ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2-98-389

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ODD INVARIANT SEMIDENSITY AND DIVERGENCE-LIKE OPERATORS ON AN ODD SYMPLECTIC SUPERSPACE

Submitted to «Communications in Mathematical Physics»

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1998

1. Introduction

In this paper we construct a differential first order divergence-like operator on a superspace endowed with an odd symplectic structure. This operator is applied for constructing invariant differential objects in this superspace. In particular an odd semidensity invariant under the transformations preserving the odd symplectic structure of the superspace and the volume form is constructed in a geometrically clear way.

The odd symplectic structure plays essential role in Lagrangian formulation of the BRST formalism (Batalin-Vilkovisky formalism) [2]. In supermathematics it is a natural counterpart of even symplectic structure [3,10]. On the other hand it has some "odd" features which have no analogues in usual mathematics. Canonical transformations preserving odd symplectic structure (non-degenerate odd Poisson bracket) do not preserve any volume form. This fact, indeed, is the reason why in the case of odd symplectic structure some invariant geometrical objects have no natural homologues as it is for the even symplectic structure, for which the superconstructions are rather straightforward generalizations of those for the symplectic structures in usual spaces. We consider some examples.

In order to construct geometrical integration objects one needs to consider a pair, the volume form and the odd symplectic structure. This pair is in fact the geometrical background for the formulation of Batalin-Vilkovisky formalism (See [2,5,13,7]). The so called Δ operator which plays essential role in this formalism can be defined in the following way: its action on the function f is equal to the divergence with respect to the given volume form (defined by the action of the theory) of the Hamiltonian vector field, corresponding to the function f. It is a second order differential operator. In the case of even symplectic structure there exists the volume form which naturally corresponds to this structure, and Δ operator is evidently equal to zero (Liouville Theorem). Even if the volume form is arbitrary, one arrives at a first order differential operator [5].

The second example is the invariant volume form (density) which can be defined on the Lagrangian supersurfaces embedded in the superspace endowed with an odd symplectic structure and a volume form. It is nothing but the integrand for the partition function in the space of fields and antifields in the BV formalism [13,7].

We focus on the third example, on the problem of finding an analogue of Poincaré-Cartan integral invariants for the odd symplectic structure. In usual mathematics to the Poincaré-Cartan integral invariant (invariant volume form on the embedded surfaces) there corresponds the wedged power of a differential two-form which defines the symplectic structure. In case of even symplectic structure in spite of the fact that differential form has nothing in common with invariant integration objects, the superdeterminant of the two-form induced on the embedded supersurface from the two-form which defines the symplectic structure leads us to the Poincaré–Cartan invariant [8]. For the odd symplectic structure the situation is essentially different. In [6] there was considered the problem of constructing of the invariant densities for the superspace endowed with an odd symplectic structure.

Density is the object which defines a volume form on embedded (super)surfaces.– If $z = z(\zeta)$ is the local parametrization of a (super)surface then a density A is a function of $z(\zeta)$, $\frac{\partial z}{\partial \zeta}$, $\frac{\partial^2 z}{\partial \zeta \partial \zeta}$, ... $A = A\left(z(\zeta), \frac{\partial z}{\partial \zeta}, \frac{\partial^2 z}{\partial \zeta \partial \zeta}, \ldots, \frac{\partial^k z}{\partial \zeta \ldots \partial \zeta}\right)$ subject to the condition that under reparametrization $\zeta \to \zeta(\tilde{\zeta})$, $A \to A \cdot sdet\left(\frac{\partial \zeta}{\partial \tilde{\zeta}}\right)$. A is a density of the rank k if it depends on the tangent vectors of the order $\leq k$ (i.e. on the derivatives of the order $\leq k$). In usual mathematics densities are the natural generalization of differential forms (if $k = 1$). In supermathematics even in the case of $k = 1$ this object is of more importance, since differential forms in supermathematics are no more integration objects [3,4,1,14].

In the case of symplectic structure in usual mathematics as well as in the case of even symplectic structure invariant densities are exhausted by densities of the rank $k = 1$ which correspond to Poincaré–Cartan integral invariants [8]. This is not the case for the odd symplectic structure. One can show that on $(p.p)$-dimensional non-degenerate supersurfaces embedded in a superspace $E^{n.n}$ which is endowed with a volume form and an odd symplectic structure there are no invariant densities of the rank $k = 1$ (except of the volume form itself), and in the class of densities of the rank $k = 2$ and of the weight σ (i.e. which are multiplied by the σ-th power of the superdeterminant of the reparametrization) there exists a unique (up to multiplication by a constant) semidensity ($\sigma = \frac{1}{2}$) in the case of $p = n - 1$ [6]. In fact, in [6] this semidensity was constructed in a non-explicit way in terms of dual densities: If $(n - 1.n - 1)$-dimensional supersurface M is given by the equations $f = 0, \varphi = 0$, where f is even function and φ is an odd function then to this semidensity there corresponds the function

$$\tilde{A}\Big|_{f=\varphi=0} = \frac{1}{\sqrt{\{f,\varphi\}}}\left(\Delta f - \frac{\{f,f\}}{2\{f,\varphi\}}\Delta\varphi - \frac{\{f,\{f,\varphi\}\}}{\{f,\varphi\}} - \frac{\{f,f\}}{2\{f,\varphi\}^2}\{\varphi,\{f,\varphi\}\}\right) \tag{1.1}$$

which depends on the second derivatives ($k = 2$), is invariant under the transformations preserving the odd symplectic structure and the volume form, and is multiplied by the square root of the corresponding Berezinian (superdeterminant) under the transformation $f \to af + \alpha\varphi$, $\varphi \to \beta f + b\varphi$, which does not change the supersurface M. The semidensity (1.1) takes odd values. It is an exotic analogue of Poincaré–Cartan invariant: the corresponding density $\tilde{A}^2 = 0$, so it cannot be integrated nontrivially over supersurfaces.

The formula (1.1) was obtained in [6] by straightforward calculations. To clarify its geometrical meaning in this paper a special geometrical object for odd symplectic superspace is considered.

As it was mentioned above, to a symplectic structure in usual space there corresponds a volume form (Liouville form):

$$dv = \rho(x)dx^1 \ldots dx^{2n} = \sqrt{det\,(\Omega_{ik})}dx^1 \ldots dx^{2n}\,, \tag{1.2}$$

where $\Omega = \Omega_{ik}dx^i \wedge dx^k$ is the closed non-degenerated two-form which defines the symplectic structure.

The volume form (1.2) is preserved under canonical transformations (i.e., the transformations preserving the two-form Ω). If $\mathbf{X} = X^i \frac{\partial}{\partial x^i}$ is an arbitrary vector field one can consider its divergence:

$$div\mathbf{X} = \frac{\mathcal{L}_{\mathbf{X}} dv}{dv} = \frac{\partial X^i}{\partial x^i} + X^i \frac{\partial \log \rho}{\partial x^i}\,. \tag{1.3}$$

In a symplectic space the canonical transformations preserve not only the volume form (1.2) but also its projection on an arbitrary symplectic subspace. Moreover, if $\hat{L}(z)$ is a projector-valued function such that $\mathbf{Im}\hat{L}(z)$ is a symplectic subspace in T_zE then it is easy to see that to $\hat{L}(z)$ one can associate divergence-like invariant operator whose action on an arbitrary vector field $\mathbf{X}$ is given by the expression

$$\partial(\hat{L}, \mathbf{X}) = \left(\frac{\partial X^k}{\partial x^i} + \frac{1}{2}X^m \frac{\partial \Omega_{ip}}{\partial x^m}\Omega^{pk}\right) L^i_k\,. \tag{1.4}$$

(Compare with (1.3) in case of $\hat{L} = \mathbf{id}$).

