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RENORM-GROUP, CAUSALITY AND NON-POWER PERTURBATION EXPANSION IN QFT

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1 Introduction

In papers [1, 2] the procedure of **Causal Analytization** of perturbative quantum chromodynamics (pQCD) has been elaborated. It implements a **combining** of two ideas: the **RG summation** of leading UV logs **with** spectral representation imposed by **causality**. This combination was first proposed and devised [3] in the QED context about forty years ago.

For the QCD invariant (running) coupling $a(Q^2/\Lambda^2) = \beta_1\alpha_s(Q^2)/4\pi$; $\beta_1(n) = 11-(2n)/3$ (defined in the space-like region), it results in a specific transformation into a form $a_{an}(Q^2/\Lambda^2)$ free of ghost singularities. Here, by construction,
the analytic coupling is defined via the Källén–Lehmann representation

$$a_{\rm an}(x) = \frac{1}{\pi}\int\limits_0^\infty \frac{\rho(\sigma)\,d\sigma}{\sigma+x} \quad \textit{with the spectral density,} \quad \rho(\sigma) = \Im\, a(-\sigma)\,, \quad (1)$$

calculated on the basis of "initial" RG-summed perturbation expression $a(x)$.

Generally, $a_{\rm an}(x)$ differs from its "original input" $a(x)$ by nonperturbative[1] additive terms, which "subtract" unphysical singularities like ghost pole – see, below, Eqs. (2) and (6).

A detailed analysis revealed [1, 2] that the analytic coupling $a_{\rm an}(x)$ obeys several important properties. It turns out to be remarkably stable in the IR region, at $Q < \Lambda$, with respect to higher-loop contribution and to renormalization scheme dependence. Its IR limiting value $a_{\rm an}(0)$ is universal in this sense. We review this subject quite shortly in Sections 2.1 — 2.3.

On the other hand, the Causal Analytization of a physical amplitude $F(Q,\alpha)$ is not a strightforward procedure. A few different scenarios are possible. In papers [5, 6], a particular version, the Analytic Perturbation Theory (APT), has been proposed and elaborated. Here, due to specific analytization ansatz, instead of the power perturbation series common for theoretical physics and QFT, an analytic amplitude $\mathcal{F}(x)$ is presented in a form of an asymptotic expansion of a more general form, the expansion over an asymptotic set of functions $\mathcal{A}_n(x) = [a^n(x)]_{\rm an}$, the "n-th power of $a(x)$ analytized as a whole". In the APT approach, the drastic reduction of loop and renormalization scheme sensitivity for several observables has been found – see Refs.[5]–[7].

[1]On a deep connection between renormalization invariance plus causality and nonanalyticity in α, see our papers [4].

To understand the nature of the "APT's loop and scheme immunity", in Section 3 we study properties of a nonpower asymptotic set $\{\mathcal{A}_n(x)\}$ emerging from the APT recipe of analytization.

In Section 4, we analyse the structure of possible variants of analytization of expression for an observable and discuss the danger of inconsistency (more precisely — incompatibility with the inner structure of RG) for some of them.

2 RG solution and analytization

2.1 One–Loop Analytization of a(x)

At the one-loop level, the invariant coupling $a(x) = 1/\ln x$ suffers from a pole singularity at $x = 1$ incompatible with the spectral representation (1). Here, analytization consists of analytic continuation of the $1/\ln x$ expression into the negative x region and defining the spectral density via its imaginary part. The resulting spectral integral (1) with $\rho(\sigma) = \pi \cdot \left(\ln^2\sigma + \pi^2\right)^{-1}$ can be calculated explicitly

$$a(x) = \frac{1}{\ln x} \Rightarrow a_{an}(x) = \mathcal{A}(x) = \frac{1}{\ln x} - \frac{1}{x-1}. \tag{2}$$

The second term, precisely compensating the ghost pole, has a nonperturbative nature. At small $a_\mu = a(Q^2 = \mu^2)$ it is not "visible" in Taylor series as it behaves like $\exp(-1/a_\mu)$. To see this clearly, one should return from Λ–parameterization to the one in terms of a_μ, the renormalized coupling constant, and μ^2, the reference momentum squared.

