98-348

СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

98-348

E2-98-348

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NAMBU–POISSON REFORMULATION OF THE FINITE DIMENSIONAL DYNAMICAL SYSTEMS

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1998

1. Introduction

The Hamiltonian mechanics (HM) is in the ground of mathematical description of the physical theories, [1]. But HM is in a sense blind, e.g., it does not make difference between two opposites: the ergodic Hamiltonian systems (with just one integral of motion) and integrable Hamiltonian systems (with maximal number of the integrals of motion).

By our proposal, Nambu's mechanics (NM) [2] is proper generalization of the HM, which makes difference between dynamical systems with different numbers of integrals of motion explicit.

In this paper we introduce a system of nonlinear ordinary differential equations which in a particular case reduces to Volterra's system, [3] and integrate this system using Nambu–Poisson formalism, [2, 4].

In Sec.2 of this paper we introduce the dynamical system. In Sec.3 and Sec.4 we construct a complete set of integrals of motion in two particular cases for which we found the general solutions in quadratures. In Sec.5 we found some integrals of motion in the general case and present our conclusions.

2. The system

In this section we introduce the following dynamical system

$$\begin{aligned} &\dot{x}_n = \gamma_n \sum_{m=1}^{p} (e^{x_{n+m}} - e^{x_{n-m}}), \\ &1 \leq n \leq N,\ 1 \leq p \leq [(N-1)/2],\ 3 \leq N, \\ &x_{n+N} = x_n, \end{aligned} \tag{1}$$

where γ_n are real numbers, and $[a]$ means the integer part of a.
The system, (1) for $\gamma_n = 1$, $p = 1$ and $x_n = ln v_n$, becomes Voltera's system

$$\dot{v}_n = v_n(v_{n+1} - v_{n-1}), \tag{2}$$

then it is connected also to the Toda's lattice system, [5]

$$\dot{y}_n = e^{y_{n+1} - y_n} + e^{y_n - y_{n-1}}.$$

Indeed if

$$x_n = y_n - y_{n-1},$$

then

$$\dot{x}_n = e^{x_{n+1}} - e^{x_{n-1}}.$$

If $\gamma_n = 1$ and $p \geq 1$, the system (1) reduces to the so-called Bogoiavlensky lattice system, [6]

$$\dot{v}_n = v_n \sum_{m=1}^{p} (v_{n+m} - v_{n-m}). \tag{3}$$

For $N = 3$, $p = 1$ and arbitrary γ_n, (1) is connected to the system of three vortexes of two-dimensional ideal hydrodynamics, [7, 8].

3. The case of $N = 3, p = 1$

It is well known that the system of N vortexes can be described by the following system of differential equations, [7]

$$\dot{z}_n = i \sum_{m \neq n}^{N} \frac{\gamma_m}{z_n^* - z_m^*}, \tag{4}$$

where $z_n = x_n + iy_n$ are complex coordinate of the centre of n-th vortex.

For $N = 3$, it is easy to verify that the quantities

$$\begin{aligned} x_1 &= ln|z_2 - z_3|^2, \\ x_2 &= ln|z_3 - z_1|^2, \\ x_3 &= ln|z_1 - z_2|^2 \end{aligned} \tag{5}$$

satisfy the following system

$$\begin{aligned} \dot{x}_1 &= \gamma_1(e^{x_2} - e^{x_3}), \\ \dot{x}_2 &= \gamma_2(e^{x_3} - e^{x_1}), \\ \dot{x}_3 &= \gamma_3(e^{x_1} - e^{x_2}) \end{aligned} \tag{6}$$

after change of the time parameter as

$$dt = \frac{e^{(x_1+x_2+x_3)}}{4S} d\tau = e^{(x_1+x_2+x_3)/2} R d\tau, \tag{7}$$

where S is the area of the triangle with vertexes in the centres of the vortexes and R is the radius of the circle with the vortexes on it.

The system (6) has two integrals of motion

$$H_1 = \sum_{i=1}^{3} \frac{e^{x_i}}{\gamma_i}, \tag{8}$$

$$H_2 = \sum_{i=1}^{3} \frac{x_i}{\gamma_i}$$

and can be presented in the Nambu–Poisson form, [8]

$$\begin{aligned} \dot{x}_i &= \omega_{ijk} \frac{\partial H_1}{\partial x_j} \frac{\partial H_2}{\partial x_k} \\ &= \{x_i, H_1, H_2\} = \omega_{ijk} \frac{e^{x_j}}{\gamma_j} \frac{1}{\gamma_k}, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \omega_{ijk} &= \epsilon_{ijk} \rho, \\ \rho &= \gamma_1 \gamma_2 \gamma_3 \end{aligned} \tag{10}$$

and the Nambu–Poisson bracket of the functions A, B, C on the three-dimensional phase space is

$$\{A, B, C\} = \omega_{ijk} \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial x_j} \frac{\partial C}{\partial x_k}. \tag{11}$$

