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ON THE INFRARED LIMIT OF UNCONSTRAINED $SU(2)$ YANG-MILLS THEORY

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The conventional perturbative treatment of gauge theories works successfully for the description of high energy phenomena, but fails in applications in the infrared region. Several different approaches [1]-[4] have been proposed for the nonperturbative reduction of gauge theories to the equivalent unconstrained system. The guideline of these investigations has been the search for a representation of the gauge invariant variables which are adapted to the study of the low energy phase of Yang-Mills theory. An alternative and very interesting approach has been proposed very recently by [5] where a topological soliton model with features relevant for the low energy region is argued to be extendable to full $SU(2)$ Yang-Mills theory. We shall discuss in this work how an effective low energy theory can be obtained directly from the unconstrained sytem.

In previous work [6] we obtained the unconstrained system equivalent to the degenerate $SU(2)$ Yang-Mills theory following the method of Hamiltonian reduction ([7, 8] and references therein) in the framework of the Dirac constraint formalism [9]. It has been shown that $SU(2)$ Yang-Mills theory can be reduced to the corresponding unconstrained system describing the dynamics of a positive definite symmetric 3×3 matrix Q^*. In this letter we separate the six physical dynamical field variables into three rotation invariant and three rotational ones. We shall obtain an effective low energy theory involving only two of the three rotational fields, summarized in a unit vector, and one of the tree scalar fields, and shall discuss its possible relation to the effective soliton Lagrangian proposed recently in [5].

As derived in [6] the dynamics of the physical variables of $SU(2)$ Yang-Mills theory can be described by the nonlocal Hamiltonian

$$H = \frac{1}{2}\int d^3x\Big[\,\mathrm{Tr}(P^*)^2 + \mathrm{Tr}(B^2(Q^*)) \; + \frac{1}{2}\vec{E}^2(Q^*,P^*)\Big] \qquad (1)$$

in terms of the unconstraint canonical pairs, the positive definite symmetric 3×3 matrices Q^* and P^*. The first term is the conventional quadratic "kinetic" part, the second the trace of the square of the non-Abelian magnetic field

$$B_{sk}(Q^*) = \epsilon_{klm}(\partial_l Q^*_{sm} + \frac{g}{2}\epsilon_{sbc} Q^*_{bl} Q^*_{cm}) \ . \qquad (2)$$

The third term in the Hamiltonian is the square of the vector $\vec{E}$ given as solution of the differential equation

$$\left[\gamma_{ik}(Q^*) - \frac{1}{g}\epsilon_{ikl}\partial_l\right] E_k = \mathcal{S}_i \qquad (3)$$

with $\gamma_{ik} := Q^*_{ik} - \delta_{ik}\mathrm{Tr}(Q^*)$ and the source term

$$\mathcal{S}_k(x) := \epsilon_{klm}\left(P^* Q^*\right)_{lm} - \frac{1}{g}\partial_l P^*_{kl} \ , \qquad (4)$$

which coincides with the spin density part of the Noetherian angular momentum up to divergence terms.

The solution $\vec{E}$ of the differential equation (3) can be expanded in $1/g$. The zeroth order term is

$$E^{(0)}_s = \gamma^{-1}_{sk}\epsilon_{klm}\left(P^* Q^*\right)_{lm} \ , \qquad (5)$$

and the first order term is determined via

$$E^{(1)}_s := \frac{1}{g}\gamma^{-1}_{sl}\left[(\mathrm{rot}\ \vec{E}^{(0)})_l - \partial_k P^*_{kl}\right] \qquad (6)$$

from the corresponding zeroth order term. The higher terms are obtained via the simple recurrence relations

$$E^{(n+1)}_s := \frac{1}{g}\gamma^{-1}_{sl}(\mathrm{rot}\ \vec{E}^{\ (n)})_l \ . \qquad (7)$$