The formulae (1.2)–(1.4) have the straightforward generalization to the case of even symplectic structure in superspace (by changing determinant to superdeterminant and adding the powers of (-1) wherever necessary). It is not the case for the odd symplectic structure, where there is no any invariant volume form related with this structure.

Nevertheless, it turns out that the analogue of the formula (1.4) can be considered for odd symplectic structure in a case where $\hat{L}$ is a projector on (1.1)-dimensional symplectic subspace and $\mathbf{X}$ is the odd vector field which belongs to this subspace and is symplectoorthogonal to itself. In the next two sections we perform the corresponding constructions which are essentially founded on the following remark. Let $E^{1.1}$ be (1.1)-dimensional odd symplectic superspace and (x, θ) be Darboux coordinates on it: $\{x, \theta\} = 1$, $\{x, x\} = 0$, where $\{\,,\,\}$ is the odd Poisson bracket (Buttin bracket) corresponding to this symplectic structure. Let $\mathbf{\Psi}$ be an odd vector field in this superspace which is equal to $\Psi(x, \theta)\partial/\partial_\theta$ in these Darboux coordinates. Then this vector field has the same form and its divergence $(\partial\Psi(x, \theta)/\partial\theta)$ remains the same in arbitrary Darboux coordinates, in spite of the fact that there is no invariant volume form. (See Example 2.)

In the 4-th Section we consider a $(n-1.n-1)$-dimensional supersurface embedded in an odd symplectic superspace $E^{n.n}$ endowed with a volume form. The differential operator described above can be naturally applied to the odd vector field which is defined only at the points of this supersurface and is symplectoorthogonal to itself and to this supersurface. In spite of the fact that this field is not defined on the whole superspace one can define the invariant "truncated divergence" of this vector field. The analogue of this construction for usual symplectic structure is trivial. On the other hand this construction can be considered as an analogue of the corresponding operator acting on the vector field which is defined in a Euclidean (Riemannian) space at the points of embedded surface. But the essential difference is that the group of transformations which preserve metrics in Euclidean (Riemannian) space is exhausted by linear transformations (the linear part of transformation defines uniquely all higher terms) and this is not true for symplectic case where the group of canonical transformations is infinite-dimensional.

In the 5-th Section we apply our geometrical construction to obtain the formula for odd semidensity (1.1) in a geometrically clear way: it turns out that on $(n-1.n-1)$-dimensional supersurfaces embedded in a $(n.n)$-dimensional odd symplectic superspace endowed with a volume form one can define in a natural way the odd semidensity whose values are odd vectors, symplectoorthogonal to itself and to these supersurfaces. The "truncated divergence" of this vector-valued semidensity is invariant semidensity (1.1).

It has to be noted that our formula for this semidensity is very similar to the formula for the density corresponding to the mean curvature of hypersurfaces in the Euclidean space. In the last Section to discuss this point we consider invariant operators in Riemannian geometry which can be treated as analogues of the geometrical constructions of this paper.

2. Odd symplectic superspace

Let $E^{n.n}$ be a superspace with coordinates $z^A = x^1, \dots, x^n, \theta^1, \dots, \theta^n$; $p(x^i) = 0, p(\theta^j) = 1$, where p is a parity. We say that this superspace is odd symplectic superspace if it is endowed with odd symplectic structure, i.e., if an odd closed non-degenerate 2-form

$$\Omega = \Omega_{AB}(z)dz^A dz^B, \qquad (p(\Omega) = 1, \quad d\Omega = 0) \tag{2.1}$$

is defined on it [3,10].

To the differential form (2.1) on the superspace $E^{n.n}$ one can relate a function which at every point defines the following skewsymmetric (in a supersense) odd bilinear form on tangent vectors:

$$\begin{gathered}
\Omega(\mathbf{X}(z), \mathbf{Y}(z)) = -\Omega(\mathbf{Y}(z), \mathbf{X}(z))(-1)^{\tilde{\mathbf{X}}\tilde{\mathbf{Y}}}, \\
\Omega(\mathbf{X}(z), \lambda\mathbf{Y}(z)) = \Omega(\mathbf{X}(z)\lambda, \mathbf{Y}(z)), \\
\Omega(\lambda\mathbf{Y}(z) + \mu\mathbf{Z}(z), \mathbf{X}(z)) = \lambda\Omega(\mathbf{Y}(z), \mathbf{X}(z)) + \mu\Omega(\mathbf{Z}(z), \mathbf{X}(z)), \\
p(\Omega(\mathbf{X}, \mathbf{Y})) = 1 + p(\mathbf{X}) + p(\mathbf{Y}).
\end{gathered} \tag{2.2}$$

More precisely, a point of superspace $E^{n.n}$ is Λ-point– $2n$-plet $(a^1, \dots, a^n, \alpha^1, \dots, \alpha^n)$, where $(a^1, \dots, a^n)$ are even and $(\alpha^1, \dots, \alpha^n)$ are odd elements of an arbitrary Grassmann algebra Λ. (We use the most general definition of superspace suggested by A.S. Schwarz as the functor on the category of Grassmann algebras [12].)

In the coordinates:

$$\Omega_{AB} = -\Omega_{BA}(-1)^{\tilde{A}\tilde{B}} = \Omega\left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right), (p(\Omega_{AB}) = 1 + p(A) + p(B)). \tag{2.3}$$

We use the notation $\tilde{}$ for the parity of corresponding object. $\mathbf{X}(z)$, $\mathbf{Y}(z)$, $\mathbf{Z}(z)$ are the vector fields $X^A(z)\frac{\partial}{\partial z^A}$, $Y^A(z)\frac{\partial}{\partial z^A}$, $Z^A(z)\frac{\partial}{\partial z^A}$, the left derivations of functions on $E^{n.n}$.

(Differential form is usually considered as an element of an algebra generated by z^A and dz^A, where the parity of dz^A is opposite to the parity of z^A. In this case $\Omega_{AB} = \Omega_{BA}(-1)^{(\tilde{A}+1)(\tilde{B}+1)}$ instead of (2.3). The slight difference is eliminated by the transformation $\Omega_{AB} \to \Omega_{AB}(-1)^{\tilde{B}}$).

From (2.2, 2.3) it follows that

$$\Omega\left(X^A(z)\frac{\partial}{\partial z^A}, Y^B(z)\frac{\partial}{\partial z^B}\right) = X^A(z)\Omega_{AB}(z)Y^B(z)(-1)^{\tilde{\mathbf{Y}}\tilde{B}+\tilde{\mathbf{Y}}}. \tag{2.4}$$

In the same way as in the standard symplectic calculus one can relate to the odd symplectic structure (2.1) the odd Poisson bracket (Buttin bracket) [3,10]:

$$\{f, g\} = \frac{\partial f}{\partial z^A}(-1)^{\tilde{f}\tilde{A}+\tilde{A}}\Omega^{AB}\frac{\partial g}{\partial z^B}, \tag{2.5}$$

where $\Omega^{AB} = -\Omega^{BA}(-1)^{(\tilde{A}+1)(\tilde{B}+1)} = \{z^A, z^B\}$ is the inverse matrix to Ω_{AB} : $\Omega^{AC}\Omega_{CB} = \delta^A_B$.

