangian L is closed and it depends only on first derivatives, $d_{E.L} L = 0, L \in \Lambda_1^q$, then it corresponds to a closed differential form up to a constant

$$L = L_w + c,\ dw = 0. \tag{2.9}$$

In the case when L in (2.9) is a density then one has $c = 0$.

The 2-nd statement immediately follows from (2.6) and from the definition of the density. We do not need the first one here and we omit its proof.

We use this proposition to consider the following subcomplex $(\mathcal{C}^*, d_{E.L})$ of the complex (2.7), which will be of use in this paper:

$$(\mathcal{C}^*, d_{E.L}): \quad \Lambda^0(M) \xrightarrow{d_{E.L}} \Lambda_1^1(M) \xrightarrow{d_{E.L}} d_{E.L}\Lambda_1^1(M) \longrightarrow 0, \tag{2.10}$$

where, like in (2.7), $\mathcal{C}^0 = \Lambda^0(M)$ is the space of functions on M, $\mathcal{C}^1 = \Lambda_1^1(M)$ is the space of Lagrangians $L(q^\mu, \dot{q}^\mu)$ of classical mechanics defined on the configuration space M, $\mathcal{C}^2$ is the subspace of coboundaries in Λ_2^2. It contains elements corresponding to equations of motion of some Lagrangian from Λ_1^1: $a \in \mathcal{C}^2 = d_{E.L}\Lambda_1^1(M)$ iff there exists a Lagrangian L such that $a = d_{E.L} L$.

From the 2-nd statement of Proposition 1 it follows that the cohomology of this truncated complex is strictly related to the de Rham cohomology:

$$H^0(\mathcal{C}^*, d_{E.L}) = H^0(M),\ H^1(\mathcal{C}^*, d_{E.L}) = H^1(M) + \mathbf{R},\ H^2(\mathcal{C}^*, d_{E.L}) = 0. \tag{2.11}$$

For our purposes, it is also useful to consider the following modification of the complex (2.7). We consider the spaces $\{\overline{\Lambda^j}\}$, where $\overline{\Lambda^j} = \Lambda^j/\mathbf{R}$, if $j \geq 1$ and $\overline{\Lambda^0} = \Lambda^0 = \Omega^0(M)$. Elements of $\overline{\Lambda^j}$ ($j \geq 1$) are j-Lagrangians defined up to constants. We denote by $\overline{L}$ the equivalence class of Lagrangian L in $\overline{\Lambda}$. Instead the complex (2.7), one can consider the complex

$$\{\overline{\Lambda^j}(M), \overline{d}_{E.L}\}: \quad \Lambda^0(M) \xrightarrow{\overline{d}_{E.L}} \overline{\Lambda^1}(M) \xrightarrow{\overline{d}_{E.L}} \overline{\Lambda^2}(M) \xrightarrow{\overline{d}_{E.L}} \dots \tag{2.12}$$

and respectively the double complex $\{C^i(\mathcal{G}, \overline{\Lambda^j}), \overline{d}_{E.L}, \overline{\delta}\}$ of i-cochains on $\mathcal{G}$ with values in $\overline{\Lambda^j}$. The differentials $\overline{d}_{E.L}$ and $\overline{\delta}$ are well defined in a natural way: $\overline{d}_{E.L}\overline{\lambda} = \overline{d_{E.L}\lambda}$ and $\overline{\delta}\overline{\lambda} = \overline{\delta\lambda}$ where $\overline{\lambda}$ is the equivalence class of the cochain λ in $C^*(\mathcal{G}, \overline{\Lambda^*})$. The differential $\overline{d}_{E.L}$ does not differ essentially from $d_{E.L}$: If λ is a cochain with values in Lagrangians, then it is easy to see that

$$\overline{d}_{E.L}\overline{\lambda} = 0, \iff d_{E.L}\lambda = 0. \tag{2.13}$$

To (2.10) there corresponds the subcomplex

$$(\overline{\mathcal{C}^*}, d_{E.L}): \quad \Lambda^0(M) \xrightarrow{\overline{d}_{E.L}} \overline{\Lambda_1^1}(M) \xrightarrow{\overline{d}_{E.L}} (d_{E.L}\overline{\Lambda_1^1}(M)) \longrightarrow 0 \tag{2.14}$$

of the complex (2.12). From (2.13) it follows that for the truncated complex $\overline{\mathcal{C}^*}$ one has

$$H^0(\overline{\mathcal{C}^*}, d_{E.L}) = H^0(M),\ H^1(\overline{\mathcal{C}^*}, d_{E.L}) = H^1(M),\ H^2(\overline{\mathcal{C}^*}, d_{E.L}) = 0. \tag{2.15}$$

Non-pleasant constants do not arise here like in (2.11). The difference between the complexes $\{C^i(\mathcal{G},\overline{\Lambda^j}),\overline{d}_{E.L},\overline{\delta}\}$ and $\{C^i(\mathcal{G},\Lambda^j),d_{E.L},\delta\}$ becomes non-trivial on the level of 1-cochains at least. It corresponds to the difference between time-independent and time-dependent Noether charges. (See, e.g., Example 1 in Section 5.)

Finally, we want to note that to every Lagrangian L on M there corresponds a density A_L on the space $\hat{M} = M \times \{\text{space of parameters}\}$. To the functional (2.3) there corresponds the integral of the density over the graph of the map (2.2). For example to Lagrangian $L(q^\mu, \frac{dq^\mu}{dt})$ of classical mechanics one can put into correspondence the density

$$A_L\left(q^\mu, \frac{dq^\mu}{d\tau}, \frac{dt}{d\tau}\right) = L\left(q^\mu, \frac{\frac{dq^\mu}{d\tau}}{\frac{dt}{d\tau}}\right)\cdot\frac{dt}{d\tau}; \qquad \text{if } \tau \to \tau'(\tau), \text{ then } A_L \to \frac{d\tau}{d\tau'}A_L. \tag{2.16}$$

To a path $q^\mu(t)$ there corresponds the curve $(q^\mu(\tau), t(\tau))$ and one has $S_L([q(t)]) = S_{A_L}([q(\tau),t(\tau)])$ for any parametrization $q(\tau)$. (This transformation is useful in the formalism where fields and space variables are on an equal footing [22].)

It is easy to see that for densities A_L the difference between complexes (2.10) and (2.14) is removed. To redefinition of Lagrangian L by a constant c there corresponds redefinition of A_L by the form $c\,dt$.

3 Cohomology of Lagrangians Double Complex and its Spectral Sequences.

Using now the complexes constructed in the previous section we investigate systematically the problem which we considered in the Introduction.

We study simultaneously two double complexes, the double complex$(E^{*.*}, d_{E.L},\delta)$ of cochains on $\mathcal{G}$ with values in the spaces of the complex $\mathcal{C}^*$ defined by (2.10), $\{E^{i.j}, d_{E.L},\delta\} = \{C^i(\mathcal{G},\mathcal{C}^j),\delta,d_{E.L})\}$, and the double complex $(\overline{E^{*.*}},\overline{d}_{E.L},\overline{\delta})$ of cochains on $\mathcal{G}$ with values in the spaces of the complex $\overline{\mathcal{C}}^*$ defined by (2.14), $\{\overline{E^{i.j}},\overline{d}_{E.L},\overline{\delta}\} = \{C^i(\mathcal{G},\overline{\mathcal{C}^j}),\overline{d}_{E.L},\overline{\delta})\}$.

The complex $(E^{*.*}, d_{E.L},\delta)$ is as follows

$$\begin{array}{ccccccc}
\Lambda^0(M) & \xrightarrow{d_{E.L}} & \Lambda^1_1(M) & \xrightarrow{d_{E.L}} & d_{E.L}\Lambda^1_1(M) & \xrightarrow{d_{E.L}} & 0\\
\delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
C^1(\mathcal{G},\Lambda^0(M)) & \xrightarrow{d_{E.L}} & C^1(\mathcal{G},\Lambda^1_1(M)) & \xrightarrow{d_{E.L}} & C^1(\mathcal{G},d_{E.L}\Lambda^1_1(M)) & \xrightarrow{d_{E.L}} & 0\\
\delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
C^2(\mathcal{G},\Lambda^0(M)) & \xrightarrow{d_{E.L}} & C^2(\mathcal{G},\Lambda^1_1(M)) & \xrightarrow{d_{E.L}} & C^2(\mathcal{G},d_{E.L}\Lambda^1_1(M)) & \xrightarrow{d_{E.L}} & 0\\
\delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
\cdots & & \cdots & & \cdots & &
\end{array} \tag{3.1}$$

and the complex $(\overline{E^{*.*}}, d_{E.L},\delta)$ is represented in a similar way (by putting the "bar"s in corresponding places).

The differential of the complex $E^{*.*}$ is equal to

$$Q = (-1)^j\delta + d_{E.L}, \tag{3.2}$$

and $\overline{Q} = (-1)^j\overline{\delta} + \overline{d}_{E.L}$ for the complex $\overline{E^{*.*}}$.

The problem of weakly invariant Lagrangians classification can be now reformulated in terms of these double complexes.

To do this we consider their spectral sequences $\{E_r^{*.*}\}, \{\overline{E_r^{*.*}}\}$ and the transposed spectral sequences $\{{}^tE_r^{*.*}\}, \{\overline{{}^tE_r^{*.*}}\}$. The relations between $\{{}^tE_r^{*.*}\}$ and $\{E_r^{*.*}\}$ lead to the hierarchy in the space of weakly $\mathcal{G}$-invariant Lagrangians with time-independent Noether charges, the relations between $\{\overline{{}^tE_r^{*.*}}\}$ and $\{\overline{E_r^{*.*}}\}$ lead to the hierarchy in the space of weakly $\mathcal{G}$-invariant Lagrangians with time-dependent Noether charges, and the relations between $\{E_r^{*.*}\}$ and $\{\overline{E_r^{*.*}}\}$ lead to the relations between these two hierarchies.

We denote by $\mathcal{V}_{0.0}$ (see Introduction) the subspace of weakly $\mathcal{G}$-invariant Lagrangians in the space $E^{0.1}$, i.e., Lagrangians of classical mechanics on M, whose motions equations l.h.s. are $\mathcal{G}$-invariant:

$$\mathcal{V}_{0.0} = \{L\colon L \in \Lambda^1_1 \quad\text{and}\quad \delta d_{E.L}L = 0\}. \tag{3.3}$$

One can see that the cochain $\mathbf{f} = (d_{E.L}L,0,0)$ is a cocycle of the differential Q iff $L \in \mathcal{V}_{0.0}$. The cohomology class $[(d_{E.L}L,0,0)]$ of this cocycle belongs to $H^2(Q)$. If we express the cohomology of differential Q via the stable terms of transposed spectral sequence $\{{}^tE_r^{*.*}\}$, i.e. calculating $H^*(Q)$ in perturbation theory, considering in (3.1) the differential δ as zeroth order approximation for the differential Q, we see that $[d_{E.L}\mathcal{V}_{0.0}]_\infty = {}^tE_\infty^{0.2}$ is the subspace of $H^2(Q)$. On the other hand if we express the cohomology of differential Q via the stable terms of spectral sequence $\{E_r^{*.*}\}$, i.e. calculating $H^*(Q)$ in perturbation theory, considering in (3.1) the differential $d_{E.L}$ as zeroth order approximation, we express $H^2(Q)$ in terms of $\{E_\infty^{i.2-i}\}$. The relations between the space ${}^tE_\infty^{0.2}$ and the spaces $\{E_\infty^{i.2-i}\}$ lead to the relations between the space of weakly $\mathcal{G}$-invariant Lagrangians and cohomologies groups of $\mathcal{G}$ and M.

The technique of spectral sequences calculations see for example in [16] or in Appendix 2.

The spaces $\{E_r^{i.j}\}$ and $\{\overline{E_r^{i.j}}\}$

We pay more attention to the calculations of the spaces $\{E_r^{*.*}\}$. The calculations of the spaces $\{\overline{E_r^{*.*}}\}$ can be performed in a similar way. The spaces $\{E_1^{i.j}\}$ are equal to the cohomologies of operator $d_{E.L}$: $E_1^{i.j} = H(d_{E.L},E^{i.j})$. From (2.11) and (2.15) it immediately follows that

$$E_1^{*.*}:\quad \begin{array}{ccc}
\mathbf{R} & H^1(M)\oplus\mathbf{R} & 0\\
C^1(\mathcal{G}) & C^1(\mathcal{G},H^1(M)\oplus\mathbf{R}) & 0\\
C^2(\mathcal{G}) & \cdots & 0\\
\cdots & \cdots & 0
\end{array}
\qquad
\overline{E_1^{*.*}}:\quad \begin{array}{ccc}
\mathbf{R} & H^1(M) & 0\\
C^1(\mathcal{G}) & C^1(\mathcal{G},H^1(M)) & 0\\
C^2(\mathcal{G}) & \cdots & 0\\
\cdots & \cdots & 0
\end{array} \tag{3.4}$$

Hereafter we identify the differential forms with Lagrangians corresponding to them by (2.4) and the differential $d_{E.L}$ on these Lagrangians with the differential d on forms.

The operator d_1 acts in the columns of $E_1^{*,*}$ and is generated by δ. By definition $E_2^{i,j} = H(E_1^{i,j}, d_1)$. It is easy to see that $E_2^{i,0} = H^i(\mathcal{G})$ is i-th cohomology group of the Lie algebra $\mathcal{G}$ with coefficients in $\mathbf{R}$.

