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# HAMILTONIAN REDUCTION OF $SU(2)$ YANG-MILLS FIELD THEORY

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The degenerate character of the conventional Yang-Mills action for $SU(2)$ gauge fields $A_\mu^a(x)$

$$S[A] = -\frac{1}{4} \int d^4x \; F_{\mu\nu}^a F^{a\mu\nu} \tag{1}$$

leads to a restriction of the corresponding phase space spanned by the canonical variables $(A_0^a, P_0^a)$ as well as $(A_{ai}, E_{ai})$ due to the primary constraints $P_0^a(x) = 0$ and the secondary constraints, the non-Abelian Gauss law

$$\Phi_a := \partial_i E_{ai} + g\epsilon_{abc} A_{ci} E_{bi} = 0 \ , \tag{2}$$

Since they are first class

$$\{\Phi_a(x), \Phi_b(y)\} = g\epsilon_{abc}\Phi_c \delta(x-y) \ , \tag{3}$$

the dynamics of the system is not uniquely predictible. The main problem in the Hamiltonian formulation of Yang-Mills theories is to find the projection from the initial phase to the phase space of unconstrained gauge invariant variables with uniquely predictable dynamics. The conventional perturbative gauge fixing method [1] for solving this problem works successfully for the description of high energy phenomena, but fails in applications in the infrared region. The correct nonperturbative reduction of gauge theories [2]-[13], on the other hand, leads to representations for the unconstrained Yang-Mills systems which are valid also in the low energy region but unfortunately are very complicated for practical calculations. The problem is to state some practical form of the theory preserving all main properties of initial gauge theory which can applied directly to the solution of infrared problems. With this aim we follow the method of Hamiltonian reduction ([14] and references therein) in the framework of the the Dirac constraint formalism [15, 16]. In previous work [17] devoted to the case of the mechanics of spatially constant $SU(2)$ Dirac Yang-Mills fields we obtained the corresponding unconstrained system desribing the dynamics of a symmetric second rank tensor under spatial rotations.

In this letter we generalize our approach to field theory. We give a Hamiltonian formulation of classical $SU(2)$ Yang-Mills field theory entirely in terms of gauge invariant variables.

The non-Abelian character of the secondary constraints (3) is the main obstacle for the corresponding projection to the unconstrained phase space. The way to avoid this difficulty is to replace the non-Abelian constraints (3) by a new set of Abelian constraints $\Psi_\alpha$ which describe the same constraint surface[1]. For the new Abelian constraints $\Psi_\alpha$ the projection to the reduced phase space can be simply achieved in the following two steps. One performs a canonical transformation to new variables such that part of the new momenta $\overline{P}_\alpha$ coincide with the constraints $\Psi_\alpha$. After the projection onto the constraint shell, i.e. putting in all expressions $\overline{P}_\alpha = 0$, the coordinates canonically conjugate to the $\overline{P}_\alpha$ drop out from the physical quantities. The remaining canonical pairs are then gauge invariant and form the basis for the unconstrained system.

The problem of Abelianization is considerably simplified when studied in terms of coordinates adapted to the action of the gauge group. The knowledge of the local gauge transformations of the Yang-Mills action (1), $A_\mu \rightarrow A'_\mu = U^{-1}(x)\left(A_\mu - \frac{1}{g}\partial_\mu\right)U(x)$, directly promts us with the choice of adapted coordinates by using the following point transformation to the new set of Lagrangian coordinates $\overline{Q}_i$ $(i = 1, 2, 3)$ and $Q^*$

$$A_{ai}\left(\overline{Q}, Q^*\right) = O_{ak}\left(\bar{Q}\right) Q^*_{ki} - \frac{1}{2g}\epsilon_{abc}\left(O\left(\overline{Q}\right)\partial_i O^T\left(\overline{Q}\right)\right)_{bc}, \tag{4}$$

where $O$ is an orthogonal $3\times 3$ matrix and $Q^*$ is a positive definite symmetric $3\times 3$ matrix. [2] The first term on the right hand side of (4) corresponds to the wellknown polar representation for an arbitrary quadratic matrix [18]. The transformation (4) induces a point canonical transformation linear in the new canonical momenta $P^*_{ik}$ and $\overline{P}_i$. Using the corresponding generating functional depending on the old momenta and the new coordinates

$$F_3\left[E;\, \overline{Q}, Q^*\right] := \int d^3z\; E_{ai}(z) A_{ai}\left(\bar{Q}(z), Q^*(z)\right) \tag{5}$$

one can obtain the transformation to new canonical momenta $\overline{P}_i$ and $P^*_{ik}$

$$\overline{P}_j(x) \;:=\; \frac{\delta F_3}{\delta \overline{Q}_j(x)} = -\frac{1}{g}\Omega_{jr}\left(D_i(Q^*)O^T E\right)_{ri}, \tag{6}$$

$$P^*_{ik}(x) \;:=\; \frac{\delta F_3}{\delta Q^*_{ik}(x)} = \frac{1}{2}\left(E^T O + O^T E\right)_{ik}. \tag{7}$$

