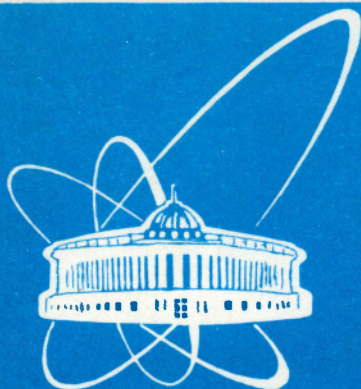


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98-285

E2-98-285
ITP-UH-23/98
solv-int/9810009

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FERMIONIC FLOWS AND TAU FUNCTION
OF THE $N=(1|1)$ SUPERCONFORMAL
TODA LATTICE HIERARCHY

Submitted to «Nuclear Physics B»

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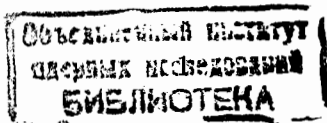
1. Introduction. Recently the $N=(1|1)$ supersymmetric generalization of the Darboux transformation was proposed, and an infinite class of bosonic solutions of its symmetry equation was constructed in [1]. These solutions generate bosonic flows of the $N=(1|1)$ supersymmetric Toda lattice hierarchy in the same way as their bosonic counterparts – solutions of the symmetry equation of the Darboux transformation [2] – produce the flows of the bosonic Toda lattice hierarchy. However, due to the supersymmetry it is obvious that besides bosonic flows the supersymmetric hierarchy also possesses fermionic ones. A natural question arises about finding solutions of the symmetry equation which are responsible for the fermionic flows.

The issue of fermionic flows is also interesting from a slightly different side. It is a well-known fact by now that particular reductions of the tau function of the bosonic Toda lattice hierarchy reproduce partition functions of some matrix models (for review, see e.g. [3, 4, 5] and references therein). It is reasonable to suspect that the tau function of the $N=(1|1)$ supersymmetric Toda lattice hierarchy may also be relevant in this respect. In view of the longstanding yet unsolved problem of constructing supermatrix models which differ non-trivially from bosonic ones, the knowledge of the super tau function might be an important advance. In order to derive the complete tau function it is clearly necessary to know both bosonic and fermionic flows, leading us again to the topic of fermionic flows.

The present paper addresses both above-stated problems. In section 2 we construct an infinite class of fermionic flows of the $N=(1|1)$ supersymmetric Toda lattice hierarchy and derive their algebraic structure. In section 3 we present the general solution of its reduction – the $N=(1|1)$ semi-infinite Toda lattice hierarchy –, and derive a superdeterminant representation for its tau function. We also analyze the tau function of the semi-infinite $N=(1|1)$ Toda chain hierarchy and discuss its bosonic limit.

2. Fermionic flows of the $N=(1|1)$ Toda lattice hierarchy. In this section we construct an infinite class of fermionic flows of the $N=(1|1)$ superconformal Toda lattice hierarchy and produce their algebraic structure.

Our starting point is the $N=(1|1)$ supersymmetric generalization of the



Darboux transformation [1],

$$\overleftarrow{u} = \frac{1}{v} , \quad \overleftarrow{v} = v(D_- D_+ \ln v - uv) , \quad (1)$$

where $u \equiv u_j(x^+, \theta^+; x^-, \theta^-)$ and $v \equiv v_j(x^+, \theta^+; x^-, \theta^-)$ are bosonic $N=(1|1)$ superfields defined on the lattice, $j \in \mathbb{Z}$, and D_{\pm} are the $N=1$ supersymmetric fermionic covariant derivatives

$$D_{\pm} = \frac{\partial}{\partial \theta^{\pm}} + \theta^{\pm} \frac{\partial}{\partial x^{\pm}} , \quad D_{\pm}^2 = \frac{\partial}{\partial x^{\pm}} \equiv \partial_{\pm} , \quad \{D_+, D_-\} = 0 . \quad (2)$$

To simplify our formulae, we use the following convenient notation:

$$\overleftarrow{u} \equiv u_{j+1} , \quad \overleftarrow{u}^k \equiv u_{j+k} , \quad \overrightarrow{u} \equiv u_{j-1} , \quad \overrightarrow{u}^k \equiv u_{j-k} \quad (3)$$

Equations (1) are invariant under global $U(1)$ transformations which rotate the superfields u and v in opposite directions. The composite superfield

$$b \equiv uv \quad (4)$$

of length dimension

$$[b] = -1 \quad (5)$$

satisfies the $N=(1|1)$ superconformal Toda lattice equation

$$D_- D_+ \ln b = \overleftarrow{b} - \overrightarrow{b} . \quad (6)$$

For this reason, the hierarchy of equations invariant under the Darboux transformation (1) we call the $N=(1|1)$ superconformal Toda lattice hierarchy.