Now we prove that $E_1^{0,1} = E_2^{0,1}$. Indeed if $c \in E_1^{0,1}$ is a constant ($c \in \mathbf{R}$), then $d_1 c$ is evidently equal to zero. To prove that $d_1 H^1(M) = 0$ we consider the following homomorphism π from the space of differential 1-forms into the space of 1-cochains on $\mathcal{G}$ with values in functions on M (in the space $\Lambda^0(M)$):

$$\pi w(h) = w \rfloor \tilde{h}\,, \tag{3.5}$$

where $\tilde{h}$ is the fundamental vector field Φh corresponding by (2.1) to the element h of the Lie algebra $\mathcal{G}$. From the standard formulae of differential geometry it follows that

$$\text{if} \quad dw = 0 \quad \text{then} \quad \delta w = d\pi w \quad \text{and} \quad \delta\pi w = 0\,. \tag{3.6}$$

Hence for the cohomology class $[w]$ in $H^1(M)$ one has $d_1[w] = [\delta w] = [d\pi w] = 0$ in $E_1^{1,1}$. Consequently $E_1^{0,1} = E_2^{0,1}$.

Now we calculate $E_2^{1,1}$. If $[c]_1 \in E_1^{1,1}$ then

$$c = \sum_\lambda t^{(\lambda)} \otimes w^{(\lambda)} + t' + d\alpha\,, \tag{3.7}$$

where t, t' belong to $C^1(\mathcal{G})$ (are constants), the set $\{w^{(\lambda)}\}$ of closed differential 1-forms constitutes a basis in the space $H^1(M)$ of 1-cohomology and α is an element of $E^{1,0}$. Straightforward calculations using (3.5, 3.6) give

$$d_1[c]_1 = \sum_\lambda [\delta t^{(\lambda)} \otimes w^{(\lambda)} + \delta t' + d(\ldots)] = 0 \Rightarrow \delta t^{(\lambda)} = 0 \text{ and } \delta t' = 0\,. \tag{3.8}$$

On the other hand, coboundaries in $E_1^{1,1}$ are equal to zero because $E_1^{0,1} = E_2^{0,1}$. Hence from eq.(3.7) it follows that $E_2^{1,1} = H^1(M) \otimes H^1(\mathcal{G}) \oplus H^1(\mathcal{G})$. (In the case of complex $\overline{E_1^{1,1}}$, t' in (3.7) is equal to zero and from (2.13) it follows that (3.8) holds also.)

We arrive at the following tables

$E_2^{*,*}$

$\mathbf{R}$	$H^1(M) \oplus \mathbf{R}$	0
$H^1(\mathcal{G})$	$H^1(\mathcal{G}) \otimes H^1(M) \oplus H^1(\mathcal{G})$	0
$H^2(\mathcal{G})$	$\cdots$	0
$H^3(\mathcal{G})$	$\cdots$	0

$\overline{E_2^{*,*}}$

$\mathbf{R}$	$H^1(M)$	0
$H^1(\mathcal{G})$	$H^1(\mathcal{G}) \otimes H^1(M)$	0
$H^2(\mathcal{G})$	$\cdots$	0
$H^3(\mathcal{G})$	$\cdots$	0

(3.9)

One can show that the spaces $\{E_2^{i,j}\}$ in (3.9) as well as $\{\overline{E_2^{i,j}}\}$, which are of interest for us ($i + j \leq 2$) are stable: $E_2^{i,j} = E_3^{i,j} = \ldots = E_\infty^{i,j}$, $\overline{E_2^{i,j}} = \overline{E_3^{i,j}} = \ldots = \overline{E_\infty^{i,j}}$. It is evident without any calculations for the spaces $E_2^{0,0}$, $E_2^{1,0}$, because differentials d_2 on these spaces go out of the table and the boundaries are zero by the same reasons. The spaces $E_2^{0,1}$ and $E_2^{2,0}$ are stable, because the differential d_2: $E_2^{0,1} \to E_2^{2,0}$ is trivial. It follows from eq.(3.5): $d_2[w] = [Q(w, \pi w)] = [\delta\pi w] = 0$. The same arguments lead to the stability of the space $E_2^{1,1}$. One can perform similar considerations for the spaces $\{\overline{E_2^{i,j}}\}$.

Hence, the tables (3.9) establish the relations between the spaces $H^m(Q)$, $H^m(\overline{Q})$ ($m = 0, 1, 2$) and the spaces $E_\infty^{i,m-i}$, $\overline{E_\infty^{i,m-i}}$ respectively (see Appendix 2).

Evidently $H^0(Q) = H^0(\overline{Q}) = \mathbf{R}$. Considering the terms $\{E_\infty^{0,1}, E_\infty^{1,0}\}$ in (3.9), we see that

$$H^1(\mathcal{G}) \subseteq H^1(Q) \quad \text{and} \quad H^1(M) \oplus \mathbf{R} = H^1(Q)/H^1(\mathcal{G})\,. \tag{3.10}$$

according to eq. (A2.11). These relations define a canonical projection p_1 of $H^1(Q)$ on $H^1(M) \oplus \mathbf{R}$ and an isomorphism ι_1 of kerp_1 on $H^1(\mathcal{G})$: If $\mathbf{L} = (L, \alpha)$ is a cocycle of Q, then $L = w + c$, where w is a closed form, c is a constant, and $p_1([\mathbf{L}]) = [w] + c$. If $c = 0$ and $w = df$, then $\alpha - \delta f$ is a 1-cocycle in constants which is equal to $\iota_1([\mathbf{L}])$.

Using the homomorphism (3.5) one can establish also the isomorphism

$$H^1(M) \oplus H^1(\mathcal{G}) \oplus \mathbf{R} \longrightarrow H^1(Q): \quad [w] + t + c \mapsto [w + c, t + \pi w] \tag{3.11}$$

which corresponds to (3.10) and splits $H^1(Q)$ on components.

Similar considerations for the table $\overline{E_2^{*,*}}$ lead to analogous conclusions: $H^1(\mathcal{G}) \subseteq H^1(\overline{Q})$ and $H^1(M) = H^1(\overline{Q})/H^1(\mathcal{G})$; $H^1(M) \oplus H^1(\mathcal{G}) = H^1(\overline{Q})$.

Considering in the same way the terms $\{E_\infty^{0,2}, E_\infty^{1,1}, E_\infty^{2,0}\}$ in (3.9), we see that

$$H^2(\mathcal{G}) \subseteq H^2(Q) \quad \text{and} \quad H^1(M) \otimes H^1(\mathcal{G}) \oplus H^1(\mathcal{G}) = H^2(Q)/H^2(\mathcal{G})\,. \tag{3.12}$$

These relations define the canonical projection

$$p_2\colon \quad H^2(Q) \longrightarrow H^1(M) \otimes H^1(\mathcal{G}) \oplus H^1(\mathcal{G})\,, \tag{3.13}$$

while on the kernel of p_2 we have the isomorphism

$$\iota_2\colon \quad \text{ker}p_2 \longrightarrow H^2(\mathcal{G})\,. \tag{3.14}$$

We consider now (3.13) and (3.14) component-wise.

Let $\mathbf{f} = [\mathcal{F}, \lambda, f] \in H^2(Q)$ be a cohomology class of the cocycle $(\mathcal{F}, \lambda, f)$: $Q(\mathcal{F}, \lambda, f) = 0$. $d_{E.L}\lambda = -\delta\mathcal{F}, \delta\lambda = df, \delta f = 0$, $\mathcal{F} \in E^{0,2}, \lambda \in E^{1,1}, f \in E^{0,2}$. The space $E^{0,2}$ contains coboundaries only, so cocycle $(\mathcal{F}, \lambda, f)$ is cohomological to $(0, \lambda', f)$ where $\lambda' = \lambda + \delta L$ and L: $\mathcal{F} = d_{E.L} L$. Since $d_{E.L}\lambda' = 0$, from Proposition 1 it follows that 1-cochain λ' takes values in closed differential 1-forms + constants:

$$\forall h \in \mathcal{G} \quad \lambda'(h) = w(h) + t(h)\,. \tag{3.15}$$

Using (3.7, 3.8) we see that to λ' there corresponds an element of $H^1(M) \otimes H^1(\mathcal{G}) \oplus H^1(\mathcal{G})$ which is nothing but $p_2(\mathbf{f})$.

In the case $p_2(\mathrm{f}) = 0$ it means that $\lambda' = d\alpha$, where $\alpha \in E^{1.0}$ and the cocycle $(0, \lambda', f)$ is cohomological to $(0, 0, f - \delta\alpha)$. Since $d(f - \delta\alpha) = 0$ so $f - \delta\alpha$, is cocycle in $Z^2(\mathcal{G})$. The cohomology class of $f - \delta\alpha$ in $H^2(\mathcal{G})$ is $\iota_2(\mathrm{f})$.

Similar considerations for the second "antidiagonal" in the table $\overline{E_2^{*.*}}$ lead to the analogous conclusions for $H^2(\overline{Q})$: $H^2(\mathcal{G}) \subseteq H^2(\overline{Q})$ and $H^1(M) \otimes H^1(\mathcal{G}) = H^2(\overline{Q})/H^2(\mathcal{G})$; $\overline{p_2}$: $H^2(\overline{Q}) \longrightarrow H^1(M) \otimes H^1(\mathcal{G})$. An isomorphism $\overline{\iota_2}$: $\ker\overline{p_2} \longrightarrow H^2(\mathcal{G})$ is defined on the kernel of $\overline{p_2}$.

From the considerations above we see that natural relations between complexes $(E^{*.*}, d_{E.L}, \delta)$, $(\overline{E^{*.*}}, \overline{d}_{E.L}, \overline{\delta})$ lead to the isomorphisms

$$H^1(Q) = H^1(\overline{Q}) \oplus \mathbf{R}, \quad H^2(Q) = H^2(\overline{Q}) \oplus H^1(\mathcal{G}). \tag{3.16}$$

The decomposition of $H^2(Q)$ defines the projection

$$\sigma\colon H^2(Q) \to H^1(\mathcal{G}). \tag{3.17}$$

Here $\sigma(\mathrm{f})$ is equal to the element of $H^1(\mathcal{G})$ in the r.h.s. of eq. (3.15). This projection will be useful for extracting Lagrangians whose Noether charges are time-independent in the space $\mathcal{V}_{0.0}$ of weakly invariant Lagrangians.

The spaces $\{{}^tE_r^{i.j}\}$ and $\{\overline{{}^tE_r^{i.j}}\}$

Now we return again to the complex (3.1) and express the cohomologies $H(Q)$ and $H(\overline{Q})$ in terms of the transposed spectral sequences $\{{}^tE_r^{*.*}\}$ and $\{\overline{{}^tE_r^{*.*}}\}$.

For constructing ${}^tE_1^{*.*}$ and $\overline{{}^tE_1^{*.*}}$ we have to consider the cohomology of vertical differential δ as zeroth order approximation: ${}^tE_1^{*.*} = H(E^{*.*}, \delta)$ and $\overline{{}^tE_1^{*.*}} = H(\overline{E^{*.*}}, \delta)$. We arrive at the tables

$$
{}^tE_1^{*.*} = \begin{array}{ccc}
\Lambda^0_{inv} & \Lambda^1_{1inv} & d_{E.L}\mathcal{V}_{0.0} \\
H^1(\mathcal{G}, \Lambda^0(M)) & H^1(\mathcal{G}, \Lambda^1_1) & \cdots \\
H^2(\mathcal{G}, \Lambda^0(M)) & \cdots & \cdots
\end{array}
$$

and

$$
\overline{{}^tE_1^{*.*}} = \begin{array}{ccc}
\Lambda^0_{inv} & \overline{\Lambda^1_{1inv}} & \overline{d}_{E.L}\overline{\mathcal{V}}_{0.0} \\
H^1(\mathcal{G}, \Lambda^0(M)) & H^1(\mathcal{G}, \overline{\Lambda^1_1}) & \cdots \\
H^2(\mathcal{G}, \Lambda^0(M)) & \cdots & \cdots \\
\cdots & \cdots & \cdots
\end{array} \tag{3.18}
$$

Here $\Lambda^0_{inv} = C^0(\mathcal{G}, \Lambda^0(M))$ is the space of the functions on M invariant under the action of the Lie algebra $\mathcal{G}$ and Λ^1_{1inv} is the space of $\mathcal{G}$-invariant Lagrangians from Λ^1_1. The space $\overline{\Lambda^1_{1inv}}$ contains the classes (Lagrangians quotiened by constants) whose variation under symmetry transformations lying in $\mathcal{G}$ produces $\mathcal{G}$-cochains with values in constants: $\overline{\Lambda} \in \overline{\Lambda^1_{1inv}} \Leftrightarrow \overline{\delta\Lambda} = 0 \Leftrightarrow \delta_i\Lambda = t_i$. These Lagrangians have linear time dependent Noether charges (see (1.13)). The space $d_{E.L}\mathcal{V}_{0.0}$ is the image of the subspace $\mathcal{V}_{0.0}$ of weakly $\mathcal{G}$-invariant Lagrangians under the differential $d_{E.L}$ (see (3.3)). From (2.13) it also follows that $\overline{{}^tE^{0.2}} = \overline{d_{E.L}\mathcal{V}_{0.0}}$.

The differential ${}^td_1$ generated by $d_{E.L}$ acts in rows of the table ${}^tE_1^{*.*}$ (compare with the table (3.4)). For ${}^tE_2^{*.*} = H({}^tE_1^{*.*}, {}^td_1)$ we obtain

$$
{}^tE_2^{*.*} = \begin{array}{ccc}
\mathbf{R} & H^1_{inv}(M) \oplus \mathbf{R} & d_{E.L}\mathcal{V}_{0.0}/(d_{E.L}\Lambda^1_{1inv}) \\
{}^tE_2^{1.0} & \cdots & \cdots \\
\cdots & \cdots & \cdots
\end{array} \tag{3.19}
$$

Here $H^1_{inv}(M)$ is the space of closed $\mathcal{G}$-invariant differential 1-forms quotiened by the differentials of $\mathcal{G}$-invariant functions.

One can consider a similar table for $\overline{{}^tE_2^{*.*}}$.