Note that a relation between Λ/μ and a_μ in the course of analytization transformation changes. Instead of the usual expression $\Lambda^2 = \mu^2\exp(1/a_\mu)$, according to Eq.(2), we have the transcendental relation

$$\Lambda^2 = \frac{\mu^2}{f(a)}\,; \quad a = \frac{1}{\ln f(a)} + \frac{1}{1-f(a)}.$$

Here, at small a (as well as in (2) at large x), one can neglect the second, nonperturbative, term. Meanwhile, at $a \simeq 1$ ($x \le 1$) this term dominates, providing the IR fixed point at $a = 1$.

The analytic coupling (2) is a monotonous function in the whole interval $(0, \infty)$ with the finite IR limit. The second term quickly diminishes as $x \to \infty$: it contributes about 5% at $Q = 10\Lambda$ and only 1% at $Q = 25\Lambda$.

The whole set of solutions (2) with various Λ values, considered at the Q^2 scale, forms a bunch with a common limiting point $a(0) = 1$. This value, corresponding to $\alpha_{\rm an}(0) = 4\pi/\beta_1{}^2$, turns out to be universal. It does not change in the two- and three-loop approximation as well – see Refs. [1, 2, 9].

2.2 Two- and three-loop cases

For the two-loop case, the invariant coupling has to be defined by the transcendental relation

$$\frac{1}{a^{(2)}(x)} - b\ln\left(1 + \frac{1}{ba^{(2)}(x)}\right) = \ln x\,; \; b = \frac{\beta_2}{\beta_1^2} \; \left(= \frac{64}{81} \text{ at } n = 3\right) \tag{3}$$

resulting from integration of the two-loop RG differential equation.

The iterative procedure yields the explicit approximate solution

$$\bar{a}^{(2)}_{\rm iter}(x) = \frac{1}{\ell + b\ln(1 + \ell/b)}\,, \qquad \ell = \ln x \tag{4}$$

used in our previous papers.

The exact solution to Eq.(3) can also be expressed [8, 9]

$$a^{(2)}(x) = -\frac{1}{b}\cdot\frac{1}{1+\mathcal{W}(x)}\,; \; \mathcal{W}(x) = W_{-1}(z)\,; \; z = -e^{-\ln x/b - 1} \tag{5}$$

in terms of a special function W, the Lambert function (we use the notation of paper [10])

$$W(z)e^{W(z)} = z\,,$$

with an infinite number of branches $W_n(z)$. Most part of the physical region $x > 1$ corresponds to the particular branch $W_{-1}(z)$ that is real and monotonous for $z < -1/e$. The relation between z and x in Eq.(5) is normalized to correspond with approximation (4) at $x \gg 1$. The branch $\mathcal{W}(x) = W_{-1}(z)$ is complex below the ghost singularity at $x = 1$, $z = -1/e$.

The qualitative analysis of Eq.(3) or its exact solution shows that at $x = 1 + |\varepsilon|$ the ghost singularity has a form of the square-root type branch point

$$a^{(2)}(x \simeq 1) = \frac{1}{\sqrt{2b(x-1)}} - \frac{1}{3b} + O(\sqrt{x-1})$$

[2] The effective flavour number at residue of the pole, evidently is $n = 3$. This gives $\alpha_{\rm an}(0) = 1.396$.

that yields an unphysical cut between this point and the origin.

To illustrate, proceeding from the two-loop solution (5), we define the analytic coupling by the spectral representation (1) with spectral density defined in terms of $\mathcal{W}$, that is equivalent to subtraction of the "cut integral" related to the square-root singularity

$$\mathcal{A}_2(x) = -\frac{1}{b(1+\mathcal{W}(x))} - \frac{1}{\pi}\int_0^1 \frac{\mathcal{R}(\sigma)\,d\sigma}{\sigma - x}. \qquad (6)$$

Note that iterative approximation (4) has a slightly different structure of ghost singularities.