To a function f via (2.5) there corresponds the Hamiltonian vector field

$$\mathbf{D}_f = \{f, z^A\}\frac{\partial}{\partial z^A} \quad \text{and} \quad \mathbf{D}_f(g) = \{f, g\}, \quad \Omega(\mathbf{D}_f, \mathbf{D}_g) = -\{f, g\}. \tag{2.6}$$

The condition of the closedness of the form (2.1) leads to the Jacoby identities:

$$\{f, \{g, h\}\}(-1)^{(\tilde{f}+1)(\tilde{h}+1)} + \text{cycl. permutations} = 0. \tag{2.7}$$

Using the analog of Darboux Theorem [11] one can consider the coordinates in which the symplectic structure (2.1) and the corresponding Buttin bracket have the canonical expressions. We call the coordinates w^A=$(x^1, \dots, x^n, \theta^1, ..., \theta^n)$ Darboux coordinates if in these coordinates holds

$$\Omega = I_{AB}dw^A dw^B: \quad \Omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = 0, \Omega\left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j}\right) = 0, \Omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \theta^j}\right) = -\delta_{ij}, \tag{2.8}$$

respectively

$$\{x^i, x^j\} = 0, \{\theta^i, \theta^j\} = 0, \{x^i, \theta^j\} = \delta^{ij}, \{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial \theta^i} + (-1)^{\tilde{f}}\frac{\partial f}{\partial \theta^i}\frac{\partial g}{\partial x^i}\right). \tag{2.9}$$

Now on the odd symplectic superspace $E^{n.n}$ endowed with an odd symplectic structure (2.1) we consider the following geometrical constructions: Let $\mathbf{\Psi}(z)$ be an odd non-degenerate vector field symplectoorthogonal to itself:

$$p(\mathbf{\Psi}(z)) = 1, \quad \Omega(\mathbf{\Psi}(z), \mathbf{\Psi}(z)) = 0. \tag{2.10}$$

(We call the vector non-degenerate if at least one of its components is not nilpotent.)

For example, to the even function f such that $\{f,f\}=0$ there corresponds the Hamiltonian vector field $\mathbf{D}_f$ defined by (2.6), subject to condition (2.10).

Let $\mathbf{\Pi}(z)$ be a field of (1.1)-dimensional subspaces ($\mathbf{\Pi}(z)\in TE_z^{n.n}$) which contain the vector field $\mathbf{\Psi}(z)$ and the symplectic structure induced on these subspaces is not degenerate. It means that there exists an even vector field $\mathbf{H}(z)$ such that

$$\mathbf{\Psi}(z)\,,\mathbf{H}(z)\in\mathbf{\Pi}(z)\quad\text{and}\quad\Omega(\mathbf{H}(z),\mathbf{\Psi}(z))=1. \tag{2.11}$$

To this field of subspaces $\mathbf{\Pi}(z)$ there corresponds the symplectoorthogonal projector $\hat{\Pi}(z)$ of the vectors in the tangent space on these subspaces:

$$\hat{\Pi}:\quad T_zE\to\mathbf{\Pi}(z),\quad \hat{\Pi}\big|_{\mathbf{\Pi}}(z)=\mathbf{id},\quad \hat{\Pi}\mathbf{X}=0\quad\text{if}\quad\Omega(\mathbf{X},\mathbf{\Pi})=0\,. \tag{2.12}$$

In the coordinates $\{z^A\}$ to the projector $\hat{\Pi}$ there corresponds the matrix-valued function

$$\Pi_A^B(z):\quad \hat{\Pi}\left(\frac{\partial}{\partial z^A}\right)=\Pi_A^B(z)\frac{\partial}{\partial z^B}\quad\text{so}\quad\hat{\Pi}\left(X^A(z)\frac{\partial}{\partial z^A}\right)=X^A(z)\Pi_A^B(z)\frac{\partial}{\partial z^B}\,. \tag{2.13}$$

Later on we call $(\mathbf{\Pi}(z),\mathbf{\Psi}(z))$ or equivalently $(\hat{\Pi}(z),\mathbf{\Psi}(z))$ an *odd normal pair* if $\mathbf{\Psi}(z)$ and $\mathbf{\Pi}(z)$ are defined by (2.10—2.12).

3.Divergence-like operator on odd normal pairs

In this section for an odd normal pair $(\mathbf{\Pi}(z),\mathbf{\Psi}(z))$ in an odd symplectic superspace we construct a first-order divergence-like differential operator which transforms it to a function on this superspace.

Let in a superspace E, $\mathbf{X}(z)$ and $\hat{L}(z)$ be a vector field and a linear operators field respectively, defined on T_zE. If $\{z^A\}$ are arbitrary coordinates then for the pair $(\hat{L},\mathbf{X})$ one can consider the function which depends on the coordinate system $\{z^A\}$:

$$\partial(\hat{L},\mathbf{X})^{\{z\}}=\frac{\partial X^A(z)}{\partial z^B}L_A^B(z)(-1)^{\tilde{X}\tilde{B}+\tilde{B}}\,. \tag{3.1}$$

Expression (3.1) is invariant under linear transformations of the coordinates $\{z^A\}$. In the general case if $\{w^A\}$ and $\{z^A\}$ are two different coordinate systems on E then for the pair $(\hat{L},\mathbf{X})$ we consider

$$\Gamma(\hat{L},\mathbf{X})_{\{z\}}^{\{w\}}=\partial(\hat{L},\mathbf{X})^{\{w\}}-\partial(\hat{L},\mathbf{X})^{\{z\}}\,. \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$\begin{aligned}&\Gamma(\hat{L},\mathbf{X})_{\{z\}}^{\{w\}}=X^Q(z)\Gamma_{QB}^A(z\,|\{w\},\{z\})L_A^B(z)\,,\\ &\Gamma_{BC}^A(z\,|\{w\},\{z\})=\tfrac{\partial^2w^K(z)}{\partial z^B\partial z^C}\tfrac{\partial z^A(w)}{\partial w^K}(-1)^{\tilde{C}}\,.\end{aligned} \tag{3.3}$$

(In (3.3) the components of $\hat{L}$ and $\mathbf{X}$ are in the coordinates $\{z^A\}$.)

From definition (3.2) of the $\Gamma(\hat{L},\mathbf{X})_{\{z\}}^{\{w\}}$ it follows that for three different coordinate systems $\{w^A\}$, $\{z^A\}$ and $\{u^A\}$

$$\Gamma(\hat{L},\mathbf{X})_{\{z\}}^{\{w\}}+\Gamma(\hat{L},\mathbf{X})_{\{u\}}^{\{z\}}+\Gamma(\hat{L},\mathbf{X})_{\{w\}}^{\{u\}}=0\,. \tag{3.4}$$

Let $\mathcal{F}$ be a class of some coordinate systems such that for a given pair $(\hat{L},\mathbf{X})$

$$\forall\{w\},\{\tilde{w}\}\in\mathcal{F}\quad\Gamma(\hat{L},\mathbf{X})_{\{w\}}^{\{\tilde{w}\}}=0\,. \tag{3.5}$$

Then to the class $\mathcal{F}$ one can relate the first-order divergence-like differential operator $\mathcal{D}$:

$$\mathcal{D}(\hat{L},\mathbf{X})=\partial(\hat{L},\mathbf{X})^{\{z\}}+\Gamma(\hat{L},\mathbf{X})_{\{z\}}^{\{w\}}\,, \tag{3.6}$$

where $\{z^A\}$ are arbitrary coordinates on E and $\{w^A\}$ are arbitrary coordinates from the class $\mathcal{F}$. From (3.2, 3.5) it follows that the r.h.s. of (3.6) does not depend on the choice of these coordinates.

Before going to the considerations for an odd symplectic superspace we consider an example where we come to the standard definition of the divergence in superspace using (3.1—3.6).