The space ${}^tE_2^{1.0}$ in (3.19) is the subspace of $H^1(\mathcal{G}, \Lambda^0(M))$. It consists of the classes $[\alpha] \in H^1(\mathcal{G}, \Lambda^0(M))$ such that the equation $d\alpha = \delta L$ is solvable. (Compare with (1.2)). We see that the table (3.19) is not stable in the terms which we are interested in, because the differential ${}^td_2$ acting from ${}^tE_2^{1.0}$ in ${}^tE_2^{0.2}$ is not trivial: ${}^td_2[\alpha] = {}^t[d_{E.L}L]_2$. The next table ${}^tE_3^{*.*} = H({}^tE_2^{*.*}, {}^td_2)$ is stable in the terms we are interested in:

$$
{}^tE_3^{*.*} = \begin{array}{ccc}
\mathbf{R} & H^1_{inv}(M) \oplus \mathbf{R} & \dfrac{d_{E.L}\mathcal{V}_{0.0}/(d_{E.L}\Lambda^1_{1inv})}{\Im({}^td_2\,{}^tE_2^{1.0})} \\
{}^tE_3^{0.1} & \cdots & \cdots \\
\cdots & \cdots & \cdots
\end{array} \tag{3.20}
$$

From the general properties of spectral sequences it follows that in (3.20) ${}^tE_3^{0.2} = {}^tE_\infty^{0.2}$ *is the subspace in* $H^2(Q)$ and the space ${}^tE_3^{1.0} = {}^tE_\infty^{1.0}$ (which is a subspace of ${}^tE_2^{1.0}$) is the quotient space of $H^1(Q)$ by the space ${}^tE_3^{0.1} = H_{inv}(M) \oplus \mathbf{R}$ (compare with (3.10)). Hence from the decomposition (3.11) of $H^1(Q)$ it follows that

$${}^tE_3^{1.0} = (H^1(M) \oplus H^1(\mathcal{G}))/H^1_{inv}(M). \tag{3.21}$$

$H^1_{inv}(M)$ is embedded in $H^1(M) \oplus H^1(\mathcal{G})$ via a monomorphism: $[w]_{inv} \mapsto ([w], -\pi w)$, where π is defined by (3.5). If $[w]_{inv} \neq 0$ in $H^1_{inv}(M)$ and $[w] = 0$ in $H^1(M)$, then $\pi w = \delta f \neq 0$ in $H^1(\mathcal{G})$, where $w = df$.

On the other hand the element $[\alpha]$ in $H^1(\mathcal{G}, \Lambda^0(M))$ belongs to the subspace ${}^tE_3^{1.0}$ if $d\alpha = \delta L$ such that $dL = 0$. Hence from Proposition 1 and (3.6) it follows that for this element there exist such $[w] \in H^1(M)$ and $t \in H^1(\mathcal{G})$ that

$$[a] = [\pi w + t]. \tag{3.22}$$

The homomorphism $([w], t) \mapsto [\pi w + t] \in H^1(\mathcal{G}, \Lambda^0(M))$ relates (3.21) with (3.22). One can see also that in the case $t = 0$ this map induces an isomorphism

$$H^1(M)/H^1_{inv}(M)_* = {}^tE_3^{1.0}/H^1(\mathcal{G})_*. \tag{3.23}$$

Here $H^1_{inv}(M)_*$ and $H^1(\mathcal{G})_*$ are the images of natural homomorphisms $H^1_{inv}(M) \to H^1(M)$ and $H^1(\mathcal{G}) \to {}^tE_3^{1.0}$ respectively.

For the table $\overline{{}^tE_3^{*.*}}$, one has $\overline{{}^tE_3^{0.1}} = H^1(M)_{inv}$, the spaces ${}^tE_1^{1.0}$ and $\overline{{}^tE_1^{1.0}}$, as well as the spaces ${}^tE_3^{1.0}$ and $\overline{{}^tE_3^{1.0}}$ coincide, but on the other hand ${}^tE_2^{1.0} \subseteq \overline{{}^tE_2^{1.0}}$.

In the tables (3.18)–(3.20), every space ${}^{t}E_r^{1.0}$ is a subspace of the previous one and respectively every space ${}^{t}E_r^{0.2}$ is the quotient space of previous one. We denote by Π_r the homomorphism which puts into correspondence to every weakly $\mathcal{G}$-invariant Lagrangian its equivalence class in the space ${}^{t}E_r^{0.2}$:

$$\Pi_r\colon\ \mathcal{V}_{0.0}\to {}^{t}E_r^{0.2},\quad \Pi_r(L)={}^{t}[d_{E.L}L]_r,\quad \Im\Pi_3\subseteq H^2(Q). \tag{3.24}$$

Similarly $\overline{\Pi}_r\colon\ \overline{\mathcal{V}}_{0.0}\to\overline{{}^{t}E_r^{0.2}}$.

Comparing the content of the spaces $\{{}^{t}E_r^{1.0}\}$ and $\{{}^{t}E_r^{0.2}\}$ in the transposed spectral sequences (3.18)–(3.20) with the above results for the spectral sequence $\{E_r^{*,*}\}$, we come to

Proposition 2

a) To weakly $\mathcal{G}$-invariant Lagrangians there correspond elements of the space ${}^{t}E_3^{0.2}$, i.e., of $H^2(Q)$. Thus to these Lagrangians there correspond elements in $E_2^{1.1}$ or in $E_2^{2.0}$ via homomorphisms p_2 and ι_2 defined by (3.13), (3.14).

b) To weakly $\mathcal{G}$-invariant Lagrangians whose images in the space ${}^{t}E_3^{0.2}$ vanish, $\Pi_3(L)=0$, there correspond elements of ${}^{t}E_2^{0.2}$ which belong to the image of the differential ${}^{t}d_2$. Thus to these Lagrangians there correspond elements in ${}^{t}E_2^{1.0}$ defined up to the space ${}^{t}E_3^{1.0}$ (see 3.21), which is the kernel of this differential.

c) The space ${}^{t}E_3^{1.0}$ is related to weakly $\mathcal{G}$-invariant Lagrangians whose image in the space ${}^{t}E_2^{0.2}$ is equal zero: $\Pi_2(L)=0$.

A similar statement is valid for the spaces $\{\overline{{}^{t}E_r^{*,*}}\}$. In the next section, using this Proposition, we establish a hierarchy in the space of weakly $\mathcal{G}$-invariant Lagrangians.

4 The calculation of the hierarchy

Now using the calculations of the previous Section for a given pair $[\mathcal{G},M]$, we establish a hierarchy in the space of weakly $\mathcal{G}$-invariant Lagrangians of classical mechanics.

Let $\mathcal{U}$ be an arbitrary subspace in the space $\Lambda_1^1(M)$ of the classical mechanics Lagrangians on M. Let $\mathcal{U}_{0.0}$ be the subspace of weakly $\mathcal{G}$-invariant Lagrangians in $\mathcal{U}$: $\mathcal{U}_{0.0}=\mathcal{V}_{0.0}\cap\mathcal{U}$, where $\mathcal{V}_{0.0}$ is the subspace (3.3) of all weakly $\mathcal{G}$-invariant Lagrangians in $\Lambda_1^1(M)$. From Proposition 1 and (3.3) it follows that for an arbitrary L in $\mathcal{U}$ the condition $L\in\mathcal{U}_{0.0}$ is equivalent to the condition that the cochain δL takes values in closed differential forms + constants:

$$\delta d_{E.L}L=0\Leftrightarrow\delta_iL=w_{i\mu}\dot q^\mu+t_i \text{ and } dw_i=dt_i=0\,. \tag{4.1}$$

(Compare with (1.11)). Here δ_iL is the value of the cochain δL on a basis vector e_i of the Lie algebra $\mathcal{G}$. (As always, we identify differential forms with Lagrangians corresponding to them by (2.4).)

Using the homomorphism Π_3 defined by (3.24) and the projection homomorphism (3.17) σ_2 of $H^2(Q)$ on $H^1(\mathcal{G})$, we consider the homomorphism

$$\Psi=\sigma\circ\Pi_3\colon\ \mathcal{U}_{0.0}\to H^2(Q)\to H^1(\mathcal{G})$$

and we denote

$$K_0=H^1(\mathcal{G})\,. \tag{4.2}$$

Component-wise according to (3.15), $\Psi_i(L)=t_i$, where t_i is defined by (4.1). We denote by $\mathcal{U}_{0.1}$ the kernel of this homomorphism. In the case $\mathcal{U}=\Lambda_1^1(M)$ it is just the space $\mathcal{V}_{0.1}$ in (1.14) defined by the condition (1.12).

Now using Proposition 2 we define recurrently the homomorphisms $\{\phi_s\}$ and respectively $\{\overline{\phi}_s\}$ on the subspaces of $\mathcal{U}_{0.1}$ and on the subspaces of $\mathcal{U}_{0.0}$, such that every homomorphism is defined on the kernel of the previous one. Moreover, the domains for these homomorphisms will be related via the homomorphism Ψ.

Using the statement a) of Proposition 2, we consider the compositions

$$\phi_1=p_2\circ\Pi_3\colon\ \mathcal{U}_{0.0}\to {}^{t}E_3^{0.2}\subseteq H^2(Q)\to E_2^{1.1}=H^1(M)\otimes H^1(\mathcal{G})\oplus H^1(\mathcal{G})\,,$$
$$\overline{\phi}_1=\overline{p}_2\circ\overline{\Pi}_3\colon\ \mathcal{U}_{0.0}\to\overline{{}^{t}E_3^{0.2}}\subseteq H^2(\overline{Q})\to\overline{E_2^{1.1}}=H^1(M)\otimes H^1(\mathcal{G})$$

and we denote

$$K_1=H^1(\mathcal{G})\otimes H^1(M)\,. \tag{4.3}$$

From (3.15), (3.16) it follows that the restriction of $\overline{\phi}_1$ on the subspace $\mathcal{U}_{0.1}$ coincides with ϕ_1. We denote by $\mathcal{U}_{1.0}$ the kernel of the homomorphism $\overline{\phi}_1$ and by $\mathcal{U}_{1.1}$ the kernel of the homomorphism ϕ_1. The space $\mathcal{U}_{1.1}$ is also the kernel of homomorphism Ψ restricted on $\mathcal{U}_{1.0}$. Using again the statement a) of Proposition 2 (see also (3.14)) we consider the compositions

$$\phi_2=\iota_2\circ\Pi_3\colon\ \mathcal{U}_{1.1}\to {}^{t}E_3^{0.2}\subseteq H^2(Q)\to E_2^{0.2}=H^2(\mathcal{G})\,,$$
$$\overline{\phi}_2=\overline{\iota}_2\circ\overline{\Pi}_3\colon\ \mathcal{U}_{1.0}\to\overline{{}^{t}E_3^{0.2}}\subseteq H^2(\overline{Q})\to\overline{E_2^{0.2}}=H^2(\mathcal{G})$$

and we denote

$$K_2=H^2(\mathcal{G})\,. \tag{4.4}$$

The homomorphism $\overline{\phi}_2$ evidently coincides with ϕ_2 on $\mathcal{U}_{1.1}$.

For example, if the condition (4.1) is satisfied, for a Lagrangian L in $\mathcal{U}$, i.e., $L\in\mathcal{U}_{0.0}$, then $\overline{\phi}_1(L)$ is equal to the cohomology class of $w_{i\mu}dq^\mu$ in $H^1(\mathcal{G})\otimes H^1(M)$ defined by (3.7); $L\in\mathcal{U}_{1.0}$ iff $\{w_{i\mu}dq^\mu\}$ are exact forms. In this case $\overline{\phi}_2(L)$ is equal to the cohomology class of the cocycle $f_{ij}=(\delta\alpha)_{ij}$ in $H^2(\mathcal{G})$, where $d\alpha_i=w_i$. If also $t_i=0$, then $L\in\mathcal{U}_{1.1}$.

We denote by $\mathcal{U}_{2.0}$ the kernel of the homomorphism $\overline{\phi}_2$ and by $\mathcal{U}_{2.1}$ the kernel of the homomorphism ϕ_2. It is easy to see that $\mathcal{U}_{2.1}=\ker\Psi|_{\mathcal{U}_{2.0}}$.

For every Lagrangian $L\in\mathcal{U}_{2.1}$ one has $\Pi_3(L)=0$. From statement b) of Proposition 2 it follows that one can consider the homomorphism

$$\phi_3=({}^{t}d_2)^{-1}\circ\Pi_2\colon\ \mathcal{U}_{2.1}\to {}^{t}E_2^{0.2}\to {}^{t}E_2^{1.0}/{}^{t}E_3^{1.0}\subseteq {}^{t}E_1^{1.0}/{}^{t}E_3^{1.0}\,.$$

Using (3.18) and (3.21) we denote

$$K_3={}^{t}E_1^{1.0}/{}^{t}E_3^{1.0}=\frac{H^1(\mathcal{G},\Lambda^0(M))}{(H^1(M)\oplus H^1(\mathcal{G}))/H^1_{inv}(M)}\,. \tag{4.5}$$

By similar considerations for the space $\mathcal{U}_{2.0}$, we can define the homomorphism $\overline{\phi}_3 = (\overline{{}^t d_2})^{-1} \circ \overline{\Pi}_2$ of the space $\mathcal{U}_{2.0}$ into the same space K_3.

One can see that in this case, as in the previous ones, $\overline{\phi}_3|_{\mathcal{U}_{2.1}} = \phi_3$ and $\mathcal{U}_{3.1} = \ker\Psi|_{\mathcal{U}_{3.0}}$, where by $\mathcal{U}_{3.1}$, $\mathcal{U}_{3.0}$ we denote the kernels of ϕ_3 and $\overline{\phi}_3$ respectively.

For example, in the case when L in (4.1) belongs to $\mathcal{U}_{2.1}$, one can choose α_i such that $d\alpha_i = w_{i\mu} dq^\mu$ and $(\delta\alpha)_{ij} = 0$, because $\phi_2(L) = 0$. The equivalence class of α_i in K_3 is $\phi_3(L)$.