Nethertheless, in Ref.[8] it has been demonstrated that the analytized iterative solution is numerically very close[3] to the analytized exact one, Eq.(6). As a practical result, this means that for the two-loop $a_{\rm an}(x)$ one can use an expression in the form Eq.(1) with spectral density

$$\rho^{(2)}_{RG}(L) = \frac{I(L)}{R^2(L) + I^2(L)}\,; \quad L = \ln\frac{\sigma}{\Lambda^2}, \qquad (7)$$

$$R(L) = L + b\ln\sqrt{\left(1+\frac{L}{b}\right)^2 + \left(\frac{\pi}{b}\right)^2}\,, \quad I(L) = \pi + b\arccos\frac{b+L}{\sqrt{(b+L)^2+\pi^2}}\,.$$

At the same time, in Ref. [9] it has been shown that the exact three-loop solution (with Padé transformed beta–function) can also be expressed in terms of the Lambert function

$$a_3(x) = -\frac{1}{b}\cdot\frac{1}{1-c+W(z)}\,;\; z = -e^{-\ln x/b + c - 1}\;;\; c = \frac{\beta_3\beta_1}{\beta_2^2}. \qquad (8)$$

In what follows, referring to the three-loop case, we shall imply the $\overline{\rm MS}$ scheme with $\beta_3^{\overline{\rm MS}} = 2857/2 - 5033n/18 + 325n^2/54$. Here, at $n = 3$, we have $\beta_3 = 3863/6 = 643.833$; $c = 1.415$.

2.3 Stability of analytic coupling in the IR region

The analytic coupling $\mathcal{A}(x)$ obeys several important properties in the IR region. It is quite stable, at $Q < \Lambda$, with respect to the higher loop contribution and renormalization scheme dependence. Its IR limiting value

[3] The $3 \div 4$ per cent difference can be "compensated" by an 8% correction of the Λ value.

$$\mathcal{A}(0) = 1 \qquad (9)$$

(corresponding to $\alpha_{s,\rm an}(0) = 4\pi/\beta_1(3) \simeq 1.4$) is universal in this sense.

This remarkable property of universality was first noticed in our starting up paper [1]. Later on, a detailed analysis [2, 11, 8, 9] has revealed that the $\mathcal{A}(0)$ value turns out to be insensitive not only to higher loop terms in the beta-function but to the precise structure of the ghost singularity (removed by analytization) as well. This structure depends on approximation.

For instance, instead of the square-root singularity of the two-loop exact solution (5), an iterative approximate solution, Eq.(4) obeys a pole at $x = 1$ and a log's branch point at $x_* = e^{-b}$. In the three loop case, the Padé–approximated solution obeys a pole and a branch point (discussed in detail in the paper [9]), to be compared with the $(\ln x)^{-1/3}$ singularity of the solution with a non-transformed beta-function.

Nevertheless, in all these cases, the final analytic results for $\mathcal{A}(x)$ obey the property (9) and their IR behavior in the interval $(0,1)$ is very close [7] to each other. The analytization procedure "smoothing over all sharp angles" makes all them equal.

One can also say that the causality bounds from above the invariant coupling by this maximal value (9), "keeping it reasonably small". For instance, a usual beta-function in the $\overline{\rm MS}$ scheme for $n = 3$,

$$\beta(a) = -a^2(1 + 0.790a + 0.883a^2) \sim -\alpha^2(1 + 0.566\alpha + 0.453\alpha^2) \qquad (10)$$

numerically looks quite good: its higher terms are reasonably decreasing in the usual physical region with $\alpha \leq 0.4$ ($a \leq 0.3$). It is worthwhile here to introduce the "beta-function for the analytic coupling"

$$\frac{d\mathcal{A}(x)}{d\ln x} = \beta_{\rm an}\left(\mathcal{A}(x)\right) \qquad (11)$$

which is a transcendental nonanalytic function $\beta_{\rm an}(\mathcal{A})$ of its only argument (with the IR fixed point at $\mathcal{A} = 1$). In the one-loop case this function can be analyzed rather simply – see below Eqs.(14) and (15). It turns out that its maximum value $\beta_{\rm an}(1/2) = 1/12$ is of the same order of magnitude (this statement remains valid in the higher loop case) as expression (10) taken at $a = 0.3$.

Let us remind to the reader that the connection between analyticity of a function in the cut complex plane and its boundness were discussed several decades ago (see, e.g., Refs.[12]) in the context of a low-energy behavior of hadron scattering amplitudes.