Whereas the gauge fields transform as vectors under spatial rotations, the unconstrained fields Q^* and P^* transform as second rank tensors under spatial rotations.[1] In order to separate the three fields which are invariant under spatial rotations from the three rotational degrees of freedom we perform the following main axis transformation of the original positive definite symmetric 3×3 matrix field $Q^*(x)$

$$Q^*\left(\chi,\phi\right) = R^T(\chi(x))\mathcal{D}\left(\phi(x)\right) R(\chi(x)), \qquad (8)$$

with the orthogonal matrix $R(\chi)$ and the positive definite diagonal matrix

$$\mathcal{D}\left(\phi\right) := \mathrm{diag}\left(\phi_1,\phi_2,\phi_3\right) \ , \quad \phi_i > 0 \quad i = 1,2,3 \ , \qquad (9)$$

the unconstrained Hamiltonian can be written in the form

$$H = \frac{1}{2}\int d^3x \left(\sum_{i=1}^{3}\pi_i^2 + \frac{1}{2}\sum_{i=1}^{3}\mathcal{P}_i^2 + \frac{1}{2}\vec{\mathcal{E}}^{\ 2} + V\right) . \qquad (10)$$

Here the fields π_i are the canonically conjugate momenta to the diagonal fields ϕ_i and

$$\mathcal{P}_i(x) := \frac{\xi_i(x)}{\phi_j(x) - \phi_k(x)}, \quad (\textit{cyclic permut. } i \neq j \neq k) \qquad (11)$$

with the $SO(3)$ left-invariant Killing vectors

$$\xi_k(x) := \mathcal{M}(\theta,\psi)_{kl}p_{\chi_l} \ , \qquad (12)$$

where in terms of the Euler angles

$$\mathcal{M}(\theta,\psi) := \begin{pmatrix} \sin\psi/\sin\theta, & \cos\psi, & -\sin\psi\cot\theta \\ -\cos\psi/\sin\theta, & \sin\psi, & \cos\psi\cot\theta \\ 0, & 0, & 1 \end{pmatrix} . \qquad (13)$$

The electric field vector $\vec{\mathcal{E}}$ is given by an expansion in $1/g$ with the zeroth order term

$$\mathcal{E}^{(0)}_i := -\frac{\xi_i}{\phi_j + \phi_k} \quad (\textit{cycl. permut. } i \neq j \neq k) \ , \qquad (14)$$

[1] Note that for a complete analysis it is necessary to investigate the transformation properties of the field Q^* under the whole Poincaré group. We shall limit ourselves here to the isolation of the scalars under spatial rotations and treat Q^* in terms of "nonrelativistic spin 0 and spin 2 fields" in accordance with the conclusions obtained in the work [2].

the first order term given from $\mathcal{E}^{(0)}$ via

$$\mathcal{E}_i^{(1)} := -\frac{1}{g}\frac{1}{\phi_j+\phi_k}\left[\left((\nabla_{X_j}\mathcal{E}^{(0)})_k - (\nabla_{X_k}\mathcal{E}^{(0)})_j\right) + \Xi_i\right] \tag{15}$$

with cyclic permutations of $i \neq j \neq k$ and the higher order terms of the expansion determined via the recurrence relations

$$\mathcal{E}_i^{(n+1)} := -\frac{1}{g}\frac{1}{\phi_j+\phi_k}\left((\nabla_{X_j}\mathcal{E}^{(n)})_k - (\nabla_{X_k}\mathcal{E}^{(n)})_j\right) . \tag{16}$$

Here the components of the covariant derivatives ∇_{X_k} along the vector fields $X_k := R_{ki}\partial_i$

$$(\nabla_{X_k}\vec{\mathcal{E}})_b := X_k\mathcal{E}_b + \Gamma^d{}_{kb}\mathcal{E}_d \tag{17}$$

are determined by the connection

$$\Gamma^b{}_{ia} := \left(RX_iR^T\right)_{ab} = -\epsilon_{abs}(\mathcal{M}^{-1})_{sk}X_i\chi_k . \tag{18}$$

The "source" terms Ξ_k are given as

$$\begin{aligned}\Xi_1 &= \Gamma^1{}_{22}(\pi_1-\pi_2) + \frac{1}{2}X_1\pi_1 - \Gamma^2{}_{23}\mathcal{P}_2 - \Gamma^1{}_{23}\mathcal{P}_1 \\ &\quad -2\Gamma^1{}_{12}\mathcal{P}_3 + X_2\mathcal{P}_3 \; + \; (2 \leftrightarrow 3)\end{aligned} \tag{19}$$

and its cyclic permutaions Ξ_2 and Ξ_3.