In the case when $L \in \mathcal{U}_{3.1}$, one has $\Pi_2(L) = 0$. It means that the value of the homomorphism Π_1 (see (3.24)) at this Lagrangian is equal to the value of this homomorphism at some $\mathcal{G}$-invariant Lagrangian: $\Pi_1(L) = d_{E.L} L = d_{E.L} L_{inv}$. From Proposition 1 it follows that $L = L_{inv} + w$, where the closed differential 1-form w is defined uniquely up to a closed $\mathcal{G}$-invariant form and an exact form. This defines the homomorphism

$$\phi_4(L)\colon \mathcal{U}_{3.1} \to H^1(M)/(H^1_{inv(M)})_*,$$

which can be considered as taking values in the space ${}^tE_3^{1.0}/H^1(\mathcal{G})_*$, according to eq.(3.23). We denote

$$K_4 = H^1(M)/(H^1_{inv(M)})_* = {}^tE_3^{1.0}/H^1(\mathcal{G})_*. \tag{4.6}$$

One can define the homomorphism $\overline{\phi}_4(L)\colon \mathcal{U}_{3.0} \to K_4$ in a similar way.

Similarly to the previous cases $\overline{\phi}_4|_{\mathcal{U}_{3.1}} = \phi_4$ and $\mathcal{U}_{4.1} = \ker\Psi|_{\mathcal{U}_{4.0}}$, where by $\mathcal{U}_{4.1}$, $\mathcal{U}_{4.0}$ we denote the kernels of ϕ_3 and $\overline{\phi}_3$ respectively.

From the definitions of ϕ_4 and $\overline{\phi}_4$ it is evident that Lagrangians belonging to $\mathcal{U}_{4.1}$ can be reduced to $\mathcal{G}$-invariant by the redefinition on exact form (total derivative).

The spaces $\mathcal{U}_{s.\sigma}$ constructed here ($s = 0,1,2,3,4, \sigma = 0,1$) coincide with the spaces $\mathcal{V}_{s.\sigma}$ considered in the Introduction (see (1.14), (1.15)) in the case $\mathcal{U} = \Lambda_1^1(M)$.

These considerations can be summarized in

THEOREM

Let $\mathcal{U}$ be an arbitrary subspace in the space of classical mechanics Lagrangians for a given pair $[\mathcal{G}, M]$. Let $\mathcal{U}_{0.0}$ be the subspace of $\mathcal{U}$ defined by (4.1) which contains weakly $\mathcal{G}$-invariant Lagrangians in $\mathcal{U}$. Then the following relations establishing the classification (hierarchy) in the space $\mathcal{U}_{0.0}$ are satisfied

$$\begin{array}{ccccccc}
 & & \mathcal{U}_{4.1} & \subseteq & \mathcal{U}_{4.0} & & \\
 & & \cap\!\!| & & \cap\!\!| & & \\
K_4 & \xleftarrow{\phi_4} & \mathcal{U}_{3.1} & \subseteq & \mathcal{U}_{3.0} & \xrightarrow{\overline{\phi}_4} & K_4 \\
 & & \cap\!\!| & & \cap\!\!| & & \\
K_3 & \xleftarrow{\phi_3} & \mathcal{U}_{2.1} & \subseteq & \mathcal{U}_{2.0} & \xrightarrow{\overline{\phi}_3} & K_3 \\
 & & \cap\!\!| & & \cap\!\!| & & \\
K_2 & \xleftarrow{\phi_2} & \mathcal{U}_{1.1} & \subseteq & \mathcal{U}_{1.0} & \xrightarrow{\overline{\phi}_2} & K_2 \\
 & & \cap\!\!| & & \cap\!\!| & & \\
K_1 & \xleftarrow{\phi_1} & \mathcal{U}_{0.1} & \subseteq & \mathcal{U}_{0.0} & \xrightarrow{\overline{\phi}_1} & K_1 \\
 & & & & \Psi\downarrow & & \\
 & & & & K_0 & &
\end{array} \tag{4.7}$$

The spaces $\mathcal{U}_{s.\sigma}$ are intersections of $\mathcal{U}$ with the spaces $\mathcal{V}_{s.\sigma}$ defined in Introduction (see (1.6)–(1.15)); the spaces K_s and homomorphisms $\Psi, \phi_s, \overline{\phi}_s$ ($s = 0,1,2,3,4$,) are defined by the eqs. (4.2)—(4.6), the double filtration $\{\mathcal{U}_{s.\sigma}\}$ is subordinated to these homomorphisms:

$$\mathcal{U}_{s.0} = \ker(\overline{\phi}_s\colon \mathcal{U}_{s-1.0} \to K_s), \quad \mathcal{U}_{s.1} = \ker(\phi_s\colon \mathcal{U}_{s-1.1} \to K_s),$$
$$\mathcal{U}_{s.1} = \ker(\Psi\colon \mathcal{U}_{s.0} \to K_0), \quad \overline{\phi}_s|_{\mathcal{U}_{s-1.1}} = \phi_s.$$

We denote the diagram (4.7) by $\mathcal{D}([\mathcal{G}, M], \mathcal{U})$ and call it the hierarchy diagram for the subspace $\mathcal{U}$. In the case when $\mathcal{U} = \Lambda_1^1(M)$ is the space of all Lagrangians of classical mechanics on M, we denote the diagram $\mathcal{D}([\mathcal{G}, M], \mathcal{U})$ shortly by $\mathcal{D}([\mathcal{G}, M])$.

The diagram $\mathcal{D}([\mathcal{G}, M], \mathcal{U})$ measures the differences in the spaces $\{\mathcal{U}_{s.\sigma}\}$ for an arbitrary subspace $\mathcal{U}$.

We say that a weakly $\mathcal{G}$-invariant Lagrangian $L \in \mathcal{U}$ is on the floor s if $L \in \mathcal{U}_{s.0}$ and $L \notin \mathcal{U}_{s+1.0}$. All Lagrangians from $\mathcal{U}_{4.0}$ are on the fourth floor.

We say that a weakly $\mathcal{G}$-invariant Lagrangian L is on the floor s_+ if this Lagrangian is on the floor s and it belongs to $\mathcal{U}_{s.1}$. All other Lagrangians from the floor s are on the floor s_-.

All Lagrangians which are on the "plused" floors have time-independent Noether charges, except Lagrangians on the zeroth floor.

The Lagrangians which are on the floor s have non-trivial images in the space K_{s+1} in (4.7). A Lagrangian on the floor s_- have also non-trivial image in K_0 under the homomorphism Ψ.

The hierarchy diagram will be called trivial, if all the spaces K_s vanish.

Returning to the table (1.6) in the Introduction, we can conclude that a Lagrangian which possesses the property s in (1.6) and which does not possesses the property $s+1$ in (1.6) has non-trivial image in the space K_{s+1}.

An evident but important corollary of the hierarchy diagram is that a floor is empty, if the corresponding space K_s is trivial. For example, in the case when the first de Rham cohomology of the configuration space is trivial then $K_1 = K_4 = 0$, and the zeroth and the third floors are empty. In the case when the algebra $\mathcal{G}$ is semisimple, only the floors $2_+, 3_+, 4_+$ can be nonempty, because in this case $H^1(\mathcal{G}) = H^2(\mathcal{G}) = 0$ and hence $K_0 = K_1 = K_2 = 0$.

In general, the inverse statement is not valid. From the fact that the space K_s is not trivial does not follow that the floor $s-1$ is not empty, because the homomorphisms in (4.7) are not surjective in general. For example, homomorphism ϕ_3 is not surjective in general because the map ${}^td_2$ which induces this homomorphism is defined on the subspace ${}^tE_2^{0.1}$ of the space $H^1(\mathcal{G}, \Lambda^0(M))$.

We say that the diagram $\mathcal{D}([\mathcal{G}, M], \mathcal{U})$ is *full* on the floor s_+ ($s < 4$) if ϕ_{s+1} is an epimorphism onto the space K_s, we say that this diagram is full on the floor s_-, if the restriction of Ψ on $\mathcal{U}_{s.0}$ is an epimorphism. In the case if the diagram is full on the floors s_+ and s_- we say that it is full on the floor s.

For a given pair $[\mathcal{G}, M]$, two subspaces $\mathcal{U}$ and $\mathcal{U}'$ in the space $\Lambda_1^1(M)$ of classical mechanics Lagrangians on M will be called equivalent with respect to the hierarchy, if the images of all the homomorphisms $\phi_s, \overline{\phi}_s$, $\Psi|_{\mathcal{U}_{s.0}}$ for the diagram $\mathcal{D}([\mathcal{G},M],\mathcal{U}')$ coincide with the images of the corresponding homomorphisms for the diagram $\mathcal{D}([\mathcal{G},M],\mathcal{U})$. It is evident that in this case for arbitrary $L \in \mathcal{U}$ there exists $L' \in \mathcal{U}'$ such that $L' - L$ belongs to the space $\mathcal{U}_{4.1}$, i.e.,

$$L' = L + L_{inv} + \text{total derivative}. \qquad (4.8)$$

This construction can be used for defining in the space $\mathcal{U}_{0.0}$ a grading corresponding to the filtration (4.7) (see the examples in the next section).

Now we use it for simplifying the diagram (4.7) for physically important subspace $\mathcal{U}^{pol}$ of Lagrangians which are polynomial in velocities. Let $\mathcal{U}^f = \Omega^1(M)$ be the subspace of formal Lagrangians in $\mathcal{U}^{pol}$ corresponding to differential forms by (2.4), and $\mathcal{U}^{sc} = \Lambda^0(M)$ be the subspace of formal Lagrangians in $\mathcal{U}^{pol}$ which are functions on M.

One can see that the space $\mathcal{U}^{pol}$ is equivalent to the space $\mathcal{U}^f \oplus \mathcal{U}^{sc}$ with respect to the hierarchy.

To prove it, we note that every L in $\mathcal{U}^{pol}$ can be represented as

$$L(q,\dot q) = \sum_{n\geq 0} L_n(q,\dot q) = \sum_{n\geq 2} L_n(q,\dot q) + A_\mu(q)\dot q^\mu + \varphi(q), \qquad (4.9)$$

where $L_n(q,\dot q)$ is the polynomial on $\dot q$ of order n. Using the fact that the Lie derivative does not change the order of a polynomial: $(\delta L)_n = \delta(L_n)$ one can see from (4.9) and the definitions of the homomorphisms $\Psi, \phi_s, \overline{\phi}_s$ that $\Psi(L) = \Psi(\varphi), \phi_s(L) = \overline{\phi}_s(L) = \phi_s(A_\mu \dot q^\mu)$. This proves the equivalence.

The homomorphism Ψ in this case takes values in the subspace of $H^1(\mathcal{G})$ isomorphic to the cohomologies of $H^1_{inv(M)}$ which are trivial in $H^1(M)$: If $\delta\varphi \in H^1(\mathcal{G})$, then $d\varphi \in H^1_{inv}(M)$, if $w \in H^1_{inv}(M)$ and $w = d\varphi$, then $\delta\varphi \in H^1(\mathcal{G})$.

From these facts it follows that for the diagram $\mathcal{D}([\mathcal{G},M],\mathcal{U}^{pol})$ the following additional relations are satisfied:

$$\mathcal{U}^{pol}_{s.0} = \mathcal{U}^{pol}_{s.1} \oplus B, \quad \mathcal{U}^{pol}_{s.1} = \mathcal{U}_{4.1} \oplus A_s. \qquad (4.10)$$

Here the quotient space B is equal to $\mathcal{U}^{sc}_{0.0}/\mathcal{U}^{sc}_{0.1}$, where $\mathcal{U}^{sc}_{0.0}$ is the space of functions in $\Lambda^0(M)$ whose $\mathcal{G}$-symmetry variation is constant and $\mathcal{U}^{sc}_{0.1} = \Lambda^0_{inv}(M)$ is the space of $\mathcal{G}$-invariant functions. Respectively $A_s = \mathcal{U}^f_{s.1}$ are the corresponding subspaces of the space $\Omega^1(M)$ of differential 1-forms.

Weakly $\mathcal{G}$-invariant Lagrangians which belong to the space $\mathcal{U}^{pol}$ differ from the Lagrangians in $\mathcal{U}^{pol}_{4.1}$ ($\mathcal{G}$-invariant Lagrangians up to a total derivative) by the interaction with "electromagnetic" field whose field strength is $\mathcal{G}$-invariant. In particular a Lagrangian on the floor s_- differs from a Lagrangian on the floor s_+ by the interaction with "electrical field–1-form $E_\mu = \partial\varphi/\partial q^\mu$. The value of this 1-form on every symmetry vector field is constant: $E_\mu(q)e^\mu_i(q) = t_i$, where $\{e^\mu_i(q)\}$ are fundamental vector fields corresponding to the basis $\{e_i\}$ in the Lie algebra $\mathcal{G}$ via the map (2.1). The time dependence of the corresponding Noether charge is proportional to t_i.

In general, for an arbitrary Lagrangian in $\mathcal{V}$, these properties are not satisfied (see, e.g., Example 1 in Section 5).

The second physically important example is the subspace $\mathcal{U}^{dens}$ of Lagrangians on M which are *densities* (see the Remark in Section 2).

It is easy to see that $\mathcal{U}_{s.1} = \mathcal{U}_{s.0}$ in this case, i.e., all the floors s_- are empty, because the homomorphism Ψ is trivial. It follows from the definition of the homomorphism Ψ and the considerations in the end of Section 2.

We do not consider here systematically general methods to handle with calculations of the spaces K_s and corresponding homomorphisms for an arbitrary pair $[\mathcal{G},M]$, but we note only some points which can be useful for analyzing the content of the space K_3 in the hierarchy diagram and the groups $H^1(\mathcal{G},\Lambda^0(M))$ which generate these spaces.

First we note that the basic example of $[\mathcal{G},M]$ pair is provided by the following construction. Let $M \subseteq N$ be a subspace of the space N and the action of a Lie group G be defined on N. The action of G on N determines a pair $[\mathcal{G},N]$ as well as a pair $[\mathcal{G},M]$, where $\mathcal{G} = \mathcal{G}(G)$ is the Lie algebra of the group G. This pair in general cannot be generated by a group action on M.