2.4 Analytization of observables

In papers [5, 6] a specific recipe for analytization of an observable $M(s)$ has been introduced. First, one should relate $M(s)$ possibly given in the time-like region, to some auxiliary function $f(Q^2)$ of a space-like argument[4] which obeys the analyticity property in the Q^2 plane compatible with a spectral representation of the Källén–Lehmann type. This function f, after usual RG machinery, acquires a form of perturbative power series

$$F(a(x)) = \sum_n f_n a^n(x) . \tag{12}$$

By prescription first used in paper [5], the causal analytization means

$$F(a(x)) \Rightarrow \mathcal{F}(x) = \sum_n f_n \mathcal{A}_n(x) \tag{13}$$

with $\mathcal{A}_n(x) = [a^n(x)]_{\rm an}$ being "the n-th power of $a(x)$ analytized as a whole". Here, each $\mathcal{A}_n(x)$ satisfies the Källén–Lehmann representation with the spectral density $\rho_n(\sigma)$ defined as $\Im a^n(-\sigma)$.

Note that expansion (13) is not a power one as $\mathcal{A}_n(x) \neq [\mathcal{A}(x)]^n$ at $n \neq 1$. The recipe (12)$\Rightarrow$ (13) changes the nature of expansion !

The surprise feature of this particular recipe called the "Analytic Perturbation Theory" (APT), is the remarkable stability of its results with respect to a higher loop contribution [5, 6] and, in turn, with respect to a scheme dependence [7]. We are going to show that the origin of these physically important properties lies in the change of the type of perturbation expansion.

The non-power set $\{\mathcal{A}_n(x)\}$ is an asymptotic one at UV as

$$\mathcal{A}_{n+1}(x) \simeq [\ln x]^{-(n+1)} = o(\mathcal{A}_n(x)) , \quad x \to \infty .$$

This means that the series (13), generally, should be treated as an asymptotic expansion of the function $\mathcal{F}(x)$ over an asymptotic set $\{\mathcal{A}_n(x)\}$. Its convergence features are determined, on the one hand, by the coefficients f_n calculated on the basis of n-loop Feynman diagrams and, on the other, by the property of the set $\{\mathcal{A}_n(x)\}$. Turn to the discussion of this set(s).

[4] As, e.g., the e^+e^- annihilation cross-section ratio $R(s)$ is related to the Adler function $D(Q^2)$.

3 . Set of expansion functions

3.1 One-loop expansion functions

The simplest set $\{[a^n(x)]_{\rm an}\} = \{\mathcal{A}_n(x)\}$ consists of analytized powers of the one-loop pQCD expansion parameter $a(x)$. Besides (2) it includes

$$\mathcal{A}_2(x) = \frac{1}{\ln^2 x} - y(x) ; \quad y(x) = \frac{x}{(1-x)^2} = y\left(\frac{1}{x}\right) ; \tag{14}$$

$$\mathcal{A}_3(x) = \frac{1}{\ln^3 x} + \frac{x}{(1-x)^3} - \frac{1}{2}\frac{x}{(1-x)^2} ; \; \mathcal{A}_4(x) = \frac{1}{\ln^4 x} - y^2(x) - \frac{y(x)}{6} ; \; \ldots .$$

These "analytized powers" obey a specific symmetry

$$\mathcal{A}_n(x) = (-1)^n \mathcal{A}_n(1/x) \; ; \; (n > 1)$$

and are related by recursion relation with the help of the operator $\mathcal{D} = -x(d/dx)$

$$\mathcal{A}_{n+1}(x) = \frac{1}{n}\mathcal{D}\mathcal{A}_n(x) = \frac{1}{n!}\mathcal{D}^{n+1} \ln \mathcal{A}_0(x) \quad \text{with} \quad \mathcal{A}_0(x) = \frac{x-1}{x \ln x} \tag{15}$$

being the generating function.

In particular,

$$\frac{d\, a_{an}(x)}{d \ln x} = -\mathcal{A}_2(x) \equiv \beta_{an}^{(1)}(a_{an}(x)) = \beta_{an}^{(1)}(a_{an}(1/x)) . \tag{16}$$

Here, $\beta_{an}^{(1)}(a)$ is a non-analytic function of its argument.