The magnetic part V of the potential is

$$V[\phi,\chi] = \sum_{i=1}^{3} V_i[\phi,\chi] \tag{20}$$

with

$$\begin{aligned}V_1[\phi,\chi] &= \left(\Gamma^1{}_{12}(\phi_2-\phi_1) - X_2\phi_1\right)^2 + \left(\Gamma^1{}_{13}(\phi_3-\phi_1) - X_3\phi_1\right)^2 \\ &+ \left(\Gamma^1{}_{23}\phi_3 + \Gamma^1{}_{32}\phi_2 - g\phi_2\phi_3\right)^2\end{aligned} \tag{21}$$

and its cyclic permutations .

In the strong coupling limit the expression (10) for the unconstrained Hamiltonian reduces to [2]

$$H_S = \frac{1}{2}\int\left(\sum_{i=1}^{3}\pi_i^2 + \sum_{cycl.}\xi_i^2\frac{\phi_j^2+\phi_k^2}{(\phi_j^2-\phi_k^2)^2} + V[\phi,\chi]\right) \tag{22}$$

For the further investigation of the low energy properties of $SU(2)$ field theory a thorough understanding of the properties of the term in (20) containing no derivatives

$$V_{\rm hom}[\phi_i] = g^2[\phi_1^2\phi_2^2 + \phi_2^2\phi_3^2 + \phi_3^2\phi_1^2] \tag{23}$$

is crucial. The classical absolute minima of energy correspond to vanishing of the positive definite kinetic term in the Hamiltonian (22). The stationary points of the potential term (23) are

$$\phi_1 = \phi_2 = 0 , \quad \phi_3 \text{ arbitrary} \tag{24}$$

and its cyclic permutations. Analysing the second order derivatives of the potential at the stationarity points one can conclude that they form a continous line of degenerate absolute minima at zero energy. In other words the potential has a "valley" of zero energy minima along the line $\phi_1 = \phi_2 = 0$. They are the unconstrained analogs of the toron solutions [10] representing constant Abelian field configurations with vanishing magnetic field in the strong coupling limit. The special point $\phi_1 = \phi_2 = \phi_3 = 0$ corresponds to the ordinary perturbative minimum.

For the investigation of the configurations of higher energy it is necessary to include the rotational term in (22). Since the singular points of the rotational term just correspond to the absolute minima of the potential there will a competition between an attractive and a repulsive force. At the balance point we will have a local minimum corresponding to a classical configuration with higher energy.

[2] For spatially constant fields the integrand of this expression reduces to the Hamiltonian of $SU(2)$ Yang-Mills mechanics considered in previous work [8].

The above representation (8) in terms of scalar and rotational fields gives us furthermore the possibility to analyse the wellknown exact, but nonnormalizable, zero energy groundstate wave functional of $SU(2)$ gluodynamics [11] in the strong coupling limit. For the corresponding unconstrained Hamiltonian (1) it has been discussed in [6] and has the form

$$\Psi[Q^*] = \exp\left(-8\pi^2 W[Q^*]\right) \qquad (25)$$

with the winding number functional [12] $W[Q^*] := \int d^3x\ K_0(x)$ in terms of the zero component of the Chern-Simons vector [13], $K_0(Q^*) := -(16\pi^2)^{-1}\epsilon^{ijk}\mathrm{Tr}\left(F_{ij}Q_k^* - \frac{2}{3}gQ_i^*Q_j^*Q_k^*\right)$, written in terms of $Q_i^* := Q_{il}^*\tau_l$ with the Pauli matrices τ_i. In the strong coupling limit the groundstate wave functional (25) reduces to the very simple form

$$\Psi[\phi_1, \phi_2, \phi_3] = \exp\left[-g\phi_1\phi_2\phi_3\right] \ . \qquad (26)$$