We say that a pair $[\mathcal{G},M]$ is transitive, if fundamental vector fields span the tangent bundle TM: $\forall q \in M \quad \Im\Phi|_q = T_qM$ (Φ defines the action of $\mathcal{G}$ on M by (2.1)). For example this is the case, if a Lie algebra action on M is generated by a transitive action of Lie group.

For a given $[\mathcal{G},M]$, we can consider the stability subalgebra $\mathcal{G}_{st}(q)$ for every point $q \in M$: $\mathcal{G}_{st}(q) = \{\mathcal{G} \ni x\colon \Phi(x)|_q = 0\}$. In the case the pair $[\mathcal{G},M]$ is generated by the action of a group G, $\mathcal{G}_{st}(q)$ is isomorphic to the Lie algebra of stability subgroup for any point q_0.

Let $[\mathcal{G},M]$ be a transitive pair (the constructions below can be generalized on non-transitive case also).

If α is a cocycle representing a cohomology class in $H^1(\mathcal{G},\Lambda^0(M))$, then at an arbitrary point q_0 it vanishes on the vectors in the commutant $[\mathcal{G}_{st}(q_0),\mathcal{G}_{st}(q_0)]$. If this cocycle is generated by a 1-form w via the homomorphism π, defined by (3.5) ($\alpha = \pi w$), then it vanishes at an arbitrary point q_0 on all the vectors in $\mathcal{G}_{st}(q_0)$. Moreover, πw is a coboundary iff w is a coboundary. Thus for any point $q \in M$ one can consider the homomorphisms

$$H^1(M) \xrightarrow{[\pi]} H^1(\mathcal{G},\Lambda^0(M)) \xrightarrow{\rho_q} H^1(\mathcal{G}_{st}(q)), \qquad (4.11)$$

which obey to the following conditions: $[\pi]$ is the monomorphism and $\rho_q \circ [\pi] = 0$. If for every q, $\rho_q([\alpha]) = 0$ then $[\alpha] = \pi[w]$. For example $H^1(M) = H^1(\mathcal{G},\Lambda^0(M))$ if $H^1(\mathcal{G}_{st}(q)) \equiv 0$. In the case when the pair $[\mathcal{G},M]$ is generated by a transitive action of a Lie group G (on $N \supseteq M$), then the image of the monomorphism $[\pi]$ coincides with the kernel of ρ_q for an arbitrary point q, because the homomorphisms ρ_q, for different points q, are related by the adjoint action of the group transformation:

$$\forall (q,q_0), \forall \xi \in \mathcal{G}_{st}(q_0)\ \alpha(q,\mathbf{Ad}_g\xi) = \alpha(q_0,\xi) \text{ if } q = g\circ q_0. \qquad (4.12)$$

Hence in this case K_3 can be embedded into the quotient space of $H^1(\mathcal{G}_{st}(q))$ for any q:

$$K_3 \subseteq H^1(\mathcal{G}_{st}(q))/\Im\rho_q|_{H^1(\mathcal{G})}. \tag{4.13}$$

It gives an upper estimate for the dimension of the space K_3.

The formula (4.13) follows from the definition (4.5) of K_3, the explicit realization (3.22) of elements of ${}^tE_3^{1.0}$ and the properties of the homomorphism ρ_q mentioned above.

One can say more in the case when the pair $[\mathcal{G}, M]$ is generated by the transitive action of the compact connected Lie group on the same space M. In this case, taking the average of the group action on a cocycle one comes to the monomorphism of $H^1(\mathcal{G}, \Lambda^0(M))$ into $H^1(\mathcal{G})$:

$$H^1(\mathcal{G}, \Lambda^0(M)) \ni [a] \mapsto \frac{1}{\mathrm{Vol}(G)} \int \alpha^g d\mu_G \in H^1(\mathcal{G}), \tag{4.14}$$

where $d\mu_G$ is invariant measure on G.

For example, if the pair $[\mathcal{G}, M]$ is transitive and is generated by the action of semisimple compact connected Lie group on the space M, then all K_s vanish and the hierarchy diagram is trivial. Indeed $K_0 = K_1 = K_2 = 0$ since for semisimple algebra $H^1(\mathcal{G}) = H^2(\mathcal{G}) = 0$. From (4.14) and (4.11) it follows that $H^1(\mathcal{G}, \Lambda^0(M)) = H^1(M) = 0$, because $H^1(\mathcal{G}) = 0$. Hence $K_3 = K_4 = 0$ too.

We wish to note that from (4.11) it follows that even if G is a semisimple algebra in general $H^1(\mathcal{G}, \Lambda^0(M))$ is not trivial, inspite the first cohomology group with coefficients in an arbitrary finite-dimensional module over semisimple algebra is trivial (Whitehead lemma [16]).

The constructions above indicate that it is the interplaying of de Rham and symmetry algebra cohomologies which leads to the nonemptness of the second floor ($K_3 \neq 0$) of the hierarchy diagram in the case of semisimple symmetry algebra. (See Example 3.)

5 Examples

In this section, using the hierarchy diagram (4.7) and considerations below, we consider some examples of weakly $\mathcal{G}$-invariant Lagrangians classification.

Example 1

This example is a model one. But here we describe in details how to use the construction (4.8) for establishing the grading corresponding to the hierarchy filtration (4.7). We consider the following pair $[\mathcal{G}, M]$. Let $\mathcal{G}$ be the Lie algebra ℓ_3 with the generators e_1, e_2, e_3 such that $[e_1, e_2] = e_3, [e_2, e_3] = [e_3, e_1] = 0$. Let a configuration space M be the cylinder: $M = \mathbf{R} \times \mathbf{S}^1$ with the coordinates (z, φ). The homomorphism Φ (see (2.1)) is defined by the relations

$$\Phi e_1 = \tilde{e}_1 = \frac{\partial}{\partial z}, \quad \Phi e_2 = \tilde{e}_2 = z\frac{\partial}{\partial \varphi}, \quad \Phi e_3 = \tilde{e}_3 = \frac{\partial}{\partial \varphi}. \tag{5.1}$$

This defines the pair $[\ell_3, S^1 \times \mathbf{R}]$. For this pair we first calculate the hierarchy diagram $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}])$. We consider as $\mathcal{U}$ the whole space $\Lambda_1^1(M)$. From (5.1) it follows that every ℓ_3-invariant Lagrangian has the form $F(\dot{z})$ where F is an arbitrary function.

Now we calculate the spaces $\{K_s\}$. $K_0 = H^1(\mathcal{G}) = \mathbf{R}^2$ is generated by the cochains e^1 and e^2 ($\{e^i\}$ are dual to $\{e_j\}$: $e^i(e_j) = \delta^i_j$). Component-wisely the elements of $H^1(\mathcal{G})$ are of the form $t_i = (a, b, 0)$. The group $H^1(M) = \mathbf{R}$ is generated by the 1-form $d\varphi$. Hence, $K_1 = \mathbf{R}^2$ is generated by cochains $(d\varphi, 0, 0)$ and $(0, d\varphi, 0)$. Now $K_2 = H^2(\mathcal{G}) = \mathbf{R}^2$ because any cochain f_{ij} is a cocycle which is the coboundary iff $f_{23} = f_{31} = 0$. It is easy to see that $H^1_{inv}(M) = \mathbf{R}$ is generated by the 1-form dz. The stability subalgebra at every point (z, φ) is generated by the vector $e_2 - ze_3$, hence from (4.11)–(4.13) and the result for $H^1(\mathcal{G})$ it follows that $K_3 = 0$. Note that the explicit calculations without using (4.11) give that $H^1(\mathcal{G}, \Lambda^0(M)) = \mathbf{R}^2$ is generated by the cocycles $\alpha_i = (0, az + b, a)$; $d(0, az + b, a) = (0, adz, 0) = \delta_i ad\varphi$, hence ${}^tE_3^{1.0} = {}^tE_2^{1.0} = H^1(\mathcal{G}, \Lambda^0(M))$ and $K_3 = 0$.

The space $K_4 = \mathbf{R}$ is generated by the form $d\varphi$. We come to the following result

$$K_0 = K_1 = K_2 = \mathbf{R}^2, \quad K_3 = 0, \quad K_4 = \mathbf{R}. \tag{5.2}$$

We saw already that the second floor of $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}])$ is empty.

Special analysis of the homomorphism ϕ_2 leads to the fact that the 1-st floor is empty too: the image of ϕ_2 in K_2 is trivial because in this special case the subspaces ${}^tE_\infty^{0.2}$ and $E_\infty^{2.0}$ of $H^2(Q)$ have zero intersection.

Now we show that the diagram $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}])$ is full on the all floors except of the first one and study the contents of the spaces $\{\mathcal{V}_{s.1}, \mathcal{V}_{s.0}\}$.

For this purpose, we consider the following 5-dimensional subspace of the formal Lagrangians, on $S^1 \times \mathbf{R}$:

$$U = \{L: \quad L = a\dot{\varphi} + bz\dot{\varphi} + cz + d\frac{\dot{\varphi}}{\dot{z}} + \frac{q}{2}\frac{\dot{\varphi}^2}{\dot{z}}\}, \tag{5.3}$$

where (a, b, c, d, q) are constants.

We shall show that the diagram $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}], U)$ is full on all the floors except for the first one. From this fact and from the emptiness of first floor for the diagram $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}])$ it follows that the whole space $\mathcal{V}$ of classical mechanics Lagrangians on M is equivalent to its subspace U with respect to the hierarchy (see (4.8)).

Straightforward calculations show that for arbitrary Lagrangian from U one has

$$\begin{gathered} \delta_1 L = \mathcal{L}_{\frac{\partial}{\partial z}} L = bd\varphi + c, \\ \delta_2 L = \mathcal{L}_{z\frac{\partial}{\partial \varphi}} L = adz + bzdz + d + qd\varphi, \ \delta_3 L = \mathcal{L}_{\frac{\partial}{\partial \varphi}} L = 0. \end{gathered} \tag{5.4}$$

Comparing (5,4) with (4.1), we see that $U = U_{0.0}$.

Let us calculate the homomorphisms $\Psi, \phi_s, \overline{\phi}_s$ for the diagram $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}], U)$ using (5.2)–(5.4). One has $\phi_2 = \overline{\phi}_2 = \phi_3 = \overline{\phi}_3 = 0$. For any $L \in U$ we have $\Psi(L) = (c, d, 0) \in K_0$. If $c = d = 0$, then $L \in U_{0.1}$ and $\phi_1(L) = \overline{\phi}_1(L) = (bd\varphi, qd\varphi, 0) \in K_1$.

If $b = q = 0$, then $L \in U_{1.0}$ and if $c = d = b = q = 0$, then $L \in U_{1.1}$. Hence, $U_{3.0} = U_{2.0} = U_{1.0}$ and respectively $U_{3.1} = U_{2.1} = U_{1.0}$. One has $\phi_4(L) = \overline{\phi}_4(L) = ad\varphi \in K_4$. If $a = b = q = 0$ then we come to $U_{4.0}$. If also $c = d = 0$ then we come to $U_{4.1} = 0$.

All these homomorphisms, except for $\phi_2, \overline{\phi}_2$, are epimorphisms. Hence, the space $\Lambda_1^1(M)$ is reduced to its subspace U with respect to the hierarchy. Moreover these homomorphisms are isomorphisms on corresponding quotient spaces: $\Im\Psi = U_{0.0}/U_{1.0}$, $\Im\phi_s = U_{s-1.1}/U_{s.1}$ and $\Im\overline{\phi}_s = U_{s-1.0}/U_{s.0}$ if $s \neq 2$.

From these considerations and from (4.8) it follows that for every weakly ℓ_3-invariant Lagrangian there exists a unique Lagrangian in U such that their difference belongs to $\mathcal{V}_{4.1}$:

$$\begin{gathered} \forall L \in \mathcal{V}_{0.0}\, \exists!(a,b,c,d,q): \\ L = F(\dot z) + \text{total derivative} + a\dot\varphi + bz\dot\varphi + cz + d\tfrac{\dot\varphi}{\dot z} + \tfrac{q}{2}\tfrac{\dot\varphi^2}{\dot z}\,. \end{gathered} \tag{5.5}$$

Finally we come to the following grading in the space $\mathcal{V}_{0.0}$ of weakly ℓ_3-invariant Lagrangians on $S^1 \times \mathbf{R}$:

$$\begin{gathered} \mathcal{V}_{3.1} = \mathcal{V}_{2.1} = \mathcal{V}_{1.1} = \mathcal{V}_{4.1} \oplus K_4 = \mathcal{V}_{4.1} \oplus \mathbf{R}, \\ \mathcal{V}_{0.1} = \mathcal{V}_{1.1} \oplus K_1 = \mathcal{V}_{1.1} \oplus \mathbf{R}^2,\ \mathcal{V}_{s.0} = \mathcal{V}_{s.1} \oplus K_0 = \mathcal{V}_{s.1} \oplus \mathbf{R}^2\,. \end{gathered} \tag{5.6}$$

We also briefly consider the diagram $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}], \mathcal{U}^{pol})$, where $\mathcal{U}^{pol}$ is the subspace of Lagrangians polynomial in velocities (see the end of Section 4). It is easy to see that $\mathcal{U}^{pol}$ is reduced to the three-dimensional space U^{pol} which is a subspace of U defined by the additional conditions $d = q = 0$ in (5.3). The diagram $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}], \mathcal{U}^{pol})$ is not full on all the floors s_- and on the floors 0_+ and 1_+. In this case $\Im\Psi = \mathbf{R} \neq K_0$ and $\Im\phi_1 = \mathbf{R} \neq K_1$. The space $\mathcal{U}^{pol}_{0.0}$ is parametrized by the three-dimensional space U^{pol}, up to $\mathcal{U}^{pol}_{4.1}$: $U^{pol} = \mathcal{U}^{pol}_{0.0}/\mathcal{U}^{pol}_{4.1}$ similar to (5.5, 5.6) with conditions $d = q = 0$.