Here $\Omega_{ji} := (1/2)\epsilon_{lim}\left(O^T\left(\overline{Q}\right)\partial O\left(\overline{Q}\right)/\partial \overline{Q}_j\right)_{lm}$ is assumed to be invertible matrix and $D_i(Q^*)$ is the corresponding covariant derivative in the adjoint representation $(D_i(Q^*))_{mn} := \delta_{mn}\,\partial_i + g\epsilon_{mkn}\,Q^*_{ki}$. A straightforward calculation based on the linear relations (6) and (7) between old and new momenta leads to the the following expression for the field strengths $E_{ai}$ in terms of the new canonical variables

$$E_{ai} = O_{ak}\left(\overline{Q}\right)\left[P^*_{ki} + \epsilon_{kis}\,{}^*D^{-1}_{sl}(Q^*)\left[\left(\Omega^{-1}\overline{P}\right)_l - \mathcal{S}_l\right]\right]. \tag{8}$$

Here ${}^*D^{-1}$ denotes the inverse of the matrix operator ${}^*D_{ik}(Q^*) := \frac{1}{2}\epsilon_{imj}D_m(Q^*)_{jk}$ and

$$\mathcal{S}_k(x) := \epsilon_{klm}\left(P^*Q^*\right)_{lm} - \frac{1}{g}\partial_l P^*_{kl}. \tag{9}$$

Up to divergence terms this vector coincides with the spin density part of the Noetherian angular momentum $S_i(x) := \epsilon_{ijk}A^a_j E_{ak}$

[1] There are known several methods of the Abelianization of constraints (see e.g. [14, 16] and references therein).

[2] The freedom to use other canonical variables in the unconstrained phase space corresponds to another fixation of the six variables $Q^*$ in the representation (4). This observation clarifies the connection with the conventional gauge fixing method. We shall discuss this point in forthcoming publications (see also ref. [6]).

after transformation to the new variables and projection onto the constraint shell.

Using the representations (4) and (8) one can easily convince oneself that the variables $Q^*$ and $P^*$ make no contribution to the Gauss law constraints (2)

$$\Phi_a = O_{as}[\bar{Q}]\Omega^{-1}_{sj}\overline{P}_j = 0 \ . \tag{10}$$

The equivalent set of constraints

$$\overline{P}_a = 0 \tag{11}$$

is Abelian due to the canonical structure of the new variables. After having rewritten the model in terms of the new canonical coordinates and after the Abelianization of the Gauss law, the construction of the unconstrained Hamiltonian system is straightforward. In all expressions we can simply put $\overline{P} = 0$. In particular, the Hamiltonian in terms of the unconstrained canonical variables $Q^*$ and $P^*$ can be represented by the sum of three terms

$$H = \frac{1}{2}\int d^3x \Big[ \operatorname{Tr}(P^*)^2 + \operatorname{Tr}(B^2(Q^*)) \ + \frac{1}{2}\vec{E}^2(Q^*,P^*)\Big]. \tag{12}$$

The first term is the conventional quadratic "kinetic" part, the second the trace of the square of the non-Abelian magnetic field

$$B_{sk}(Q^*) = \epsilon_{klm}(\partial_l Q^*_{sm} + \frac{g}{2}\epsilon_{sbc}Q^*_{bl}Q^*_{cm}) \ . \tag{13}$$

The third term in the Hamiltonian is the square of the antisymmetric part $\vec{E}$ of the electric field (8) after projection onto the constraint surface and is given as the solution of the partial differential equations

$$^*D_{ls}(Q^*)E_s = g\mathcal{S}_l \tag{14}$$

It describes a nonlocal interaction of spin densities (9).

The electric field $\vec{E}$ can be expanded $E_s = \sum_{n=0}^{\infty} E_s^{(n)}$ in $1/g$, with the zeroth order term

$$E^{(0)}_s = \gamma^{-1}_{sk}\epsilon_{klm}\,(P^*Q^*)_{lm} \ , \tag{15}$$



where $\gamma_{ik} := Q^*_{ik} - \delta_{ik}\operatorname{Tr}(Q^*)$. The first order term is determined from the corresponding zeroth order term as

$$E^{(1)}_s := \frac{1}{g}\gamma^{-1}_{sl}\left[(\operatorname{rot}\,\vec{E}^{(0)})_l - \partial_k P^*_{kl}\right] \ . \tag{16}$$

The higher terms are obtained via the simple recurrence relations [3]

$$E^{(n+1)}_s := \frac{1}{g}\gamma^{-1}_{sl}(\operatorname{rot}\,\vec{E}^{\,(n)})_l \tag{17}$$