It is nonnormalizable despite the sign definiteness of its exponent ($\phi_i > 0\ ,\ i = 1, 2, 3$). For the analysis of this wave function in the neighbourhood of the line $\phi_1 = \phi_2 = 0$ of minima of the classical potential (23), it is useful to pass from the variables ϕ_1 and ϕ_2 transverse to the valley to the new variables $\phi_\perp$ and γ via

$$\phi_1 = \phi_\perp \cos\gamma \qquad \phi_2 = \phi_\perp \sin\gamma \qquad \left(\phi_\perp \geq 0\ ,\ \ 0 \leq \gamma \leq \frac{\pi}{2}\right) \ . \qquad (27)$$

The classical potential then reads

$$V(\phi_3, \phi_\perp, \gamma) = g^2\left(\phi_3^2\phi_\perp^2 + \frac{1}{4}\phi_\perp^4 \sin^2(2\gamma)\right) \ , \qquad (28)$$

and the groundstate wave function (26) becomes

$$\Phi[\phi_3, \phi_\perp, \gamma] = \exp\left[-\frac{1}{2}g\phi_3\phi_\perp^2 \sin(2\gamma)\right] \ . \qquad (29)$$

We see that close to the bottom of the valley, for small $\phi_\perp$, the potential is that of a harmonic oscillator and the wave functional correspondingly a Gaussian with a maximum at the classical minimum line $\phi_\perp = 0$. The height of the maximum is constant along the valley. The non-normalizability of the groundstate wave function (26) is therefore due to the outflow of the wave function with constant values along the valley to arbitrarily large values of the field ϕ_3. The formation of condensates with macroscopically large fluctuations of the field amplitude might be a very interesting consequence of the properties of the classical potential. To establish the connection between this phenomenon and the model of the squeezed gluon condensate [14] will be an interesting task for further investigation.

We now would like to find the effective classical field theory to which the unconstrained theory reduces in the limit of infinite coupling constant g, if we assume that the classical system spontaneously chooses one of the classical zero energy minima of the leading order g^2 part (23) of the potential. As discussed above these classical minima include apart from the perturbative vacuum, where all fields vanish, also field configurations with one scalar field attaining arbitrary values. Let us therefore put without loss of generality (explicitly breaking the cyclic symmetry)

$$\phi_1 = \phi_2 = 0\ , \quad \phi_3\ -\ \text{arbitrary}\ , \qquad (30)$$

such that the potential (23) vanishes. In this case the part of the potential (20) containing derivatives takes the form

$$\begin{aligned} V_{\rm inh} \ = \ & \phi_3(x)^2[({\Gamma^2}_{13}(x))^2 + ({\Gamma^2}_{23}(x))^2 + ({\Gamma^2}_{33}(x))^2 + \\ & +({\Gamma^3}_{11}(x))^2 + ({\Gamma^3}_{21}(x))^2 + ({\Gamma^3}_{31}(x))^2] + \\ & +2\phi_3(x)[{\Gamma^3}_{31}(x)X_1\phi_3 + {\Gamma^3}_{32}(x)X_2\phi_3] + \\ & +[(X_1\phi_3)^2 + (X_2\phi_3)^2] \ . \end{aligned} \qquad (31)$$

Introducing the unit vector

$$n_i(\phi, \theta) := R_{3i}(\phi, \theta)\ , \qquad (32)$$

pointing along the 3-axis of the "intrinsic frame", one can write

$$V_{\rm inh} = \phi_3(x)^2\left(\partial_i\vec{n}\right)^2 + (\partial_i\phi_3)^2 - (n_i\partial_i\phi_3)^2 - (n_i\partial_i n_j)\partial_j(\phi_3^2) \ . \qquad (33)$$

Concerning the contribution from the nonlocal term in this phase, we obtain for the leading part of the electric fields

$$\mathcal{E}_1^{(0)} = -\xi_1/\phi_3 \ , \quad \mathcal{E}_2^{(0)} = -\xi_2/\phi_3 \ . \tag{34}$$