We note that in (5.5) the term $d(\dot\varphi/\dot z)$ responsible for time-dependent Noether charges cannot be considered as interaction with "electrical field" as in the case of Lagrangians in $\mathcal{U}^{pol}$.

We also want to note that all the considerations which lead to the formula (5.6) (except for the property of homomorphism ϕ_2) based on general relations established by the diagram (4.7).

Example 2

Let $M = \mathbf{R}^n$ be the n-dimensional linear space which acts on itself by translations. It determines the pair $[\mathbf{R}^n, \mathbf{R}^n]$ (we identify the affine space with the corresponding linear space and with abelian algebra of translations). It is easy to see that $K_0 = \mathbf{R}^n, K_2 = \mathbf{R}^n \wedge \mathbf{R}^n, K_1 = K_3 = K_4 = 0$. The space of Lagrangians on $\mathbf{R}^n$ is equivalent to the space $U = \{L\colon L = w_2(q,\dot q) + w_1(q)\}$ with respect to the hierarchy, where w_2, w_1 are 2-cocycle and 1-cocycle respectively on the Lie algebra $\mathbf{R}^n$. In the same way, like in (5.3)–(5.5) we come to the statement that every weakly $\mathcal{G}$-invariant Lagrangian in this case has the form

$$L = F(\dot q^1, \ldots, \dot q^n) + \text{total derivative} + B_{ik}q^i\dot q^k + E_i q^i\,.$$

It describes the interaction with constant "magnetic" and "electric" fields (compare with (1.5)). The corresponding Noether charges are $N_i(q,\dot q,t) = \partial F/\partial\dot q^i - 2B_{ik}q^k - E_i t$. The corresponding grading of the space $\mathcal{V}_{0.0}$ is the following:

$$\mathcal{V}_{4.1} = \mathcal{V}_{3.1} = \mathcal{V}_{2.1}, \mathcal{V}_{1.1} = \mathcal{V}_{0.1} = \mathcal{V}_{4.1} \oplus \mathbf{R}^{\frac{n(n-1)}{2}}, \mathcal{V}_{s.0} = \mathcal{V}_{s.1} \oplus \mathbf{R}^n\,.$$

This case is famous in literature as "arising of constant magnetic field as a central extension of translations algebra" [2].

Example 3. so(3) algebra.

In this example, we consider the Lie algebra $so(3)$ which is a special case of semisimple algebras. Let $M = \mathbf{R}^3$ be the 3-dimensional linear space with the Cartesian coordinates (x^1, x^2, x^3). We consider first the pairs $[so(3), \mathbf{R}^3]$ and $[so(3), S^2]$, where S^2 is the sphere $x^ix^i = 1$ in $\mathbf{R}^3$ and the action of $so(3)$ on $\mathbf{R}^3$ is generated by the standard action of the group $SO(3)$ on $\mathbf{R}^3$: if $\{e_1, e_2, e_3\}$ is a basis in $so(3)$ such that $[e_i, e_j] = \varepsilon_{ijk}e_k$, then $\Phi(e_i) = \tilde L_i = -\varepsilon_{ijk}x^j\partial/\partial x^k$. For the pair $[so(3), S^2]$ the hierarchy diagram is trivial because $SO(3)$ is a semisimple compact group (see the end of the Section 4).

Alternatively one can see it by the following explicit calculations: From the commutation relations it is evident that $H^1(so(3)) = H^2(so(3)) = 0$. Hence, $K_0 = K_1 = K_2 = K_4 = 0$. If α_i is a cocycle with values in functions on S^2, then $0 = \delta\alpha = \tilde L_i\alpha_k - \tilde L_k\alpha_i - \varepsilon_{ijk}\alpha_k$. Hence, $\tilde L^2\alpha_k = \tilde L_k(\tilde L_i\alpha_i) = \tilde L_k F$ and $\alpha_k = \delta\tilde F$ is a coboundary, where $\tilde F = \sum_l \frac{F^l}{l(l+1)}$. F^l is defined by the expansion over the spherical harmonics of F. The term F^0 vanishes because it leads to a cocycle in constants and $H^1(so(3)) = 0$. Hence $K_3 = 0$ as well.

The calculations and the result are the same for the pair $[so(3), \mathbf{R}^3]$. All weakly $so(3)$-invariant Lagrangians of classical mechanics on $\mathbf{R}^3$ and on S^2 are exhausted by $so(3)$-invariant ones (up to a total derivative).

Now bearing in mind the construction (4.11) we modify a little bit this example considering instead of the sphere S^2 the domain in it, the sphere without North pole (punctured sphere) $S^2\backslash N$ ($x^3 \neq 1$). Thus we come from the pair $[so(3), S^2]$ to the pair $[so(3), S^2\backslash N]$. In the same way, we come to the pair $[so(3), \mathbf{R}^3\backslash L_+]$, taking out the ray L_+ ($x^1 = 0, x^2 = 0, x^3 \geq 0$) from $\mathbf{R}^3$.

The essential difference of these pairs from the previous ones is that they cannot be generated by the action of the corresponding Lie group.

We perform the calculations for the diagram $\mathcal{D}([so(3), S^2\backslash N])$.

It is evident that for this diagram $K_0 = K_1 = K_2 = K_4 = 0$ as well. Now we show that for this diagram $K_3 = \mathbf{R}$ and that this hierarchy is full.

The stability algebra for this pair is one-dimensional, hence from (4.11)–(4.13) it follows that $K_3 = 0$ or $K_3 = \mathbf{R}$. It remains to prove that K_3 is not trivial.

To show it, we consider the Lagrangian L which corresponds to the differential form $A = -(1 + \cos\theta)d\varphi$ on the punctured sphere $S^2\backslash N$, θ, φ being the spherical coordinates. The 2-form $dA = \sin\theta d\theta \wedge d\varphi$ corresponding to its motion equations is $so(3)$-invariant, hence this Lagrangian is weakly $so(3)$-invariant. On the other hand it cannot be reduced to a $so(3)$-invariant one by redefinition on a total derivative df because a $so(3)$-invariant 1-form on the sphere is equal to zero. Hence, since all other spaces K_s are equal to zero, this Lagrangian belongs to the floor 2_+. We come to the following result:

$$K_3 = H^1(so(3), \Lambda^0(S^2\backslash N)) = \mathbf{R} \quad \text{and } \phi_2(A) \in K_3 \neq 0\,. \tag{5.7}$$

For this special case, the explicit realization of (4.11), (4.13) is the following: We identify the vectors with the elements of $so(3)$ by the linear map γ: $(x^1, x^2, x^3) \mapsto x^1 e_1 + x^2 e_2 + x^3 e_3$. For any point $x \in S^2$, the corresponding stability subalgebra is generated by $\gamma(x)$. To (4.11)–(4.12) there corresponds the following statement: If α is a 1-cocycle with values in functions on the punctured sphere, then

$$\begin{array}{c}\alpha(x,\gamma(x)) = x^i\alpha_i(x) \text{ is a constant on the sphere,}\\ \text{this constant is equal to zero iff this cocycle is a coboundary.}\end{array} \tag{5.8}$$

This statement can be easily proved in a straightforward way without using (4.11), (4.12).

We proved that $K_3 = \mathbf{R}$ and all other K_s are equal to zero and presented in (5.7) the Lagrangian with nontrivial image in K_3. Hence, the hierarchy diagram $\mathcal{D}([so(3), S^2\backslash N])$ is full on all the floors and the space of classical mechanics Lagrangians is equivalent to the one-dimensional space $U = \{L\colon L = -q(1+\cos\theta)\dot\varphi\}$ with respect to this hierarchy. So using (4.8) we arrive to the following statement: every weakly $so(3)$-invariant Lagrangian on the punctured sphere has the form

$$L = L_{inv} + \text{total derivatives} - g(1+\cos\theta)\dot\varphi\,. \tag{5.9}$$

In the case $g \neq 0$, it belongs to the floor 2_+ of the hierarchy.

The calculations for the diagram $\mathcal{D}[so(3), \mathbf{R}^3\backslash l_+]$ are similar and the result is the same: every weakly $so(3)$-invariant Lagrangian on $\mathbf{R}^3\backslash l_+$ has the form (5.9).

One can see that in the case when L_{inv} is the free particle Lagrangian, then (5.9) corresponds to the Lagrangian which describes the interaction of a particle with the Dirac monopole [15].

Explicit calculations for (5.7) give that $\phi_2(L)$ for the Lagrangian (5.9) is equal to the cohomology class in $H^1(so(3), S^2\backslash N)$ of the following cocycle:

$$\begin{array}{c}\alpha_1 = -g\cot\frac{\theta}{2}\cos\varphi, \quad \alpha_2 = -g cot\frac{\theta}{2}\sin\varphi, \quad \alpha_3 = g,\\ \text{and} \quad \alpha_i x^i = -g, \quad (\delta L = d\alpha,\ \delta\alpha = 0).\end{array} \tag{5.10}$$

Finally, we make the following remark about the Lagrangian (5.9). Via stereographic projection of the punctured sphere on $\mathbf{R}^2$ one comes from the pair $[so(3), S^2\backslash N]$ to the pair $[so(3), \mathbf{R}^2]$, where the fundamental vector field corresponding to e_3 is the infinitesimal rotation and fundamental vector fields corresponding to e_1, e_2 are nonlinear infinitesimal transformations. The weakly $so(3)$-invariant Lagrangian (5.9) transforms to

$$L = \frac{m(\dot u^2 + \dot v^2)}{2(1+u^2+v^2)^2} + g\frac{u\dot v - v\dot u}{1+u^2+v^2} \tag{5.11}$$

in the case when L_{inv} is the free particle Lagrangian. (u, v are the Cartesian coordinates on $\mathbf{R}^2$.)

In the case $g = 0$ the Lagrangian (5.11) is strictly related to the Lagrangian describing the interaction of a free particle in 2-dimensional plane with the Coulomb potential. To the vector fields $\tilde e_1, \tilde e_2$ there correspond so called hidden symmetries of Coulomb interaction which lead to Runge–Lentz vector [23]. So, the Lagrangian (5.11) leads to the Lagrangian which possesses essentially generalized hidden symmetries of the two-dimensional Coulomb potential. These consideration deal with the so called higher symmetries which are not in the frame of this paper.

Example 4. Galilean and Poincaré Lie algebras

We consider the action of Galilean and Poincaré algebras on the 4-dimensional space $\mathbf{R}^4$ with the Cartesian coordinates (t, x^1, x^2, x^3). The lagrangians on $\mathbf{R}^4$ are $L(t, x^i, \dot t, \dot x^i)$ $(i = 1,2,3)$ where $\dot x^i, \dot t$ means derivatives with respect to "time" τ.

To threat these algebras simultaneously, we consider a 1-parametric family of the Poincaré Lie algebras $\mathcal{G}(\mathcal{P}_c)$ (c is the "velocity of light"). Their action (2.1) on the space $\mathbf{R}^4$ is generated in a standard way via the following fundamental vector fields:

$$\tilde p_0 = \frac{\partial}{\partial t}, \quad \tilde p_i = \frac{\partial}{\partial x^i}, \quad \tilde B_i = t\frac{\partial}{\partial x^i} + \frac{1}{c^2}x^i\frac{\partial}{\partial t}, \quad \tilde L_i = -\varepsilon_{ijk}x^j\frac{\partial}{\partial x^k}, \tag{5.12}$$

which correspond to its basis. The relations (5.12) define the pair $[\mathcal{G}(\mathcal{P}_c), \mathbf{R}^4]$.

In the case $c \to \infty$, Lie algebra $\mathcal{G}(\mathcal{P}_c)$ is contracted to the Lie algebra of the Galilean group (nonrelativistic limit), which we denote also by $\mathcal{G}(\mathcal{P}_\infty)$. All the commutation relations of the basis vectors in $\mathcal{G}(\mathcal{P}_c)$ do not depend on c, except for the relations $[\tilde B_i, \tilde B_k] = -1/c^2\varepsilon_{ijk}\tilde L_k, [\tilde p_i, \tilde B_k] = -1/c^2 p_0\delta_{ik}$ which tend to zero, when c tends to infinity.

Correspondingly to (5.12), the action of the Galilean Lie algebra $\mathcal{G}(\mathcal{P}_\infty)$ on $\mathbf{R}^4$ is generated by the vector fields

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x^i}, \quad t\frac{\partial}{\partial x^i}, \quad L_i = -\varepsilon_{ijk}x^j\frac{\partial}{\partial x^k}. \tag{5.13}$$

It defines the pair $[\mathcal{G}(\mathcal{P}_\infty), \mathbf{R}^4]$. (The vector field corresponding to Lorentz boost transforms to vector field corresponding to special Galilean transformation.)

The first two cohomology groups for algebras $\mathcal{G}(\mathcal{P}_c)$ are

$$\begin{array}{lll} H^1(\mathcal{G}(\mathcal{P}_c)) = 0, & H^2(\mathcal{G}(\mathcal{P}_c)) = 0, & \text{if } c \neq \infty\\ H^1(\mathcal{G}(\mathcal{P}_\infty)) = \mathbf{R}, & H^2(\mathcal{G}(\mathcal{P}_\infty)) = \mathbf{R}. & \end{array} \tag{5.14}$$

The first and the second cohomologies groups of the Galilean Lie algebra are generated by the 1-cocycle c_1 and the 2-cocycle c_B (the Bargmann cocycle) respectively, whose nonvanishing components in the basis (5.13) are

$$c(p_0) = 1, \quad c_B(p_i, B_j) = -c_B(B_j, p_i) = \delta_{ij}\,. \tag{5.15}$$

The relations (5.14) make trivial the calculations of all the spaces K_s except for the space K_3 for the hierarchy diagram $\mathcal{D}(\mathcal{G}(\mathcal{P}_c),\mathbf{R}^4)$. From the formulae (4.5), (4.11) it follows that $K_3=0$, because the stability subalgebra of every point (t_0,x_0^i) in $\mathbf{R}^4$ is isomorphic to the subalgebra generated by the vectors (L_i,B_j) which has only trivial 1-cocycles and $H^1(\mathbf{R}^4)=0$.