One of the possible ways of constructing invariant equations is to solve a corresponding symmetry equation [6]. In the case under consideration it reads

$$\overleftarrow{U} = -\frac{1}{v^2} V , \quad \overleftarrow{V} = \frac{\overleftarrow{v}}{v} V + v(D_- D_+ (\frac{1}{v} V) - vU - uV) , \quad (7)$$

where V and U are bosonic functionals of the superfields v and u . Any particular solution V_p, U_p generates an evolution system of equations with a bosonic evolution time t_p

$$\frac{\partial}{\partial t_p} v = V_p(u, v) , \quad \frac{\partial}{\partial t_p} u = U_p(u, v) . \quad (8)$$

By construction¹, this is invariant with respect to the discrete transformation (1) and, therefore, belongs to the hierarchy as defined above. In other words, different solutions of the evolution system (8) (which, actually, are given by pairs of superfields $\{\overleftarrow{v}^k, \overleftarrow{u}^k\}$ with different values for k) are related by the discrete Darboux transformation (1). Altogether, invariant evolution systems form a *differential* hierarchy, i.e. a hierarchy of equations involving only superfields at a single lattice point². In contrast, the discrete lattice shift (the Darboux transformation), when added to the differential hierarchy, generates the *discrete* $N=(1|1)$ superconformal Toda lattice hierarchy. Thus, the discrete hierarchy appears as a collection of an infinite number of isomorphic differential hierarchies [7].

The symmetry equation (7) represents a complicated nonlinear functional equation, and its general solution is not known. For a more complete understanding of the hierarchy structure and its solutions (tau function) it seems necessary to know as many solutions of eq. (7) as possible. Ref. [1] addressed this problem and derived a wide class of bosonic solutions. However, supersymmetry suggests that eq. (7) possesses fermionic solutions as well, and that they are responsible for fermionic flows of the hierarchy. It turns out that such solutions do in fact exist. We shall demonstrate that the framework developed in [1] contains a hidden possibility for generating fermionic flows.

To explain this observation, we briefly review the approach of ref. [1].

First, the functionals V and U are consistently represented in terms of a single bosonic functional $\alpha_{(0)}$,

$$V = -v\alpha_{(0)} , \quad U = u\overrightarrow{\alpha}_{(0)} , \quad (9)$$

in terms of which the symmetry equation (7) becomes

$$D_- D_+ \alpha_{(0)} = \overleftarrow{b} (\overleftarrow{\alpha}_{(0)} - \alpha_{(0)}) + b(\alpha_{(0)} - \overrightarrow{\alpha}_{(0)}) , \quad (10)$$

where the superfield b is defined by eq. (4) and constrained by eq. (6).

¹Let us recall that eq. (7) is just a result of differentiating eq. (1) with respect to the evolution time t_p .

²In the case of the one- (two-) dimensional bosonic Toda lattice the differential hierarchy coincides with the Nonlinear Schrödinger (Davey-Stewartson) hierarchy [7, 8, 2].

Second, the following recursive chain of substitutions is introduced:

$$\alpha_{(p)}^{\pm} = \pm D_{\mp}^{-1} \left(\overleftarrow{b}^{\leftarrow{p+1}} \alpha_{(p+1)}^{\pm} + (-1)^p b \overrightarrow{\alpha}_{(p+1)}^{\pm} \right), \quad p = 0, 1, 2, \dots, \quad (11)$$

where $\alpha_{(2n)}^{\pm}$ ($\alpha_{(2n+1)}^{\pm}$) are new bosonic (fermionic) functionals of length dimensions related as

$$[\alpha_{(2p)}] = [\alpha_{(0)}^{\pm}] + p, \quad [\alpha_{(2p-1)}] = [\alpha_{(0)}^{\pm}] + p - \frac{1}{2}. \quad (12)$$

Substituting the functional $\alpha_{(0)}^{\pm}$ in terms of $\alpha_{(1)}^{\pm}$ into eq. (10), the latter reads

$$\mp D_{\pm} \alpha_{(1)}^{\pm} + \alpha_{(1)}^{\pm} D_{\mp}^{-1} (\overleftarrow{b}^{\leftarrow{2}} - b) = D_{\mp}^{-1} (\overleftarrow{b}^{\leftarrow{2}} \alpha_{(1)}^{\pm} - b \overrightarrow{\alpha}_{(1)}^{\pm}) \quad (13)$$