Example 1. Let E be a superspace with a volume form $d\mathbf{v}$ which in coordinates $\{w_0^A\}=\{x^1,\dots,x^n,\theta^1,\dots,\theta^m\}$ on E is equal to

$$d\mathbf{v}=dx^1\dots dx^nd\theta^1\dots d\theta^m. \tag{3.7}$$

We define $\mathcal{F}$ as a class of coordinate systems in which the volume form $d\mathbf{v}$ is given by (3.7):

$$\mathcal{F}=\{\,\{w\}\colon\,Ber\left(\frac{\partial w}{\partial w_0}\right)=1\,\}. \tag{3.8}$$

($Ber A$ is the superdeterminant of A.) It is easy to see that if $\hat{L}=\mathbf{id}$ is identity operator and $\mathbf{X}$ is an arbitrary vector field then in arbitrary coordinates $\{z^A\}$

$$\Gamma(\hat{L},\mathbf{X})_{\{z\}}^{\{w\}}=X^Q(z)\frac{\partial^2w^K(z)}{\partial z^Q\partial z^A}\frac{\partial z^A(w)}{\partial w^K}(-1)^{\tilde{A}}=X^Q(z)\frac{\partial\log\rho(z)}{\partial z^Q}\,, \tag{3.9}$$

where $\{w^A\}$ are arbitrary coordinates from the class (3.8) and $\rho(z)dz^1\dots dz^{m+n}$ is the volume form (3.7) in the coordinates $\{z^A\}$ ($\rho=Ber(\partial w/\partial z)$). The condition (3.5) is fulfilled and we come to the standard definition of the divergence in a superspace endowed with volume form. For the pair $(\mathbf{id},\mathbf{X})$ and for class (3.8) the operator $\mathcal{D}(\mathbf{id},\mathbf{X})$ is the divergence of the vector field $\mathbf{X}$ corresponding to the volume form $d\mathbf{v}$:

$$\mathcal{D}(\mathbf{id},\mathbf{X})=\frac{\partial X^A(z)}{\partial z^A}(-1)^{\tilde{X}\tilde{A}+\tilde{A}}+X^A\frac{\partial\log\rho(z)}{\partial z^A}=div_{d\mathbf{v}}\mathbf{X}\,. \tag{3.10}$$

Now we return to the considerations of Section 2.

For the superspace $E^{n.n}$ endowed with the odd symplectic structure we consider a field $(\hat{\Pi}(z), \Psi(z))$, where $(\hat{\Pi}(z), \Psi(z))$ is an odd normal pair in a vicinity of some point. (See the end of the previous section.) We denote by $\mathcal{F}_D$ the class of Darboux coordinates (2.8, 2.9) on $E^{n.n}$ and apply constructions (3.1—3.6) in this case.

Lemma. *If $(\hat{\Pi}(z), \Psi(z))$ is an odd normal pair in $E^{n.n}$ then*

$$\forall\{w\}, \{\tilde{w}\} \in \mathcal{F}_D, \quad \Gamma(\hat{\Pi}, \Psi)^{\{\tilde{w}\}}_{\{w\}} = 0. \tag{3.11}$$

Using the statement of the Lemma we consider the action of the operator $\mathcal{D}_{can}$ corresponding to the class $\mathcal{F}_D$ of Darboux coordinates by (3.6), on the odd normal pair $(\hat{\Pi}(z), \Psi(z))$:

$$\mathcal{D}_{can}(\hat{\Pi}, \Psi) = \left(\frac{\partial \Psi^A(z)}{\partial z^B} + \Psi^Q(z)\Gamma^A_{QB}(z\,|\{w\},\{z\})\right)\Pi^B_A(z), \tag{3.12}$$

where $\{w^A\}$ are arbitrary Darboux coordinates. From (3.6) and the Lemma follows

Theorem. *For an odd normal pair $(\hat{\Pi}(z), \Psi(z))$, $\mathcal{D}_{can}(\hat{\Pi}, \Psi)$ is an invariant geometrical object.*
In particular if $\{w^A\}$ are arbitrary Darboux coordinates then

$$\mathcal{D}_{can}(\hat{\Pi}, \Psi) = \frac{\partial \Psi^A(w)}{\partial w^B}\Pi^B_A(w) \tag{3.13}$$

does not depend on the choice of Darboux coordinates $\{w^A\}$. Before proving the Lemma we will consider

Example 2. Let $E^{1.1}$ be a (1.1)-dimensional superspace endowed with odd symplectic structure (2.1). Let $w = (x, \theta)$ be some Darboux coordinates on it: $\{x, \theta\} = 1, \{x, x\} = 0$. It is easy to see that in this case the odd vector field obeying to (2.10) is of the form

$$\Psi = \Psi(x, \theta)\frac{\partial}{\partial \theta}, \tag{3.14}$$

where $\Psi(x, \theta)$ is non-nilpotent even function. The projector operator (2.12) is evidently identity operator. Hence a normal pair is of the form $(\mathbf{id}, \Psi(x,\theta)\partial_\theta)$ and

$$\mathcal{D}_{can}(\mathbf{id}, \Psi) = \frac{\partial \Psi(x, \theta)}{\partial \theta}. \tag{3.15}$$

One can see that if x', θ' are some other Darboux coordinates then they are related with coordinates x, θ by canonical transformation

$$x' = f(x), \quad \theta' = \frac{\theta}{df(x)/dx} + \beta(x), \tag{3.16}$$

where $f(x)$ and $\beta(x)$ are even and odd functions on $E^{1.1}$ respectively. (To obtain (3.16) one has to note that in $E^{1.1}$ $\{x', x'\} = 0 \to x'_\theta = 0$.) It is easy to see from (3.16) that $\Psi = \Psi(x,\theta)\partial/\partial\theta = (\Psi(x,\theta)/f_x)\partial/\partial\theta'$ and (3.15) does not change

under transformation (3.16), so in this case the statements of the Lemma and of the Theorem hold.

Indeed, for this case one can say more about (3.15). Let (y, η) be an arbitrary coordinates on the $E^{1.1}$ and a volume form $\mathbf{dv}$ on $E^{1.1}$ is defined by the equation

$$\mathbf{dv} = \frac{dy d\eta}{\{y, \eta\}}. \tag{3.17}$$

Then one can check using (3.10) that $\mathcal{D}_{can}(\mathbf{id}, \Psi)$ in (3.15) is the divergence of the vector field Ψ by the volume form (3.17). For a vector field (3.14) and a volume form (3.17) $div_{\mathbf{dv}}\Psi$ does not depend on the choice of coordinates (y, η). *Proof* of the Lemma. Let

$$z^A = w^B L^A_B + w^B w^C T^A_{BC} + o(w^2) \tag{3.18}$$

be arbitrary canonical transformation from Darboux coordinates $\{z\}$ to Darboux coordinates $\{w\}$ in a vicinity of the point $z = 0$. Let $(\hat{\Pi}(z), \Psi(z))$ be an odd normal pair which is defined also in a vicinity of the point $z = 0$. For proving the Lemma we have to show that for transformation (3.18)

$$\Gamma(\hat{\Pi}, \Psi)^{\{w\}}_{\{z\}}\Big|_{z=0} = 0. \tag{3.19}$$

(We can consider (3.18) without loss of generality, since in Darboux coordinates the translation is obviously the canonical transformation.) We include the Darboux transformation (3.18) in the chain of Darboux transformations:

$$\{z\} \xrightarrow{\text{linear}} \{\tilde{z}\} \longrightarrow \{w\} \xrightarrow{\text{linear}} \{\tilde{w}\} \tag{3.20}$$

which obey to the following conditions:

a) The transformation $\{z\} \to \{\tilde{z}\}$ is the linear canonical transformation such that in the Darboux coordinates $\tilde{z}^A = (\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{\theta}^1, \ldots, \tilde{\theta}^n)$ $\hat{\Pi}|_{z=0}$ projects $TE^{n.n}_z|_{z=0}$ on the subspace which is generated by the vectors $\partial/\partial\tilde{x}^1, \partial/\partial\tilde{\theta}^1$:

$$\hat{\Pi}\Big|_{z=0}\frac{\partial}{\partial \tilde{x}^1} = \frac{\partial}{\partial \tilde{x}^1}, \quad \hat{\Pi}\Big|_{z=0}\frac{\partial}{\partial \tilde{\theta}^1} = \frac{\partial}{\partial \tilde{\theta}^1}. \tag{3.21}$$

b) The transformation $\{w\} \longrightarrow \{\tilde{w}\}$ is the linear canonical transformation such that

$$\tilde{w}^A = \tilde{z}^A + o(\tilde{z}). \tag{3.22}$$

From (3.2, 3.4) it follows that

$$\Gamma(\hat{\Pi}, \Psi)^{\{w\}}_{\{z\}} = \Gamma(\hat{\Pi}, \Psi)^{\{w\}}_{\{\tilde{w}\}} + \Gamma(\hat{\Pi}, \Psi)^{\{\tilde{w}\}}_{\{\tilde{z}\}} + \Gamma(\hat{\Pi}, \Psi)^{\{\tilde{z}\}}_{\{z\}}. \tag{3.23}$$