An attempt to express $\mathcal{A}_n(x)$ via $\mathcal{A}(x) = a_{an}(x)$, the analytic coupling, gives

$$\mathcal{A}_2(x) = \mathcal{A}^2(x) - \frac{2}{1-x}\mathcal{A}(x) + \frac{1}{1-x} ; \tag{17}$$

$$\mathcal{A}_3(x) = \mathcal{A}^3(x) - \frac{3}{1-x}\mathcal{A}^2(x) + \frac{3}{(1-x)^2}\mathcal{A}(x) - \frac{x+2}{2(1-x)^2}$$

—a sort of a "mixed" representation, combining polynomial and nonperturbative (via $x = Q^2/\Lambda^2$ argument) dependencies. It can be argued (see below Section 3.2) that this representation is not interesting from a pragmatic point of view.

In the two-loop case, to relate solution (6) with higher analytized powers, one can use the operator

$$\mathcal{D}_2 = \frac{1}{b}\frac{\partial}{\partial \mathcal{W}} = -\frac{1+\mathcal{W}(x)}{\mathcal{W}(x)} \cdot x\frac{\partial}{\partial x} , \quad \text{so that} \quad \mathcal{A}_{n+1}^{(2)}(x) = \frac{1}{n!}\mathcal{D}_2^n \mathcal{A}_n^{(2)}(x) . \tag{18}$$

Note also, that $\mathcal{D}$ in Eq.(15) can be treated as an operator of differentiation over an "effective time variable" $t = \ln x$. An analogous interpretation of $\mathcal{D}_2$ gives

$$\mathcal{D}_2 = \frac{1}{1+ba_2(e^t)} \cdot \frac{d}{dt} \equiv \frac{d}{d\tau}\,; \quad \tau = t + b\int_0^t a(e^{t'})dt'\,, \tag{19}$$

with $\tau(t) = t_2$, an "effective two-loop time". For large t values one has $\tau \simeq t + b\ln t$.

3.2 Subtraction Structures and behavior at "the low Q region"

The rational structures $p_n(x) = \mathcal{A}_n(x) - a^n(x)$ that subtract ghost singularities

$$p_1(x) = \frac{1}{1-x}\,;\ p_2(x) = \frac{x}{(1-x)^2}\,;\ p_3(x) = \frac{x(1+x)}{2(1-x)^3}\,;$$

$$p_4(x) = -\frac{x(x-x_+)(x-x_-)}{6(1-x)^4}\,, \quad (x_\pm = 2 \pm \sqrt{3})\ ;\ \ldots \tag{20}$$

are connected by a recursion relation analogous to (15) and, except p_1, obey (anti)symmetry under $x \to 1/x$.

As can be seen from this recursion relation, all $p_{n\geq 2}(x)$ at the origin have the first-order zero that provides a property

$$\mathcal{A}_n(0) = 0\ ; \quad n \geq 2 \tag{21}$$

valid in the higher loop case as well.

For a quantitive orientation it is useful to study the $\mathcal{A}_n(x)$ behavior around the $x = 1$. At the one-loop case this can be done explicitly with the help of (20)

$$\mathcal{A}(x = 1-\epsilon) \simeq \frac{1}{2} - \frac{\epsilon}{12}\,;\ \mathcal{A}_2(x) \simeq \frac{1}{12} + \frac{19}{60}\epsilon^2\,;$$

$$\mathcal{A}_3(x) \simeq +\frac{\epsilon}{240}\,;\ \ \mathcal{A}_4(1) \simeq -\frac{1}{720} + O(\epsilon^2)\,.$$

These numerical results are rather instructive. Together with Eq.(21) they show that in the "low Q region", at $Q \leq \Lambda$, we have

$$|\mathcal{A}_n(x)| \ll \mathcal{A}^n(x)\,, \tag{22}$$

an important estimate, which, at the very end, is responsible for a low level of sensitivity of APT results with respect to the higher loop and scheme effects – see, e.g., observations made in Refs. [5] – [7]. The last estimate is valid also in the two- and three-loop cases.