The fundamental bracket is

$$\{x_1, x_2, x_3\} = \omega_{ijk}. \tag{12}$$

Then we can again change the time parameter as

$$du = \rho d\tau \tag{13}$$

and obtain Nambu's mechanics, [8]

$$\dot{x}_i = \epsilon_{ijk} \frac{\partial H_1}{\partial x_j} \frac{\partial H_2}{\partial x_k}.$$

4. The next important case is $N = 4$ and $p = 1$,

$$\begin{aligned} \dot{x}_1 &= \gamma_1(e^{x_2} - e^{x_4}), \\ \dot{x}_2 &= \gamma_2(e^{x_3} - e^{x_1}), \\ \dot{x}_3 &= \gamma_3(e^{x_4} - e^{x_2}), \\ \dot{x}_4 &= \gamma_4(e^{x_1} - e^{x_3}). \end{aligned} \tag{14}$$

Like as $N = 3, p = 1$ case, for (14) we have two integrals of motion

$$H_1 = \frac{e^{x_1}}{\gamma_1} + \frac{e^{x_2}}{\gamma_2} + \frac{e^{x_3}}{\gamma_3} + \frac{e^{x_4}}{\gamma_4}, \tag{15}$$

$$H_2 = \frac{x_1}{\gamma_1} + \frac{x_2}{\gamma_2} + \frac{x_3}{\gamma_3} + \frac{x_4}{\gamma_4}. \tag{16}$$

For the integrability of the system (14), we need one more integral of motion, H_3. To find that integral let us take Nambu's form of the system (14)

$$\dot{x}_n = \{x_n, H_1, H_2, H_3\} = \gamma_1\gamma_2\gamma_3\gamma_4\epsilon_{nmkl}\frac{\partial H_1}{\partial x_m}\frac{\partial H_2}{\partial x_k}\frac{\partial H_3}{\partial x_l}. \tag{17}$$

We found from (17) a solution for H_3

$$H_3 = -\frac{1}{2}\left(\frac{x_1}{\gamma_1} - \frac{x_2}{\gamma_2} + \frac{x_3}{\gamma_3} - \frac{x_4}{\gamma_4}\right). \tag{18}$$

Because we already have three integrals of motion, we can integrate the system (14). From (16) and (18) we get

$$\begin{aligned} x_4 &= \gamma_4\left(\frac{H_2 + 2H_3}{2} - \frac{x_2}{\gamma_2}\right), \\ x_3 &= \gamma_3\left(\frac{H_2 - 2H_3}{2} - \frac{x_1}{\gamma_1}\right) \end{aligned} \tag{19}$$

and (15) gives us

$$\frac{e^{x_1}}{\gamma_1} + \frac{e^{x_2}}{\gamma_2} + e^{-x_1\frac{\gamma_3}{\gamma_1}}\frac{e^{\frac{\gamma_3}{2}(H_2-2H_3)}}{\gamma_3} + e^{-x_2\frac{\gamma_4}{\gamma_2}}\frac{e^{\frac{\gamma_4}{2}(H_2+2H_3)}}{\gamma_4} = H_1. \tag{20}$$

So x_2 is an implicit function of x_1, $x_2 = n_1(x_1, H_1, H_2, H_3)$. When

$$\frac{\gamma_4}{\gamma_2} = \pm 1, \pm 2, \pm 3, -4, \tag{21}$$

the function n_1 reduces to the composition of the elementary functions. When

$$\frac{\gamma_3}{\gamma_1} = \pm 1, \pm 2, \pm 3, -4, \tag{22}$$

we have x_1 as a superposition of elementary functions of x_2. Similarly we can consider the cases for the ratios $\frac{\gamma_3}{\gamma_2}$ and $\frac{\gamma_4}{\gamma_1}$.

Now we can solve the equation for x_1,

$$\dot{x}_1 = \gamma_1(e^{x_2} - e^{x_4}) = n_2(x_1), \tag{23}$$

by one quadrature,

$$N(x_1) = \int_{x_{10}}^{x_1}\frac{dx}{n_2(x)} = t - t_0. \tag{24}$$

5. Conclusions

As is well known, Nambu mechanics is a generalization of classical Hamiltonian mechanics introduced by Yoichiro Nambu, [2]. In [9, 10] it was demonstrated that several Hamiltonian systems possessing dynamical symmetries can be realized in the Nambu formalism of generalized mechanics.

In this paper we invented the system (1) and investigate the integrability properties of the particular cases of the system by elementary methods using Nambu–Poisson reformulation of Hamiltonian mechanics.

For the general case we have two integrals of motion for the system (1)

$$H_1 = \sum_{n=1}^{N}\frac{e^{x_n}}{\gamma_n}, \tag{25}$$

$$H_2 = \sum_{n=1}^{N}\frac{x_n}{\gamma_n}. \tag{26}$$

For even N, $N = 2M$, we found a third integral of motion

$$H_3 = \frac{1}{2}\sum_{n=1}^{2M}\frac{(-1)^n x_n}{\gamma_n}, \tag{27}$$

but when $N \geq 5$, for integrability, we need extra integrals of motion. The integrability properties of the system (1) in the general case are under investigation, [11].

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Received by Publishing Department
on December 16, 1998.