But $\Gamma(\hat{\Pi}, \Psi)^{\{w\}}_{\{\tilde{w}\}}$ and $\Gamma(\hat{\Pi}, \Psi)^{\{\tilde{z}\}}_{\{z\}}$ are zero because the corresponding transformations are linear. To prove that $\Gamma(\hat{\Pi}, \Psi)^{\{\tilde{w}\}}_{\{\tilde{z}\}}|_{z=0}$ is also zero we note that from (3.21) and

(2.10) follows that $\Psi^{\{\tilde{z}\}}|_{z=0} = \Psi\partial/\partial_{\tilde{\theta}^1}$. (Compare with (3.14)). Hence from (3.3), from the property (3.22) of the transformation $\tilde{w} \to \tilde{z}$ and from (2.9) follows that

$$\Gamma(\hat{\Pi}, \Psi)^{\{\tilde{w}\}}_{\{\tilde{z}\}}\Big|_{z=0} = \Psi \frac{\partial^2 x'^1}{\partial\tilde{\theta}^1 \partial\tilde{x}^1}\Big|_{z=0} = \frac{\Psi}{2}\frac{\partial}{\partial\tilde{x}^1}\{x'^1, x'^1\}\Big|_{z=0} = 0, \qquad (3.24)$$

where $\tilde{w}^A = (x'^1, \dots, x'^n, \theta'^1, \dots, \theta'^n)$. Hence (3.19) is obeyed.
Lemma is proved.

Considering a pair $(\hat{L}(z), \mathbf{X}(z))$ we constructed in this section the divergence-like operator (3.12, 3.13) in an odd symplectic superspace in the case when $\hat{L}$ is a projector on an (1.1)-dimensional symplectic subspace and $\mathbf{X}$ is an odd vector in it which is symplectoorthogonal to itself. In the case of even symplectic structure this construction can be carried out in a more general case and it is trivial because in this case there exists a volume form corresponding to the symplectic structure. Indeed if in a superspace $E^{2m.n}$ endowed with even symplectic structure there is given a pair $(\hat{L}(z), \mathbf{X}(z))$, where $\hat{L}(z)$ is a symplectoorthogonal projector on $(2p.q)$-dimensional symplectic subspaces in $T_z E^{2m.n}$ and $\mathbf{X}(z)$ is an arbitrary vector field then to the class $\mathcal{F}$ of the Darboux coordinates on this superspace there corresponds $\mathcal{D}(\hat{L}, \mathbf{X})$ defined by (3.6) which is a straightforward generalization of (1.4).

4. Truncated divergence.

We consider in this section an odd vector field which is defined at the points of $(n-1.n-1)$-dimensional nondegenerate supersurface embedded in the $(n.n)$-dimensional odd symplectic superspace with a volume form. In case of this vector field being symplectoorthogonal to this supersurface and to itself using the geometrical object $\mathcal{D}_{can}(\hat{\Pi}, \Psi)$ for an odd normal pair we define the linear operator on it (truncated divergence) whose action on this field gives the function on this supersurface.

Let M be an arbitrary nondegenerate $(n-1.n-1)$-dimensional supersurface embedded in a superspace $E^{n.n}$ which is endowed with odd symplectic structure (2.1) and the volume form dv:

$$dv = \rho(z) dz^1 \dots dz^{2n}. \qquad (4.1)$$

(The supersurface M is nondegenerate if the symplectic structure of $E^{n.n}$ induces nondegenerate symplectic structure on M.) Let $z^A = z^A(\zeta^\alpha)$ be a local parametrization of the supersurface M. The induced two-form $\Omega_{\alpha\beta} d\zeta^\alpha d\zeta^\beta$ on M according to (2.2–2.4) is given by the following equation:

$$\Omega_{\alpha\beta}(z(\zeta)) = \Omega\left(\partial_\alpha z^A \frac{\partial}{\partial z^A}, \partial_\beta z^B \frac{\partial}{\partial z^B}\right) = \partial_\alpha z^A(\xi)\Omega_{AB}(z(\xi))\partial_\beta z^B(\xi)(-1)^{\tilde{B}\tilde{\beta}+\tilde{\beta}}. \qquad (4.2)$$

Hereafter we use notations $\partial_\alpha z^A = \frac{\partial z^A}{\partial \zeta^\alpha}, \partial_\alpha f = \frac{\partial f}{\partial \zeta^\alpha}, \dots$ for derivatives along supersurface.

The induced Poisson bracket structure on M $\{\,,\}_M$ according to (2.5, 2.6) is defined by the matrix $\Omega^{\alpha\beta}$ which is inverse to the matrix $\Omega_{\alpha\beta}$. Using this induced symplectic structure one can construct the symplectoorthogonal projector on TM, $\hat{P}(\zeta)$: $T_{z(\zeta)}E^{n.n} \longrightarrow T_{z(\zeta)}M$ which can be expressed in terms of Ω_{AB} and $\Omega^{\alpha\beta}$:

$$P^B_A(\zeta) = \Omega_{AK}(z(\zeta)) \cdot \{z^K(\zeta), z^B(\zeta)\}_M = \Omega_{AK}(z(\zeta))\partial_\alpha z^K (-1)^{\tilde{K}\tilde{\alpha}+\tilde{\alpha}}\Omega^{\alpha\beta}\partial_\beta z^B. \qquad (4.3)$$

The operator

$$\hat{\Pi}(\zeta) = \mathbf{id} - \hat{P}(\zeta) \qquad (4.4)$$

at every point $z(\zeta)$ is a symplectoorthogonal projector on the (1.1)-dimensional subspace $\mathbf{\Pi}(z(\zeta))$ in $T_{z(\zeta)}E$ which is symplectoorthogonal and transversal to $T_{z(\zeta)}M$ because M is non-degenerate. (See 2.10–2.12.)

We consider an odd vector field $\Psi(\zeta)$ defined at the points of M which is symplectoorthogonal to M, to itself and which is non-degenerate:

$$\Psi(\zeta) \in \mathbf{\Pi}(z(\zeta)),\ \hat{P}\Psi = 0,\ \Omega(\Psi, \Psi) = 0, \qquad (4.5)$$

$$\Psi^A P^B_A = 0,\ \Psi^A \Omega_{AK}\partial_\alpha z^K (-1)^{\tilde{K}\tilde{\alpha}+\tilde{\alpha}} = 0,\ \partial_\alpha z^A P^B_A = \partial_\alpha z^B. \qquad (4.6)$$

For example if the supersurface M is defined by equations $f(z) = 0, \varphi(z) = 0$, then for arbitrary point z_0 on this supersurface, the vectors $\mathbf{D}_\varphi(z_0)$ and $\mathbf{D}_f(z_0)$ (see 2.6) are the basis vectors of the subspace $\mathbf{\Pi}(z_0)$ and the odd vector $\Psi = (2\mathbf{D}_f + (\{f,f\}/\{f,\varphi\})\mathbf{D}_\varphi)\big|_{z_0}$ is subject to the conditions (4.5).