It is not practical to use expressions like (5), (8) and (18) for explicit analysis of asymptotic sets $\{\mathcal{A}^n(x)\}$ at two- and three-loop cases. For a short quantitative discussion we rather use results of numerical calculation via an adequate spectral integral. They show that in the interval $0 \leq Q \leq \Lambda$ in addition to relation (22) we have

$$\mathcal{A}_{k+1}(x) \cong [\mathcal{A}(x)]^{1+2k}\,.$$

In other words, numerically, in the "low Q region" the real expansion parameter is closer to $\mathcal{A}^2(x)$ rather than to $\mathcal{A}(x)$ [5]. Naturally, as Q/Λ grows and power terms diminish, all $\mathcal{A}_n(x)$ tend to their natural limits $[\mathcal{A}(x)]^n \simeq a^n(x)$.

Moreover, as it follows from the representation (20) and property of singularity $\ln^{-n} x$ at $x = 0$, the expansion functions $\mathcal{A}_{n+2}(x)$ obey precisely n zeroes on the interval $(0, X(\Lambda))$ with $X(\Lambda)$ being the upper boundary of the region where a power nonperturbative correction $\sim x^{-1} = \Lambda^2/Q^2$ is essential. Hence, the set under discussion consists of quasi-oscillating functions. This feature makes the problem of estimating the resudial term (that is an error) in the asymptotic expansion Eq.(13) more complicated. Quite probably, in the IR region we have to deal here with an *asymptotic expansion à la Erdélyi.*

Note also that the sets $\{\mathcal{A}_n(x)\}$ both in the one- and two-loop cases obey a peculiar structure. Their neighbouring terms are connected by differential relations (15) and (18).

4 Discussion

We have analysed a particular version of "Causal, Q^2–analytic perturbation theory", the APT version, which, by convention first introduced in paper [5], uses a set $\{\mathcal{A}_n(x)\}$ for analytization of observables. It can be considered as a "nonpower analytization", to distinguish it from another possibility,

[5] In particular, this means that a "mixed" representation in powers of $\mathcal{A}(x)$ with nonperturbative coefficients is not reasonable due to big cancellation inside the r.h.s. of relations analogous to Eq.(17).

the "power analytization" with the help of a power asymptotic set $\{\mathcal{A}^n(x)\}$ by the recipe

$$F(a(x)) \Rightarrow F(\mathcal{A}(x)) = \sum_n f_n [\mathcal{A}(x)]^n , \qquad (23)$$

instead of (13).

Just the nonpower analytization yields intriguing results with respect to loop and scheme stability. At the same time, the power analytization, Eq.(23), results in a moderate change of usual pQCD practice mainly in the IR region.

This, technically simpler, second version has an advantage from a theoretical point of view related to the issue of *Consistency of analytization with the RG structure* – see below Section 4.2. To clarify, let us make a comment on the structure of the RG algorithm and on "noncommutativity" of analytization with some of its elements.

4.1 On ambiguity of analytization procedure

The procedure of the renorm-group method, in addition to deriving functional and differential group equations, consists in a few steps

[1] Calculating beta–function(s) and anomalous dimensions;

[2] Solving RG differential equations (RGDEs) for invariant coupling(s) $a(x)$;

[3] Solving RGDEs for other functions $f(Q^2, a)$, e.g., propagator amplitudes, effective masses and "physical amplitudes"[6] with the use of explicit expressions for beta-function of Step [1] or invariant coupling(s) $a(x)$ obtained in Step [2]. The resulting $F(a(x))$ can be expressed as a power series (starting, possibly with logarithmic term).

The invariant coupling analytization adds an additional step that follows the Step [2]:

[2a] $\qquad a(Q^2) \to \mathcal{A}(Q^2)$.

However, analytization of propagators and observables can now be performed either by modification of Step [3] –

[3m] Using explicit expression $\mathcal{A}(x)$ in the process of RGDEs for $f(Q^2, a)$ solving

or as an additional Step:

[$4_{\rm APT}$] Analytizing the result of Step [3], i.e., by applying the analytization procedure to the power series for $F(a(x)) \Rightarrow \mathcal{F}(x)$.

[6] Like , Adler functions, structure function moments, etc.

The sequence

$$[1]+[2]+[2a]+[3]+[4_{\rm APT}] = [\mathbf{APT}]$$

was used in Refs.[5]–[7]. Just this procedure yields nonpower asymptotic expansion (13).

On the other hand, in parallel with step [$4_{\rm APT}$] there exists a simpler possibility:

[$4_{\rm an}$] Substituting expression $\mathcal{A}(Q^2)$, like in (23), in the result of Step [3].