Now we study the space of weakly $\mathcal{G}(\mathcal{P}_c)$-invariant Lagrangians on $\mathbf{R}^4$. First of all we note that from (5.12) it follows that $\mathcal{G}(\mathcal{P}_c)$-invariant Lagrangian is an arbitrary function of the square of velocity in Minkovsky space if $c\neq\infty$ and $\mathcal{G}(\mathcal{P}_\infty)$-invariant Lagrangian is an arbitrary function of velocity t component:

$$L_{inv}=F(c^2\dot t^2-\sum_i \dot x^i\dot x^i)\quad \text{if } c\neq\infty,\qquad L_{inv}=F(\dot t)\quad \text{if } c=\infty. \tag{5.16}$$

1) Poincaré Lie algebra ($c\neq\infty$) In this case the hierarchy diagram $\mathcal{D}(\mathcal{G}(\mathcal{P}_c),\mathbf{R}^4)$ is trivial because all the spaces K_s are equal to zero. Every weakly $\mathcal{G}(\mathcal{P}_c)$-invariant Lagrangian is invariant one (up to a total derivative) and it belongs to the floor 4_+.

$$\mathcal{U}_{0.0}\ni L=F(c^2\dot t^2-\sum_i \dot x^i\dot x^i)+\text{total derivative}. \tag{5.17}$$

2) Galilean Lie algebra ($c=\infty$):
In this case $K_0=K_2=\mathbf{R}, K_1=K_3=0$.
We consider the following 2-dimensional subspace of Lagrangians on $\mathbf{R}^4$

$$U=\{L:\quad L=\frac{m(\sum_i \dot x^i\dot x^i)}{2\dot t}+b\dot t\}, \tag{5.18}$$

where m and b are constants. One can see by straightforward calculations that for a Lagrangian in U

$$\mathcal{L}_{p_0}L=b,\ \mathcal{L}_{p_i}L=\mathcal{L}_{L_i}L=0,\ \mathcal{L}_{B_i}L=m\dot x^i. \tag{5.19}$$

Comparing (5.19) with (4.1) we see that $U=\mathcal{U}_{0.0}$, i.e. U is the subspace of weakly $\mathcal{G}(\mathcal{P}_\infty)$-invariant Lagrangians. The calculation of homomorphisms Ψ and ϕ_2 on the diagram $\mathcal{D}([\mathcal{G}(\mathcal{P}_\infty),\mathbf{R}^4],U)$ gives

$$\Psi(L)=b[c_1],\quad \phi_2(L)=m[c_B], \tag{5.20}$$

where $[c_1],[c_B]$ are cohomological classes in K_0 and K_2 respectively of the cocylces (5.15).

Hence the hierarchy diagram $\mathcal{D}([\mathcal{G}(\mathcal{P}_\infty),\mathbf{R}^4],U)$ is full and the space U in (5.18) is equivalent to the space of all Lagrangians on $\mathbf{R}^4$ with respect to the hierarchy. From (5.18), (5.20) and (4.8) it follows that every weakly $\mathcal{G}(\mathcal{P}_\infty)$-invariant Lagrangian on $\mathbf{R}^4$ is of the form:

$$\mathcal{U}_{0.0}\ni L=F(\dot t)+\frac{m(\sum_i \dot x^i\dot x^i)}{2\dot t}+b\dot t+\text{total derivative}. \tag{5.21}$$

It belongs to the floor 1_+ if $b=0$ and $m\neq 0$.

Physically it is more interesting to consider the hierarchy diagram for the subspace $\mathcal{U}^{dens}$ of Lagrangians which are densities in R^4. The space of Lagrangians $L(t,x^i,dt/d\tau,dx^i/d\tau)$ in $\mathbf{R}^4$ is more wide than the space of classical mechanics Lagrangians $L(x^i,dx^i/dt)$ on the configuration space $\mathbf{R}^3$ with coordinates x^i ($i=1,2,3,$). To every Lagrangian in $\mathbf{R}^3$, according to (2.16), there corresponds a Lagrangian which is a density in $\mathbf{R}^4$. On the other hand, to every Lagrangian L in $\mathbf{R}^4$ which is a density and which does not depend explicitly on time, there corresponds a classical mechanics Lagrangian, if we put the parameter τ to be equal t. For example, to the Lagrangian of a free relativistic particle there corresponds the density in $\mathbf{R}^4$

$$L_{rel}(c)=-mc\sqrt{c^2\dot t^2-\sum_i \dot x^i\dot x^i}. \tag{5.22}$$

and to the Lagrangian of a free non-relativistic particle there corresponds the density

$$L_{nonrel}=\frac{m\sum_i \dot x^i\dot x^i}{2\dot t}. \tag{5.23}$$

The Lagrangian $L_{rel}(c)+mc^2\dot t$, which differs from $L_{rel}(c)$ by the total derivative, tends to L_{nonrel} when $c\to\infty$.

On the other hand the condition that weakly $\mathcal{G}(\mathcal{P}_c)$-invariant Lagrangians in (5.17) as well as in (5.21) are densities gives that they are equal to (5.22) and (5.23) respectively (up to a total derivative). Indeed if (5.21) is a density then $b=0$ and $F(\lambda\dot t)=\lambda F(\dot t)$ (see (2.16)). Hence $F(\dot t)=a\dot t$ and it is a total derivative. The similar considerations for (5.17) lead to (5.22).

We come to the following conclusion:

Every weakly $\mathcal{G}(\mathcal{P}_\infty)$-invariant Lagrangian-density in $\mathbf{R}^4$ belongs to the floor 1_+ and is equal to L_{nonrel} (up to total derivative). The value of homomorphism ϕ_2 on it is proportional to the mass of the particle. There are no non-trivial $\mathcal{G}(\mathcal{P}_\infty)$-invariant Lagrangians-densities on $\mathbf{R}^4$. The floor 4_+ contains trivial Lagrangians only.

For the Poincaré algebra every weakly $\mathcal{G}(\mathcal{P}_c)$-invariant Lagrangian–density coincides (up to a total derivative) with the $\mathcal{G}(\mathcal{P}_c)$-invariant Lagrangian $L_{rel}(c)$. On one hand, when one contracts Poincaré algebra to the Galilean algebra, the Bargmann cocycle arises. On the other hand, the unique nontrivial component $\mathcal{V}_{4.1}$ of the hierarchy diagram for the Poincaré algebra transforms to the unique nontrivial component $\mathcal{V}_{1.1}$ of the hierarchy diagram for the Galilean algebra.

Vanishing of $H^2(\mathcal{G}(\mathcal{P}_c))$ is the reason why in relativistic quantum mechanics the projective representation of Poincaré symmetries in the space of states (which are rays in a linear Hilbert space) can be reduced to linear one, while because of (5.15) it is not the case for nonrelativistic mechanics. The considerations of this example reflect this phenomenon. (The detailed physical analysis see for example in [12], [13].)

6 Discussions

The problem considered here and the technique we used to study it can be generalized in several directions. The considerations of this paper can be easily translated into Hamiltonian language. One can consider the classification of Lagrangians not only for symmetries induced by point transformations of the configuration space, but also for so-called higher symmetries. For example, it is interesting to analyze the generalized Runge–Lentz symmetries from this point of view (see the end of Example 3 in Section 5).

It is interesting to apply this method to supersymmetrical case [14]. It seems to be interesting also to analyze the phenomenon of spin-like transformations (1.9) arising for Lagrangians from the second floor of the hierarchy (4.7), in order to apply it to the Dirac monopoles [24].

We hope that a generalization of this method to field theory Lagrangians will be fruitful. From this point of view we want to note the relations of our considerations with the problem of the Ward identities anomaly absence in the case when field theory Lagrangians possess classically the given symmetry [25,10].

To develop this technique for field theory Lagrangians, the first order formalism and multisymplectic formalism become very useful [23]. We wish to develop these considerations on the firm ground of investigations by A.M. Vinogradov and his collaborators [26].

On the other hand, to our opinion, the method considered in this paper is maybe more important than the problem we applied it to.

We give here only three examples, one of them pure mathematical, where the calculations of double complex cohomology (the method we use in this paper) make a bridge between the corresponding structures.

1. *Calculation of de Rham cohomology in terms of Čhech cohomology.*

When manifold M is covered by a family $\{U_\alpha\}$ of open sets, one can consider Čhech cohomology of this covering. Then one can consider double complex of q-forms which are defined on the sets $\{U_\alpha\}$. The differential Q of this complex is the sum of the de Rham exterior differential and the Čhech differential. Considering the differential Q "perturbatively" near the Čhech differential, one arrives naturally at the de Rham cohomology of M, hence the "perturbative" calculations near the de Rham differential lead in general to calculation of spectral sequence which converges to the de Rham cohomology of M. In the case when the covering is a Leray covering, i.e., all the sets and their intersections are convex connected sets, then Čhech cohomology coincides with the de Rham one; application of the Poincaré lemma reduces spectral sequence calculations to trivial resolutions of so-called descent equations [10]. But practically it is more convenient to use for calculations a suitable covering which generally is not a Leray covering (see for details e.g. [16]).

2. *Relations between the Hamiltonian reduction method and the BRST cohomology for classical mechanics*

One can say that the relations between these two methods are encoded in the cohomology of the double complex differential $Q = \partial + \delta$, in the case when constraints form a Lie algebra (so-called closed algebras). Here ∂ corresponds to the Koszul differential of the complex generated by constraints and δ is the differential corresponding to Hamiltonian vector fields induced by these constraints. Perturbative expansion of Q near δ leads to standard Hamiltonian methods, and expansion around ∂ leads to BRST. In the case when constraints form so-called open algebra, one has to consider the corresponding filtered space instead of this double complex [3,4,6]. This approach seems to be very fruitful.

3. *Local BRST Cohomology*

Considering BRST physical observables as integrals of local functions, one comes naturally to the differential $Q = s + d$, where s is the BRST differential, acting on integrand which is a local function and d is the usual de Rham differential. It turns out that the consideration of cohomology of this double complex is a very powerful tool for BRST cohomology investigations in field theory, especially in Lagrangian framework (see [8,9,10,27] and references there). In spite of these examples, one has to note that the method of spectral sequences was not used actively in these calculations.

Maybe the method of spectral sequences was applied in physics first by J. Dixon in [8] to analyze local BRST cohomology. In series of works the so-called method of descent equations which is in fact a special case, a reminiscent of this technique was applied successfully to these problems (see the review [10] and references there). Nowadays the technique of spectral sequences seems to be not very popular in theoretical physics. We hope to attract attention to importance of this technique used here in a simple physical framework. In principal, using the method "Deus ex machina" one can formulate the hierarchy without using explicitly the method developed in this paper which indeed seems to be very tedious. But to our opinion, this method is inherent to this problem and it is the adequate technique in other important problems such as constrained dynamics theory; it may have useful applications in future.

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Appendix 1. Lie algebra cohomologies

Let $\mathcal{G}$ be Lie algebra and A be a linear space which is module on $\mathcal{G}$, i.e. the action of $\mathcal{G}$ on A which respects the structure of the Lie algebra $\mathcal{G}$ and the space A is defined:

$$\begin{array}{l} h\in\mathcal{G}, m\in A \quad (h,m)\to h\circ m\in A: \\ (\lambda h_1+\mu h_2)\circ m=\lambda(h_1\circ m)+\mu(h_2\circ m), \quad (\lambda,\mu\in\mathbf{R}) \\ h\circ(\lambda m_1+\mu m_2)=\lambda(h\circ m_1)+\mu(h\circ m_2), \\ h_1\circ(h_2\circ m)-h_2\circ(h_1\circ m)=[h_1,h_2]\circ m. \end{array} \tag{A1.1}$$

($[\,,]$ defines commutator in $\mathcal{G}$. A and $\mathcal{G}$ are linear spaces on $\mathbf{R}$).

The complex $(C^q(\mathcal{G},A),\delta)$ of cochains can be defined in the following way. Let $C^q(\mathcal{G},A)$ be a space of skewsymmetric q-linear functions on $\mathcal{G}$ (q-cochains) which take values in A (If $q=0$, $C^0(\mathcal{G},A)=A$). $\mathcal{G}$-differential δ on $\{C^q\}$ $\delta: C^q\to C^{q+1}$, $\delta^2=0$ is defined in the following way:

$$\begin{array}{ll} \delta: C^0\to C^1 & (\delta c)(h)=h\circ c, (c\in C^0=A) \\ \delta: C^1\to C^2 & (\delta c)(h_1,h_2)=h_1\circ c(h_2)-h_2\circ c(h_1)-c([h_1,h_2]), \end{array} \tag{A1.2}$$

and so on:

$$\begin{array}{ll} \delta: C^q\to C^{q+1} & (\delta c)(h_1,\dots,h_{q+1})=\sum_{1\leq i\leq q+1}(-1)^{i+1}h_i\circ c(h_1,\dots,\hat{h}_i\dots,h_{q+1})+ \\ & \sum_{1\leq i<j\leq q+1}(-1)^{i+j}c([h_i,h_j],h_1,\dots,\hat{h}_i,\dots,\hat{h}_j\dots,h_{q+1}) \end{array}$$

($\hat{h}_i$ means omitting of the variable h_i)). The cohomologies $H^q(\mathcal{G},A)$ of the complex $(\{C^q\},\delta)$ are called cohomologies of Lie algebra $\mathcal{G}$ with coefficients in the module A. (See in details for example [16].)

$$H^q(\mathcal{G},A)=\left(\ker\delta: C^q\to C^{q+1}\right)/\left(\Im\,\delta: C^{q-1}\to C^q\right).$$

If module A is $\mathbf{R}$ and $\mathcal{G}$ acts trivially on it: $h\circ\lambda=0$, $C^q(\mathcal{G},\mathbf{R})$ is denoted by $C^q(\mathcal{G})$ and correspondingly $H^q(\mathcal{G},\mathbf{R})$ is denoted by $H^q(\mathcal{G})$. In this case cochains are constant antisymmetrical tensors and $\mathcal{G}$-differential δ is expressed only via structure constants $\{t^n_{ik}\}$ of Lie algebra $\mathcal{G}$.