One can see that the vector field obeying to the conditions (4.5) is fixed uniquely up to the multiplication by an even non-nilpotent function of ζ, $\Psi(\zeta) \to f(\zeta)\Psi(\zeta)$. In other words (4.5) define linear subbundle $SN(M)$ in $TE^{n.n}|_M$.

A field Ψ obeying to these conditions and the projector $\hat{\Pi}$ form an odd normal pair $(\hat{\Pi}, \Psi)$ at the points $z(\zeta)$ of the supersurface M.

Now on odd vector fields, sections of the bundle $SN(M)$, we will define the linear operator "truncated divergence". For this purpose we consider in a vicinity of a given point $z(\zeta)$ an odd normal pair $(\tilde{\hat{\Pi}}(z), \tilde{\Psi}(z))$ in $E^{n.n}$ which is a prolongation of the odd normal pair $(\hat{\Pi}(\xi), \Psi(\xi))$: $\tilde{\Psi}|_M = \Psi, \tilde{\hat{\Pi}}|_M = \hat{\Pi}$ and define the truncated divergence in the following way:

$$Div_{trunc}(\Psi(\zeta)) = \left(div_{dv}\tilde{\Psi} - \mathcal{D}_{can}(\tilde{\hat{\Pi}}, \tilde{\Psi})\right)\Big|_M. \qquad (4.7)$$

In (4.7) $\mathcal{D}_{can}(\tilde{\hat{\Pi}}, \tilde{\Psi})$ is given by (3.12, 3.13) for the odd normal pair $(\tilde{\hat{\Pi}}, \tilde{\Psi})$, $div_{dv}\tilde{\Psi}$ is divergence (3.10) of the vector field corresponding to the volume form dv defined by (4.1).

One can see that the r.h.s. of equation (4.7) indeed does not depend on the prolongation $(\tilde{\hat{\Pi}}, \tilde{\Psi})$ of the odd normal pair $(\hat{\Pi}, \Psi)$, moreover (4.7) is zeroth order linear differential operator:

$$Div_{trunc}(f(\zeta)\Psi(\zeta)) = f(\zeta) \cdot Div_{trunc}(\Psi(\zeta)), \qquad (4.8)$$

because Ψ is symplectoorthogonal to M.

Using the relation (4.4) between operators $\hat{\Pi}$ and $\hat{P}$, the equation (4.3) for $\hat{P}$, the conditions (4.6) we arrive by straightforward calculations at the following expression for (4.7):

$$Div_{trunc}(\Psi(\zeta)) =$$

$$\Psi^A(\zeta)\cdot\frac{\partial\log\rho(w)}{\partial w^A}\Big|_{w(\zeta)} - \Psi^A(\zeta)I_{AB}\partial_\beta\partial_\alpha w^B(\zeta)\Omega^{\alpha\beta}(w(\zeta))(-1)^{\tilde{B}(\tilde{\alpha}+\tilde{\beta})+\tilde{\alpha}}. \quad (4.9)$$

Here $\Psi(\zeta) = \Psi^A(\zeta)\partial/\partial w^A$ is an odd vector field obeying to (4.5), $\{w^A\}$ are arbitrary Darboux coordinates on $E^{n.n}$, I_{AB} is the matrix of symplectic two-form in Darboux coordinates (see 2.8), $\rho(w)$ is the density of the volume form (4.1) in these coordinates and $w^A(\zeta)$ is the parametrization of non-degenerate supersurface $M^{n-1.n-1}$.

In the case if there exist Darboux coordinates $\{w^A\}$ in which the volume form (4.1) obeys to condition

$$dv = dw^1\dots dw^{2n}, \quad (4.10)$$

i.e. its density $\rho(w)\equiv 1$ then

$$Div_{trunc}(\Psi(\zeta)) = -\Psi^A(\zeta)I_{AB}\partial_\beta\partial_\alpha w^B(\zeta)\Omega^{\alpha\beta}(w(\zeta))(-1)^{\tilde{B}(\tilde{\alpha}+\tilde{\beta})+\tilde{\alpha}}. \quad (4.11)$$

The analogous construction in Riemannian geometry in the case if a vector field is tangent to surface is reduced to the divergence along a surface and in the case if a vector field is orthogonal to surface is reduced to a linear operator related with the second quadratic form of this surface. (See the 6-th Section).

5. The odd invariant semidensity.

Now we are well prepared for writing the formula for odd invariant semidensity using constructions (4.7–4.11) for truncated divergence. The constructions of this section are founded on the following remark. Let $M^{n-1.n-1}$ be an arbitrary nondegenerate $(n-1.n-1)$-dimensional supersurface embedded in the odd symplectic superspace $E^{n.n}$ and a field Ψ on this supersurface obeys to conditions (4.5). Then the r.h.s. of (4.7,4.9) by its definition is invariant under coordinate transformations of the superspace $E^{n.n}$ and does not depend on the parametrization $z^A = z^A(\zeta^\alpha)$ of the supersurface $M^{n-1.n-1}$. So the equations (4.7,4.9) define a density of the weight $\sigma = 0$ on this supersurface. Moreover, if Ψ is a density of an arbitrary weight σ which is defined on non-degenerate $(n-1.n-1)$-dimensional non-degenerate supersurfaces and takes values in the odd vector fields obeying to (4.5) then the truncated divergence (4.9) of this density is the density of the same weight σ which is defined on $(n-1.n-1)$-dimensional nondegenerate supersurfaces and takes numerical values.

Let, as in Section 4, $E^{n.n}$ be a superspace which is endowed with odd symplectic structure (2.1) and volume form (4.1), and $M^{n-1.n-1}$ be an arbitrary nondegenerate

supersurface in this odd symplectic superspace. Now we will construct the semidensity on the M which takes values in the odd vectors Ψ obeying to conditions (4.5). Let the vectors $(\mathbf{e}_1,\dots,\mathbf{e}_{n-1};\mathbf{f}_1,\dots,\mathbf{f}_{n-1})$ constitute a basis of the tangent space $T_{z(\zeta)}M$ at arbitrary point $z(\zeta)$ of the supersurface $M^{n-1.n-1}$ ($\mathbf{e}_i$ are even vectors and $\mathbf{f}_i$ are odd ones). Let $\Psi(z(\zeta))$ and $\mathbf{H}(z(\zeta))$ be respectively an odd and an even vector fields which belong to $\mathbf{\Pi}(z(\zeta))$ (see (4.5)) such that $(\mathbf{\Pi}(z(\zeta)),\Psi(z(\zeta))$ form an odd normal pair:

$$\mathbf{H},\Psi\in\mathbf{\Pi},\quad \Omega(\Psi,\Psi)=0 \quad\text{and}\quad \Omega(\mathbf{H},\Psi)=1. \quad (5.1)$$

These conditions fix the vector fields $\mathbf{H}$ and Ψ up to the transformation

$$\mathbf{H}\to\frac{1}{\lambda}\mathbf{H}+\beta\Psi,\quad \Psi\to\lambda\Psi, \quad (5.2)$$

where λ is an arbitrary even function (taking values in non-nilpotent numbers) and β is an arbitrary odd function (compare with 3.16). Using transformation (5.2) one can choose the vector field Ψ (but not the vector field $\mathbf{H}$) in the unique way by imposing the normalization condition via volume form (4.1):

$$dv(\mathbf{e}_1,\dots,\mathbf{e}_{n-1},\mathbf{H};\mathbf{f}_1,\dots,\mathbf{f}_{n-1},\Psi)=1. \quad (5.3)$$

We arrive at the function

$$\Psi=\Psi(z(\zeta),\mathbf{e}_1,\dots,\mathbf{e}_{n-1};\mathbf{f}_1,\dots,\mathbf{f}_{n-1}), \quad (5.4)$$

which depends on points $z(\zeta)$ of the supersurface $M^{n-1.n-1}$ and the basis $(\mathbf{e}_1,\dots,\mathbf{e}_{n-1};\mathbf{f}_1,\dots,\mathbf{f}_{n-1})$ in the $T_{z(\zeta)}M^{n-1.n-1}$ and which takes values in odd vectors obeying to condition (4.5). This function is defined uniquely by conditions (5.1,5.3). It is easy to see that under the change of the basis the function (5.4) is multiplied by the *square root of the corresponding Berezinian.* For example if $\mathbf{e}_1\to\lambda\mathbf{e}_1$ and $\mathbf{f}_1\to\mu\mathbf{f}_1$ then $\Psi\to\sqrt{\frac{\lambda}{\mu}}\Psi$. Hence (5.4) defines semidensity. (It is interesting to note that these considerations for obtaining the formula for invariant vector–valued semidensity are similar to the considerations for obtaining the formula for the invariant density on the Lagrangian surfaces in $E^{n.n}$ suggested by A.S. Schwarz [13].).