The initial gauge fields $A_i$ transform as vectors under spatial rotations. From the Noetherian expression of the total angular momentum in terms of the physical fields (neglecting surface terms)

$$I_i = \int d^3x\ \epsilon_{ijk}\left((Q^*P^*)_{jk} + \frac{1}{g}x_k\operatorname{Tr}\left(P^*\partial_j Q^*\right)\right) \ , \tag{18}$$

we find that the matrix fields $Q^*$ and $P^*$ transform as second rank tensors under spatial rotations. Any such tensor can be decomposed into its irreducible components, one spin-0 and the five components of a spin-2 field by extraction of its trace [19]. Decomposing the symmetric matrix $Q^*$ into the irreducible representations of the S0(3) group

$$Q^*_{ij}(x) = \frac{1}{\sqrt{2}}Y_A(x)\ T^A_{ij} + \frac{1}{\sqrt{3}}\Phi(x)\ I_{ij} \tag{19}$$

with the field $\Phi$ proportional to the trace of $Q^*$ as spin-0 field and the five-dimensional spin-2 vector $Y(x)$ with components $Y_A$ labeled by the value of spin projection on the $z$- axis $A = \pm 2, \pm 1, 0$. [4] $I$ is the $3 \times 3$ unit matrix and the five traceless $3 \times 3$ spin-2

[3]These expressions can be rewritten in terms of the covariant curl operation $\operatorname{curl} S(e_i, e_j) := \langle \nabla_{e_i} S, e_j \rangle - \langle \nabla_{e_j} S, e_i \rangle$ using the basis $e_i := \left(\gamma^{1/2}\right)_{ij}\partial_j$ such that $\gamma_{ij} := \langle e_i, e_j \rangle$.

[4]For the lowering and raising of the indices of 5-dimensional vectors the metric tensor $\eta_{AB} = (-1)^A\delta_{A,-B}$ is used.

basis matrices $T^A$ satisfying the commutator relations $[J^0, T^A]_- = A\, T^A$ with the $SO(3)$ generators $(J^a)_{ik} := i\epsilon_{iak}$ [19]. The canonical conjugate momenta $P_A(x)$ and $P_\Phi(x)$ to the fields $Y_A(x)$ and $\Phi(x)$, respectively, are the components of the corresponding expansion for the $P^*$ variable

$$P^*_{ij}(x) = \frac{1}{\sqrt{2}} P_A(x)\; T^A_{ij} + \frac{1}{\sqrt{3}} P_\Phi(x)\; I_{ij}\; . \tag{20}$$

For the magnetic field $B$ we obtain the expansion

$$B_{ij}(x) = \frac{1}{\sqrt{2}} H_A(x)\; T^A_{ij} + \frac{1}{\sqrt{2}} h_\alpha(x)\; J^\alpha_{ij} + \frac{1}{\sqrt{3}} b(x)\; I_{ij} \tag{21}$$

with the components

$$H_A := \frac{1}{2} c^{(2)}_{A\beta B} \partial_\beta Y^B + \frac{g}{\sqrt{3}} \left( \frac{1}{\sqrt{2}} {}^*Y_A - \Phi Y_A \right) , \tag{22}$$

$$h_\alpha := \frac{1}{2} d^{(1)}_{\alpha B\gamma} \partial_\gamma Y^B + \sqrt{\frac{2}{3}} \partial_\alpha \Phi\; , \tag{23}$$

$$b := \frac{g}{\sqrt{3}} (\frac{1}{2} Y_A Y^A - \Phi^2)\; . \tag{24}$$

The structure constants $c^{(2)}_{A\beta C}$ and $d^{(1)}_{\alpha B\gamma}$ are defined via the algebra $[T_A, T_B]_- = c^{(2)}_{AB\gamma} J^\gamma$ and $[J_\alpha, T_B]_+ = d^{(1)}_{\alpha\gamma B} J^\gamma$ respectively, and the five-dimensional vector

$${}^*Y_C := d^{(2)}_{CAB} Y^A Y^B \tag{25}$$

via the structure constants $d^{(2)}_{ABC}$ from $[T_A, T_B]_+ = \frac{4}{3}\eta_{AB} I + \frac{2}{\sqrt{3}} d^{(2)}_{ABC} T^C$.

Note that for a complete investigation of the transformation properties of the reduced matrix field $Q^*$ under the whole Poincaré group it is necessary also to include the Lorentz transformations. But we shall limit ourselves here to the isolation of the scalars under spatial rotations and can treat $Q^*$ in terms of "nonrelativistic spin-0 and spin-2 fields", in accordance with the conclusions obtained in the work [4].