$H^0(\mathcal{G})=\mathbf{R}$, $H^1(\mathcal{G})$ is defined by the solutions of the equation $c^m_{ik}b_m=0$ and it is nothing but the space dual to the $\mathcal{G}/[\mathcal{G},\mathcal{G}]$.

In a case if $\mathcal{G}$ is abelian $H^q(\mathcal{G})=C^q(\mathcal{G})=(\wedge\mathcal{G}^*)^q$ where $\mathcal{G}^*$ is the linear space dual to the linear space of $\mathcal{G}$.

In a case if $\mathcal{G}$ is semisimple Lie algebra then $H^1\mathcal{G}=H^2\mathcal{G}=0$. This statement is valid in a general case too. Very important Whitehead lemmas state that if $\mathcal{G}$ is semisimple Lie algebra then $H^1(\mathcal{G},A)=H^2(\mathcal{G},A)=0$ in the case if A is an arbitrary module which is *finite-dimensional* vector space on $\mathbf{R}$ [16]

Appendix 2. Double complex and its spectral sequences.

Now we give a brief sketch on the topic how to apply spectral sequences technique for calculations of cohomology of double complexes. (See for the details for example [16].)

Let $E^{**}=\{E^{p.q}\}$ $(p,q=0,1,2,...)$ be a family of abelian groups (modules, vector spaces) on which are defined two differentials ∂_1 and ∂_2 which define complexes in rows and in columns of $E^{*.*}$ and which commute with each other:

$$\partial_1: E^{p.q}\to E^{p.q+1}, \partial_1^2=0,\ \partial_2: E^{p.q}\to E^{p+1.q}, \partial_2^2=0,\ \partial_1\partial_2=\partial_2\partial_1. \tag{A2.1}$$

$\{E^{**},\partial_1,\partial_2\}$ is called double complex.

(It is convenient to consider $E^{p.q}$ for all integers p and q fixing that $E^{p.q}=0$ if $p<0$ or $q<0$.)

One can consider "antidiagonals": $\mathcal{D}^m=\{E^{p.m-p}\}$ $(p=0,1,...,m)$ which form complex with differential

$$Q=(-1)^q\partial_2+\partial_1 \tag{A2.2}$$

which evidently obeys to condition $Q^2=0$.

$$0\to\mathcal{D}^0\xrightarrow{Q}\mathcal{D}^1\xrightarrow{Q}\mathcal{D}^2\to\dots. \tag{A2.3}$$

The cohomologies $H^m(Q)$ of this complex are called the cohomologies of double complex $(E^{**},\partial_1,\partial_2)$.

The rows and the columns complexes define the cohomologies $H(\partial_1)$ and $H(\partial_2)$ of E^{**}.

One can consider the filtration corresponding to the double complex $\{E^{*.*},\partial_1,\partial_2\}$

$$\dots\subseteq X^m\subseteq X^{m+1}\subseteq\dots\subseteq X^1\subseteq X^0 \tag{A2.4}$$

$$\text{where}\qquad X^k=\bigoplus_{q\geq 0,p\geq k}E^{p.q} \tag{A2.5}$$

and sequence of the spaces $\{E^{p.q}_r\}$ $(r=0,1,2,\dots$ corresponding to this filtration

$$E^{p.q}_r=Z^{p.q}_r/B^{p.q}_r \quad (E^{p.q}_0=E^{p.q}). \tag{A2.6}$$

In (A2.6) $Z^{p.q}_r$ ("r-th order cocycles") is the space of the elements in $E^{p.q}$ which are leader terms of cocycles of the differential Q up to r–th order w.r.t. the filtration (A2.4), i.e.

$$\{Z^{p.q}_r\}=\{E^{p.q}_r\ni c:\quad \exists\tilde{c}=c(mod\,X_{p+1})\,\text{such that}\,Q\tilde{c}=0(mod\,X_{p+r})\}. \tag{A2.7}$$

It means that there exists $\tilde{c}=(c,c_1,c_2,\dots,c_{r-1})$ where $c_i\in E^{p+i.q-i}$ such that $Q(c,c_1,c_2,\dots,c_{r-1})\subseteq X_{p+r}$:

$$\partial_1 c=0, \partial_2 c=\partial_1 c_1, \partial_2 c_1=\partial_1 c_2,\dots,\partial_2 c_{r-2}=\partial_1 c_{r-1},\ \text{so}\ Q\tilde{c}=\partial_2 c_{r-1}\in X_{p+r}.$$

Correspondingly $B_r^{p.q}$ is the space of up to r-th order borders:

$$\{B_r^{p.q}\} = \{E_r^{p.q} \ni c: \quad \exists \tilde{b} \in X_{p-r+1} \text{ such that } Q\tilde{b} = c. \qquad (A2.8)$$

It means that there exist $\tilde{c} = (b_0, b_1, b_2, \ldots, b_{r-1})$ where $b_i \in E^{p-i.q+i}$ and $Q(b_0, b_1, b_2, \ldots, b_{r-1}) = c$:

$$\partial_1 b_0 + \partial_2 b_1 = c, \partial_1 b_1 + \partial_2 b_2 = 0, \partial_1 b_2 + \partial_2 b_3 = 0, \ldots, \partial_1 b_{r-1} = 0. \qquad (A2.9)$$

For example $E_1^{p.q} = H(\partial_1, E^{p.q})$.

We denote by $[c]_r$ the equivalence class of the element c in the $E_r^{p.q}$ if $c \in Z_r^{p.q}$.

It is easy to see that the sequence $\{E_r^{p.q}\}$ $r = 0, 1, 2, \ldots$ is stabilized after finite number of the steps: $(E_{r_0}^{p.q} = E_{r_0+1}^{p.q} = \ldots = E_\infty^{p.q}$, where $r_0 = max\{p+1, q+1\}$.

Let $H^m(Q, X_p)$ be cohomologies groups of double complex truncated by filtration (A2.4) (we come to $H^m(Q, X_p)$ considering $\{\mathcal{D} \cap X^p, Q\}$ as subcomplex of (A2.3), $H^m(Q) = H^m(Q, X^0)$. We denote by ${}_{(p)}H^m(Q)$ the image of $H^m(Q, X_p)$ in $H(Q)$ under the homomorphism induced by the embedding $\mathcal{D} \cup X_p \to \mathcal{D}$. The spaces ${}_{(p)}H^m(Q)$ are embedded in each other

$$0 \subseteq {}_{(m)}H^m(Q) \subseteq {}_{(m-1)}H^m(Q) \subseteq \ldots {}_{(1)}H^m(Q) \subseteq {}_{(0)}H^m(Q) = H^m(Q). \qquad (A2.10)$$

The spaces $E_\infty^{p.q}$ considered above are related with (A2.10) by the following relations:

$$E_\infty^{p.m-p} = {}_{(p)} H^m(Q) / {}_{(p+1)}H^m(Q). \qquad (A2.11)$$

In particular $E_\infty^{0.m}$ is canonically embedded in $H^m(Q)$.

The formula (A2.11) is the basic formula which expresses the cohomology $H(Q)$ of the double complex $\{E^{p.q}, \partial_1, \partial_2\}$ in terms of $\{E_\infty^{p.q}\}$. From (A2.10, A2.11) it follows that

$$H^m(Q) \simeq \bigoplus_{i=0}^{m} E^{p-i.i}. \qquad (A2.12)$$

The essential difference of (A2.12) from (A2.11) is that in (A2.12) the isomorphism of l.h.s. and of r.h.s. *is not canonical.*

The importance of the sequence $\{E_r^{*.*}\}$ $(r = 0, 1, 2, \ldots)$ is explained by the fact that its terms (and so $\{E_\infty^{*.*}\}$) can be calculated in a recurrent way. Namely one can consider differentials (See for details [16.]) d_r: $E_r^{p.q} \to E_r^{p+r.q+1-r}$ such that $\{E_r^{*.*}, d_r\}$ form spectral sequence, i.e.

$$E_{r+1}^{*.*} = H(d_r, E_r^{*.*}). \qquad (A2.13)$$

The differentials d_r are constructed in the following way: $d_0 = \partial_1$: $E^{p.q} = E_0^{p.q} \to E^{p.q+1} = E_0^{p.q+1}$.

If $c \in E^{p.q}$ and $\partial_1 c = 0 \leftrightarrow [c]_1 \in E_1^{p.q}$ then $d_1[c] = [\partial_2 c]$, d_1: $E_1^{p.q} \to E_1^{p+1.q}$.

In general case for $[c]_r \in E_r^{p.q}$ $d_r[c]_r = [Q\tilde{c}]_r$ d_r: $E_r^{p.q} \to E_1^{p+r.q+1-r}$,

where $\tilde{c}$: $c - \tilde{c} \in X^{p+r}$ (see the definition (A2.7) of $Z_r^{p.q}$).

One can show that definition of d_r is correct, $d_r^2 = 0$ and (A2.13) is obeyed [16].

Using (A2.13) one come after finite number of steps to $E_\infty^{p.q}$ calculating each $E_r^{p.q}$ as the cohomology group of the $E_{r-1}^{p.q}$: $E_1^{p.q} = H(d_0, E^{p.q})$, $E_2^{p.q} = H(d_1, E_1^{p.q}$ and so on.

The spaces $E_r^{p.q}$ can be considered intuitively as r-th order (with respect to differential ∂_2) cohomologies of differential Q. The operator ∂_1 is zeroth order approximation for differential Q. The calculations of $E_\infty^{p.q}$ via (A2.13) can be considered as perturbational calculations. One can develop this scheme considering in perturbative calculations not the operator ∂_1, but ∂_2 as zeroth order approximation.

Instead filtration (A2.4) one has consider the "transposed" filtration

$$\ldots \subseteq {}^tX^m \subseteq {}^tX^{m+1} \subseteq \ldots \subseteq {}^tX^1 \subseteq X^0 \qquad (A2.14)$$

$$\text{where} \qquad {}^tX^k = \bigoplus_{p \geq 0, q \geq k} E^{p.q}$$

and corresponding transposed spaces $\{{}^tE_r^{p.q}\}$. For example

$$E_1^{p.q} = H(\partial_1, E^{p.q}), \quad {}^tE_r^{p.q} = H(\partial_2, E^{p.q}).$$

Instead spectral sequence $\{E_r^{*.*}, d_r\}$ one has to consider transposed spectral sequence $\{{}^tE_r^{*.*}, {}^td_r\}$:

$$d_0 = \partial_1, \to {}^td_0 = \partial_2; d_1[c]_1 = [\partial_2 c]_1, \to {}^td_1[c]_1 = [\partial_1 c]_1,$$

and so on.

The relations between spaces $\{E_\infty^{p.q}\}$ and $\{{}^tE_\infty^{p.q}\}$ which express in different ways the cohomology $H(Q)$ is one of the applications of the method described here.

Example. Let $\mathbf{c} = (c_0, c_1.c_2)$ where $c_0 \in E^{0.2}, c_1 \in E^{1.1}, c_2 \in E^{2.0}$ be cocycle of the differential Q: $Q(c_0, c_1.c_2) = 0$ i.e. $\partial_1 c_0 = 0, \partial_2 c_0 = -\partial_1 c_1, \partial_2 c_1 = \partial_1 c_2$. To the leading term c_0 of this cocycle w.r.t. the filtration (A2.4) corresponds the element $[c_0]_\infty$ in $E_\infty^{0.2}$ which represents the cohomology class of the cocycle $\mathbf{c}$ in $E_\infty^{0.2}$.

In the case if the equation $(c_0, c_1.c_2) + Q(b_0, b_1) = (0, c_1', c_2')$ has a solution, i.e. the leading term c_0 of the cocycle $\mathbf{c}$ can be cancelled by changing of this cocycle on a coboundary, then the element $[c_1']_\infty \in E_\infty^{1.1}$ represents the cohomology class of the cocycle $\mathbf{c}$ in $E_\infty^{1.1}$.

In the case if the equation $(c_0, c_1.c_2) + Q(b_0, b_1) = (0, 0, \tilde{c}_2)$ have a solution, i.e. the leading term and next one both can be cancelled, by redefinition on a coboundary, then $[\tilde{c}_2]_\infty \in E_\infty^{2.0}$ represents the cohomology class of the cocycle $\mathbf{c}$ in $E_\infty^{2.0}$.

To put correspondences between the cohomology class of the cocycle $\mathbf{c}$ and corresponding elements from transposed spaces ${}^tE_\infty^{0.2}$, ${}^tE_\infty^{1.1}$ ${}^tE_\infty^{1.1}$ we have to do the same, changing only the definition of leading terms, which we have to consider now w.r.t. the filtration (A2.14).

To the leading term c_2 of this cocycle w.r.t. the filtration (A2.14) corresponds the element $[c_2]_\infty$ in ${}^tE_\infty^{2.0}$ which represents the cohomology class of the cocycle $\mathbf{c}$ in ${}^tE_\infty^{2.0}$. In the case if the equation $(c_0, c_1.c_2) + Q(b_0, b_1) = (c_0', c_1', 0)$ has a solution, i.e.

the leading term c_0 of the cocycle $\mathbf{c}$ can be cancelled by changing of on a coboundary, then the element $[c_1']_\infty$ represents the cohomology class of the cocycle $\mathbf{c}$ in ${}^tE^{1,1}_\infty$. In the case if the equation $(c_0, c_1.c_2) + Q(b_0, b_1) = (\tilde{c}_0, 0, 0)$ has a solution, then $[\tilde{c}_0]$ represents the cohomology class of the cocycle $\mathbf{c}$ in ${}^tE^{0,2}_\infty$.

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