If $z^A = z^A(\zeta^\alpha)$ is any parametrization of a supersurface $M^{n-1.n-1}$, where $\{\zeta^\alpha\} = (\xi^1,\dots,\xi^{n-1};\nu^1,\dots,\nu^{n-1})$ are even and odd parameters of this supersurface then considering as the basis vectors

$$\mathbf{e}_1=\frac{\partial z^A}{\partial\xi^1}\frac{\partial}{\partial z^A},\dots,\mathbf{e}_{n-1}=\frac{\partial z^A}{\partial\xi^{n-1}}\frac{\partial}{\partial z^A};\ \mathbf{f}_1=\frac{\partial z^A}{\partial\nu^1}\frac{\partial}{\partial z^A},\dots,\mathbf{f}_{n-1}=\frac{\partial z^A}{\partial\nu^{n-1}}\frac{\partial}{\partial z^A},$$

we come from (5.4) to odd vectors valued semidensity $\Psi(z(\zeta),\frac{\partial z}{\partial\zeta})$ of the rank $k=1$ on nondegenerate $(n-1.n-1)$-dimensional supersurfaces. The truncated divergence of this semidensity is the odd semidensity of the rank $k=2$. Using the formula (4.9) we arrive at this odd invariant semidensity:

$$A\left(w(\zeta),\tfrac{\partial w}{\partial\zeta},\tfrac{\partial^2 w^A}{\partial\zeta\partial\zeta}\right)=Div_{trunc}\Psi\left(w(\zeta),\tfrac{\partial w}{\partial\zeta}\right)=$$
$$\Psi^C\left(w(\zeta),\tfrac{\partial w}{\partial\zeta}\right)\cdot\left(-I_{CB}\partial_\beta\partial_\alpha w^B(\zeta)\Omega^{\alpha\beta}(w(\zeta))(-1)^{\tilde B(\tilde\alpha+\tilde\beta)+\tilde\alpha}+\tfrac{\partial\log\rho(w)}{\partial w^C}\Big|_{w(\zeta)}\right), \tag{5.5}$$

where $\{w^A\}$ are arbitrary Darboux coordinates in the $E^{n.n}$, $\rho(w)$ is the density of the volume form in these coordinates and $w(\zeta)$ is parametrization of supersurface $M^{n-1.n-1}$. In a case if there exist Darboux coordinates $\{w^A\}$ in which volume form $d\mathbf{v}$ obeys to condition (4.10) ($\rho(w)\equiv 1$) the formula (5.5) according to (4.11) is reduced to

$$A\left(w(\zeta),\frac{\partial w}{\partial\zeta},\frac{\partial^2 w}{\partial\zeta\partial\zeta}\right)=-\Psi^C\left(w(\zeta),\frac{\partial w}{\partial\zeta}\right)\cdot I_{CB}\partial_\beta\partial_\alpha w^B(\zeta)\Omega^{\alpha\beta}(w(\zeta))(-1)^{\tilde B(\tilde\alpha+\tilde\beta)+\tilde\alpha}. \tag{5.6}$$

The semidensity (5.5) is nothing but the semidensity (1.1) obtained in [6] because of its uniqueness. (See Ref.[6]). To compare (5.5) with (1.1) we do the following. Let M be $(n-1.n-1)$-dimensional arbitrary non-degenerate supersurface in $E^{n.n}$ and z_0 be any point in M. One can consider Darboux coordinates $\{w^A\}=(x^1,\dots,x^n,\theta^1,\dots,\theta^n)$ on $E^{n.n}$ and parameters $\{\zeta^\alpha\}=(\xi^1,\dots,\xi^{n-1},\nu^1,\dots,\nu^{n-1})$ such that this supersurface in the parametrization $w^A(\zeta^\alpha)$ is flat up to second order derivatives in a vicinity of z_0 : $x^n=o(\zeta^2),\theta^n=o(\zeta^2),x^i=\xi^i+o(\zeta^2),\theta^i=\nu^i+o(\zeta^2)$ if $1\le i\le n-1$ and $\rho(z_0)=1$. The vector valued semidensity (5.4) at the point z_0 in these coordinates and in this parametrization is equal to $\partial/\partial_{\theta^n}$, $\partial_\alpha\partial_\beta w^A|_{z_0}=0$, hence semidensity (5.5) at the point z_0 is equal to

$$\frac{\partial\log\rho}{\partial\theta^n}\Big|_{z=z_0}. \tag{5.7}$$

On the other hand to this supersurface M there correspond the functions $f=x^n+o(z^2)$ and $\varphi=\theta^n+o(z^2)$ in (1.1), hence the dual density $\tilde A$ in (1.1) at the point z_0 is equal to

$$\Delta f\Big|_{z=z_0}=div_{d\mathbf{v}}\mathbf{D}_f\Big|_{z=z_0}=\frac{\partial\log\rho}{\partial\theta^n}\Big|_{z=z_0}$$

and coincides with (5.7).

6. Discussions

In this section we consider analogues of the geometrical constructions (3.6), (4.7) and (5.5) in Riemannian geometry. 1) Let E^n be n-dimensional Riemannian space endowed with Riemannian metrics G. Then for a pair $(\hat L(x),\mathbf{h}(x))$, where $\hat L(x)$ and $\mathbf{h}(x)$ are a linear operator and vector field respectively, defined on T_xE^n, one can consider divergence-like operator

$$\mathcal{D}(\hat L,\mathbf{h})=Tr(\hat L\cdot\nabla\mathbf{h})=L^k_i\nabla_k h^i=\left(\frac{\partial h^i}{\partial x^k}+\Gamma^i_{kp}h^p\right)L^k_i, \tag{6.1}$$

where ∇ is covariant derivative corresponding to the metrics G and Γ^i_{km} are the components of the corresponding connection in the coordinates $\{x^i\}$. In a case if E is locally Euclidean, i.e., there exists a class $\mathcal{F}_{euc}$ of local coordinates in which $G_{ik}\equiv\delta_{ik}$, then it is easy to see that $\mathcal{D}(\hat L,\mathbf{h})$ in (6.1) is nothing but $\mathcal{D}_{euc}(\hat L,\mathbf{h})$ in (3.6) corresponding to the class $\mathcal{F}_{euc}$ and (3.3) defines the components of the trivial connection in the coordinates $\{x^i\}$.

Here like in a case of even symplectic structure (1.4) we do not need to put special conditions like (2.10, 2.11). 2) The construction analogous to (4.7) can be also carried out without special conditions on the dimension of embedded surface and on the vector field.

Let M be an arbitrary surface embedded in the Riemannian space E^n and $\mathbf{h}$ be an arbitrary vector field which is defined at the points of M.