Since the third component $\mathcal{E}_3^{(0)}$ and $\mathcal{P}_3$ are singular in the limit $\phi_1, \phi_2 \to 0$, it is necessary to have $\xi_3 \to 0$. The assumption of a definite value of ξ_3 is in accordance with the fact that the potential is symmetric around the 3-axis for small ϕ_1 and ϕ_2, such that the intrinsic angular momentum ξ_3 is conserved in the neighbourhood of this configuration. Hence we obtain the following effective Hamiltonian up to order $O(1/g)$

$$\begin{aligned} H_{\text{eff}} &= \frac{1}{2}\int d^3x \Big[\pi_3^2 + \frac{1}{\phi_3^2}(\xi_1^2+\xi_2^2) + (\partial_i\phi_3)^2 + \phi_3^2(\partial_i\vec{n})^2 \\ &\qquad -(n_i\partial_i\phi_3)^2 - (n_i\partial_i n_j)\partial_j(\phi_3^2)\Big] \ . \end{aligned} \tag{35}$$

After the inverse Lagrangian transformation we obtain the corresponding nonlinear sigma model type effective Lagrangian for the unit vector $\vec{n}(t,\vec{x})$ coupled to the scalar field $\phi_3(t,\vec{x})$

$$\begin{aligned} L_{\text{eff}}[\phi_3,\vec{n}] &= \frac{1}{2}\int d^3x \Big[(\partial_\mu\phi_3^2)^2 + \phi_3^2(\partial_\mu\vec{n})^2 \\ &\qquad +(n_i\partial_i\phi_3)^2 + n_i(\partial_i n_j)\partial_j(\phi_3^2)\Big] \ . \end{aligned} \tag{36}$$

In the limit of infinite coupling the unconstrained field theory in terms of six physical fields equivalent to the original $SU(2)$ Yang-Mills theory in terms of the gauge fields A_μ^a reduces therefore to an effective classical field theory involving only one of the three scalar fields and two of the three rotational fields summarized in the unit vector $\vec{n}$. Note that this nonlinear sigma model type Lagrangian admits singular hedgehog configurations of the unit vector field $\vec{n}$. Due to the absence of a scale at the classical level, however, these are unstable. Consider for example the case of one static monopole

placed at the origin,

$$n_i := x_i/r \ , \quad \phi_3 = \phi_3(r) \ , \quad r := \sqrt{x_1^2+x_2^2+x_3^2} \ . \tag{37}$$

Minimizing its total energy E

$$E[\phi_3] = 4\pi \int dr \phi_3^2(r) \tag{38}$$

with respect to $\phi_3(r)$ we find the classical solution $\phi_3(r) \equiv 0$. There is no scale in the classical theory. Only in a quantum investigation a mass scale such as a nonvanishing value for the condensate $< 0|\hat{\phi}_3^2|0 >$ may appear, which might be related to the string tension of flux tubes directed along the unit-vector field $\vec{n}(t,\vec{x})$. The singular hedgehog configurations of such string-like directed flux tubes might then be associated with the glueballs. The pure quantum object $< 0|\hat{\phi}_3^2|0 >$ might be realized as a squeezed gluon condensate [14]. Note that for the case of a spatially constant condensate,

$$< 0|\hat{\phi}_3^2|0 > =: 2m^2 = const. \ , \tag{39}$$

the quantum effective action corresponding to (36) should reduce to the lowest order term of the effective soliton Lagangian discussed very recently by Faddeev and Niemi [5]

$$L_{\text{eff}}[\vec{n}] = m^2 \int d^3x (\partial_\mu\vec{n})^2 \ . \tag{40}$$

As discussed in [5], for the stability of these knots furthermore a higher order Skyrmion-like term in the derivative expansion of the unit-vector field $\vec{n}(t,\vec{x})$ is necessary. To obtain it from the corresponding higher order terms in the strong coupling expansion of the unconstrained Hamiltonian (10) is under present investigation.

In summary we have found a representation of the physical variables which is appropriate for the study of the infrared limit of $SU(2)$ Yang-Mills theory. We have shown how in the infrared limit an effective nonlinear sigma model type Lagrangian can be derived which out of the six physical fields involves only one of three scalar fields and two rotational fields summarized in a unit vector. The study of the corresponding quantum theory as well as the consideration of higher order terms in the strong coupling expansion will be the subject of future work.

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