In summary, we have shown how to project SU(2) Yang-Mills theory onto the constraint shell defined by the Gauss law. However, several questions in connection with the global aspects of the reduction procedure are arising at ths point. It is well known that the exponentiation of infinitisimal transformations generated by the Gauss law operator can lead only to homotopically trivial gauge transformations, continuously deformable to unity. However, the initial classical action is invariant under all gauge transformations including the homotopically nontrivial ones. How does this fact reflect itself on the properties of the obtained unconstrained theory? In order to discuss the global aspects of the Hamiltonian reduction, we compare the wellknown exact zero energy solution [20] of the Schrödinger equation in the extended quantization scheme, where the Gauss law is implemented on the quantum level, with the corresponding solution of the unconstrained Schrödinger equation. For the original constrained system of $SU(2)$ gluodynamics in terms of the gauge fields $A^a_i(x)$ this exact but nonnormalizable solution $\Psi[A]$, which satisfies both the functional Schrödinger equation with zero energy eigenvalue and the Gauss law constraints is

$$\Psi[A] = \exp\left(\pm 8\pi^2 W[A]\right) , \tag{26}$$

with so-called "winding number functional [21] $W[A] := \int d^3x\, K_0(x)$ defined via the zero component of the Chern-Simons vector $K^\mu(A) := -(16\pi^2)^{-1} \epsilon^{\mu\nu\sigma\kappa} \mathrm{Tr}\left(F_{\nu\sigma} A_\kappa - \frac{2}{3} g A_\nu A_\sigma A_\kappa\right)$ . The winding number functional is known to be invariant under small but not under large gauge transformations.

In terms of the new variables $Q^*$ and $\overline{Q}$ the zero component of the the Chern-Simons vector $K^\mu$ can be written

$$\begin{aligned} K^0(A(Q^*, \overline{Q})) &= K^0(Q^*) - \frac{g}{36\pi^2} \epsilon^{ijk} \mathrm{Tr}\left(\Omega_i \Omega_j \Omega_k\right) \\ &- \frac{g}{24\pi^2} \epsilon^{ijk} \partial_i \mathrm{Tr}\left(Q^*_j \Omega_k\right)\; . \end{aligned} \tag{27}$$

Here we have used the $SU(2)$ matrix $Q^*_l := Q^*_{li}\tau_i$, with the Pauli matrices $\tau_i$, and the $SU(2)$ one-form components

$g\Omega_i(\overline{Q}) := U^{-1}(\overline{Q})\partial_i U(\overline{Q}) = \Omega_{ls}\tau^s\left(\partial\overline{Q}_l/\partial x_i\right)$, with the $SU(2)$ matrix $U(\overline{Q})$ related to the orthogonal $3\times 3$ matrix $O(\overline{Q})$ defined in the transformation (4) via $O_{ab}(\overline{Q}) = \frac{1}{2}\mathrm{Tr}(U(\overline{Q})\tau_a U^T(\overline{Q})\tau_b)$.

The wave functional $\Psi[Q^*]$ obtained from (26) by replacing $A$ by $Q^*$ is a zero energy eigenstate of the corresponding unconstrained Hamiltonian (12). This follows from two important properties of the potential terms of the Hamiltonian (12). Firstly, the reduced magnetic field $B_{ij}(Q^*)$ can be written as the functional derivative of $W[Q^*]$ Furthermore, the nonlocal part of the physical electric field in the unconstrained Hamiltonian annihilates $W[Q^*]$

$$\vec{E}^2[Q, \frac{\delta}{\delta Q^*_{ij}(x)}]W[Q^*] = 0\ . \tag{28}$$

Taking into account that the magnetic field $B_i = {}^*F_{0i}$ satisfies the Bianchi identity $D_i{}^*F_{0i} = 0$.

The second and third terms in (27) are both surface terms. The third term gives no contribution if we assume the physical variable $Q^*$ to vanish at spatial infinity. About the behaviour of the unphysical variables $\overline{Q}_i$ at spatial infinity we have no information. The requirement of the finiteness of the action usually used to fix the behaviour of the physical fields does not apply for the unphysical field $\overline{Q}$. Using the usual boundary condition $U(\overline{Q}) \longrightarrow \pm I$ at spatial infinity, the integral over the second term reduces to an integer n representing the corresponding winding of the mapping of compactified three space into $SU(2)$.

Hence we obtain the relation

$$\Psi[A] = \exp[\pm\frac{8\pi^2}{g^2}n]\Psi[Q^*] \tag{29}$$

between the groundstate wave functionals $\Psi[A]$ of the extended quantization scheme and the reduced $\Psi[Q^*]$. We find that the winding number of the original gauge field $A$ only appears as an unphysical normalization prefactor originating from the second term in (27) which depends only on the unphysical $\overline{Q}_i$. Furthermore we note that the power $8\pi^2 n/g^2$, is the classical Euclidean action of $SU(2)$ Yang-Mills theory of self-dual fields [22] with winding number $n$ .



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