It is useful to recall here the following standard formulae from Riemannian differential geometry [9]: if $\mathbf{u}_1$ and $\mathbf{u}_2$ are arbitrary vector fields which are defined on M and are tangent to M then

$$\begin{aligned}&\hat P(\nabla_{\mathbf{u}_1}\mathbf{u}_2)=(\nabla_M)_{\mathbf{u}_1}\mathbf{u}_2, &&\hat P\ \text{ is orthogonal projector } TE^n|_M\to TM,\\ &\hat\Pi(\nabla_{\mathbf{u}_1}\mathbf{u}_2)=\mathcal{A}(\mathbf{u}_1,\mathbf{u}_2), &&\hat\Pi\ \text{ is orthogonal projector } TE^n|_M\to TM^\perp,\end{aligned} \tag{6.2}$$

where ∇_M is covariant derivative on M induced from E^n and $\mathcal{A}(\mathbf{u}_1,\mathbf{u}_2)$ is bilinear symmetric form on the TM which takes values in the vectors orthogonal to M. To arbitrary vector field $\mathbf{h}$ which is defined at the points of M there correspond number–valued symmetric bilinear form $a_{\mathbf{h}}$ and related to it linear operator $A_{\mathbf{h}}$ on TM such that

$$a_{\mathbf{h}}(\mathbf{u}_1,\mathbf{u}_2)=G(\mathbf{u}_1,A_{\mathbf{h}}(\mathbf{u}_2))=G(\mathbf{h},\mathcal{A}(\mathbf{u}_1,\mathbf{u}_2)). \tag{6.3}$$

In the case if M is hypersurface and $\mathbf{n}$ is the field of unit orthogonal vectors then $a_{\mathbf{n}}$ is nothing but its second quadratic form. Trace of $A_{\mathbf{n}}$ defines the mean curvature at the points of M and determinant of $A_{\mathbf{n}}$ defines Gaussian curvature.

Using (6.1) we consider the following expression as an analogue of the formula (4.7) for truncated divergence:

$$Div_{trunc}\mathbf{h}=\left(div\tilde{\mathbf{h}}-Tr(\tilde{\hat\Pi}\cdot\nabla\tilde{\mathbf{h}})\right)\Big|_M=Tr(\tilde{\hat P}\cdot\nabla\tilde{\mathbf{h}})|_M, \tag{6.4}$$

where like in (4.7) $\tilde{\mathbf{h}},\tilde{\hat\Pi},\tilde{\hat P}$ are prolongations of $\mathbf{h},\hat\Pi,\hat P$ respectively, in a vicinity of M. From (6.4) and (6.2) it follows that

$$Div_{trunc}\mathbf{h}=\begin{cases}a) & div_M\mathbf{h}\quad\text{if } \mathbf{h} \text{ is tangent to } M\\ b) & -TrA_{\mathbf{h}}\quad\text{if } \mathbf{h} \text{ is orthogonal to } M\,.\end{cases} \tag{6.5}$$

In the components orthogonal projector $\hat P$ is given by the expression (compare it with (4.3))

$$P^k_i=G_{ij}\partial_\alpha x^j g^{\alpha\beta}\partial_\beta x^k, \tag{6.6}$$

where $x(\xi)$ is the parametrization of the surface M, $g_{\alpha\beta} = \partial_\alpha x^m G_{mn} \partial_\beta x^n$ is the metrics induced on the surface M and $g^{\alpha\beta} = (g)^{-1}_{\alpha\beta}$ is inverse metrics tensor.

So if **h** is orthogonal to M then (6.5b) in components has the form

$$Div_{trunc}(\mathbf{h}) = -h^i(\xi) G_{ij} \partial_\alpha \partial_\beta x^j(\xi) g^{\alpha\beta}(x(\xi)) \tag{6.7}$$

in the case if E^n is Euclidean and $\{x^i\}$ are cartesian coordinates on it ($G_{ij} = \delta_{ij}$). In the case if E^n is not Euclidean the eq. (6.7) also defines (6.5b) at any given point ξ_0 if coordinates $\{x^i\}$ are Euclidean at this point, i.e. $\Gamma^i_{km}\big|_{x(\xi_0)} = 0$.

We see that in case (6.5a) the truncated divergence is reduced to the induced divergence operator on M and in case (6.5b) it is reduced to zeroth order linear differential operator which corresponds to (4.9, 4.11). 3) Now we compare the odd semidensity constructed in Section 5 with the mean curvature of the hypersurfaces.

In analogy with considerations (5.1–5.4) of Section 5 one can consider on hypersurfaces in E^n invariant density $\mathbf{N}(x(\xi), \partial x/\partial\xi)$ of the rank $k = 1$ which takes values in the vectors orthogonal to hypersurfaces. By these conditions it is fixed uniquely (up to multiplication by a constant):

$$\mathbf{N}\left(x(\xi), \frac{\partial x}{\partial \xi}\right) = \mathbf{n}\left(x(\xi), \frac{\partial x}{\partial \xi}\right) \cdot \sqrt{det(g_{\alpha\beta})}\,, \tag{6.8}$$

where $\sqrt{det(g_{\alpha\beta})} = \sqrt{det(\partial_\alpha x^i G_{ik} \partial_\beta x^k)}$ is the density of the volume form dv_M induced on the hypersurface $x(\xi)$ and $\mathbf{n}(x(\xi), \frac{\partial x}{\partial \xi})$ is the unit vector orthogonal to the vectors $\partial x/\partial\xi$ which are tangent to hypersurface $x(\xi)$ at the point ξ.

Applying (6.4) to vector-valued density (6.8) and using (6.5b) we come in analogy with (5.5) to the density of the second rank corresponding to the mean curvature:

$$H\left(x(\xi), \frac{\partial x}{\partial \xi}, \frac{\partial^2 x}{\partial\xi\partial\xi}\right) = -Tr A_{\mathbf{n}} \cdot \sqrt{det(g_{\alpha\beta})}\,. \tag{6.9}$$

In the case if E^n is Euclidean and the coordinates $\{x^i\}$ are cartesian then according to (6.7)

$$H\left(x(\xi), \frac{\partial x}{\partial \xi}, \frac{\partial^2 x}{\partial\xi\partial\xi}\right) = -n^i G_{ij} \partial_\alpha \partial_\beta x^j(\xi) g^{\alpha\beta} \sqrt{det(g^{\alpha\beta})}, \quad (G_{ij} = \delta_{ij})\,. \tag{6.10}$$

(Compare it with (5.6).)

Here we want to note that in spite of the fact that odd semidensity (5.5) cannot be integrated over surfaces ($A^2 = 0$, so it leads to trivial volume form) one can consider the equation $A|_M \equiv 0$ which extracts the class of $(n-1.n-1)$ supersurfaces in an odd symplectic superspace endowed with a volume form. For example, from (5.5—5.7) it follows that to this class there belong supersurfaces which can be defined by equations $x^n = \theta^n = 0$, where $(x^1, \ldots, x^n, \theta^1 \ldots, \theta^n)$ are Darboux coordinates in

which the density of the volume form obeys to condition (4.10) (if such Darboux coordinates exist).

The analogous condition for mean curvature (6.10) $H|_M \equiv 0$ is the solution of the variational problem for minimal hypersurface.

Acknowledgments. I want to express my deep gratitude to my teacher A.S. Schwarz. Many years ago he taught me geometry and I tried to answer to some of his questions in Ref.[6] and also in this paper. I am grateful to D.A. Leites for interest to my work in odd symplectic geometry. When the paper [6] had appeared he asked me a question about the meaning of the invariant semidensity constructed therein. This paper can be considered as an answer to this question. I am deeply grateful to J-C. Hausmann for valuable discussions and to A.V.Karabegov, O.Piguet and Th. Voronov who encouraged me during the preparation of this work. The work was supported in part by Suisse National Science Foundation and by INTAS Grant No 95-0829.

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Received by Publishing Department
on December 30, 1998.