The sequence

$$[1]+[2]+[2a]+[3]+[4_{\rm an}] = [\mathbf{ICA}]$$

can also be used for analytization of observables. This procedure, involving just the 'invariant coupling analytization' (ICA), yields power asymptotic expansion (23) differing from the usual one, Eq.(12), by substitution $a(Q^2/\Lambda^2) \Rightarrow \mathcal{A}(Q^2/\Lambda^2)$ only.

We see that, generally, a causal analytization is not a unambiguous procedure. Quite remarkably, the above–mentioned ambiguity is of a functional nature (not in possibility to introduce an adjustable parameter).

4.2 Analyticity vs RG structure ?

Meanwhile, the sequence $[1]+[2]+[2a]+[\mathbf{3m}]$ contains an inner contradiction. E.g., Step [**3m**], used for the gluon propagator amplitude[7], yields an expression that, at the very end, is not compatible with the result of the previous Step [2a]. At one loop level, it gives [14]

$$d^i_{RG}(x) = [a(x)]^{\nu_i} \Rightarrow [\mathcal{A}_0(x)]^{\nu_i}$$

with $\mathcal{A}_0(x)$ defined in (15). However, as it follows from basic RG relations, the product of a vertex and appropriate powers of propagators forms an invariant coupling. In the case under consideration, one obtains $\mathcal{A}_0(x)$ rather than $a_{\rm an}(x) = \mathcal{A}(x)$ used as an input.

Quite analogously, there is a subtlety with the Step [$4_{\rm APT}$] implementation. The point is that for some objects (e.g., for propagator amplitudes) the result of Step [3] at the one loop level starts with fractional power (or logarithm) of $a(x)$ that gives rise to a branch point. The analytization of expression like $[\ln x]^{-\nu}$ is equivalent to subtraction of a cut contribution,

[7] Compare with the Step [$4_{\rm an}$] used in Ref.[8].

i.e., yields a two-term structure of a specific form[8]. It is easy to see that the appropriate product of such structures cannot give expression (2) for invariant coupling. Taken literally, this observation means that the APT procedure also faces a contradiction with the RG structure.

Conclusion

1. Our analysis in Section 3 reveals that the APT expansion, Eq.(13), for an observable function, generally, represents an asymptotic expansion over a nonpower asymptotic set $\{\mathcal{A}_n(x)\}$. The latter obeys quite different properties in various ranges of the x variable. In UV it is close to the power set $\{a^n(x)\}$, commonly used in the current practice of QFT pertirbation calculation. Hence, the APT converegence property in UV is completely determined by expansion coefficients f_n. On the other hand, in IR the asymptotic set $\{\mathcal{A}_n(x)\}$ is of a more complicated structure. In the "low Q region" the behavior of the functions $\mathcal{A}_n(x)\,; n \geq 3$ is oscillating. Due to this, the contribution of higher terms in the APT expansion is suppressed. The APT expansion, Eq.(13), in IR has a feature of asymptotic expansion à la Erdélyi. This tentative conclusion raises hopes that the pertubative approach to QCD may be fruitful in the region $Q \sim 1$ GeV where the QCD running coupling is not a small quantity.

2. In Section 4.1, we have shown that the general program of Causal Analytisation, being quite a definite procedure for effective coupling, is not "rigid" enough when applied to other objects. In particular, it contains a degree of freedom in analytizing observables.

This ambiguity together with a "proximity to contradiction", discussed in Section 4.2, poses a question of looking for an additional ansatz in the whole Causal analytic approach. The APT possibility is too interesting to be "abandoned without a struggle".

3. In our opinion, one more funny lesson of the considered nonpower construction is a semiquantitive observation that the APT approach is equivalent to the usual pQCD practice with one strange amendment: "To restrict calculation to only the leading QCD contribution"; by the way, forgetting about all headaches of higher-loop diagram calculation, scheme dependency and expansion convergence. This intriguing feature could be formulated as a suspicion that the pQCD is an effective theory[9] and its higher order contributions have no clear physical content.

[8] For explicit expressions we refer, e.g., to Refs. [3, 8].

[9] Like, e.g., higher order perturbation contributions of the effective four-fermion Fermi weak interaction in QFT and of the BCS model Hamiltonian in the theory of superconductivity.

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