Repeating the same procedure applied to the functional $\alpha_{(1)}^{\pm}$, i.e. substituting $\alpha_{(1)}^{\pm}$ in terms of $\alpha_{(2)}^{\pm}$ into eq. (13), the resulting equation for $\alpha_{(2)}^{\pm}$ takes the following form:

$$\begin{aligned} \pm D_{\pm} \alpha_{(2)}^{\pm} + \alpha_{(2)}^{\pm} D_{\mp}^{-1} (\overleftarrow{b}^{\leftarrow{3}} - \overleftarrow{b} + \overleftarrow{b}^{\leftarrow{2}} - b) = \\ D_{\mp}^{-1} (\overleftarrow{b}^{\leftarrow{3}} \alpha_{(2)}^{\pm} - \overleftarrow{b} \alpha_{(2)}^{\pm} + \overleftarrow{b}^{\leftarrow{2}} \alpha_{(2)}^{\pm} - b \overrightarrow{\alpha}_{(2)}^{\pm}) \end{aligned} \quad (14)$$

Next, the equation for $\alpha_{(3)}^{\pm}$ emerges,

$$\begin{aligned} \mp D_{\pm} \alpha_{(3)}^{\pm} + \alpha_{(3)}^{\pm} D_{\mp}^{-1} (\overleftarrow{b}^{\leftarrow{4}} - \overleftarrow{b} + \overleftarrow{b}^{\leftarrow{3}} - b) = \\ D_{\mp}^{-1} (\overleftarrow{b}^{\leftarrow{4}} \alpha_{(3)}^{\pm} + \overleftarrow{b} \alpha_{(3)}^{\pm} - \overleftarrow{b}^{\leftarrow{3}} \alpha_{(3)}^{\pm} - b \overrightarrow{\alpha}_{(3)}^{\pm}) \end{aligned} \quad (15)$$

and so on.

We now analyze the solutions of the equations arising in this iterative process. It turns out that, at any given p , those equations possess very simple solutions for $\alpha_{(2p)}^{\pm}$ which, however, translate to very non-trivial solutions for the functional $\alpha_{(0)}^{\pm}$ via relations (11). In turn, $\alpha_{(0)}^{\pm}$ yields flows via eqs. (8), (9).

Let us start from the equations for the bosonic functionals $\alpha_{(2p)}^{\pm}$. They admit the solutions [1]

$$\alpha_{(2p)}^{\pm} = 1 \quad \Rightarrow \quad [\alpha_{(2p)}^{\pm}] = 0, \quad (16)$$

and the recursive procedure may be interrupted at every even step (for the particular case of $p=1$ this solution can be seen from eq. (14)). The corresponding $\alpha_{(0)}^{\pm}$, being expressed in terms of $\alpha_{(2p)}^{\pm}$ (16) via relations (11), generates the p -th bosonic flow of the hierarchy

$$\frac{\partial}{\partial t_p^{\pm}} v = -v \alpha_{(0)}^{\pm}, \quad \frac{\partial}{\partial t_p^{\pm}} u = u \overrightarrow{\alpha}_{(0)}^{\pm} \quad \Rightarrow \quad [t_p^{\pm}] \equiv -[\alpha_{(0)}^{\pm}] = p, \quad (17)$$

where we have used eqs. (8), (9), (12), and (16). Although this $\alpha_{(0)}^{\pm}$ depends on all superfields $\overleftarrow{v}^{\leftarrow{k}}$ and $\overleftarrow{u}^{\leftarrow{k}}$ with $0 \leq k \leq 2p$, by using eq. (1) it can be expressed completely in terms of the superfields u and v without arrows. In this way the differential hierarchy of bosonic flows (17) is generated (see the discussion after eq. (8)). For illustration, we present the first two [1]:

$$\frac{\partial}{\partial t_1^+} v = v, \quad \frac{\partial}{\partial t_1^+} u = u, \quad (18)$$

$$\begin{aligned} \frac{\partial}{\partial t_2^+} v = +\partial_+^2 v - 2(D_+ v) D_-^{-1} \partial_+(uv) + \\ 2v D_-^{-1} [\partial_+(v D_+ u) + 2uv D_-^{-1} \partial_+(uv)] \quad , \\ \frac{\partial}{\partial t_2^+} u = -\partial_+^2 u - 2(D_+ u) D_-^{-1} \partial_+(uv) + \\ 2u D_-^{-1} [\partial_+(u D_+ v) - 2uv D_-^{-1} \partial_+(uv)] \quad . \end{aligned} \quad (19)$$

Concerning eqs. (13) and (15) for the fermionic functionals $\alpha_{(1)}^{\pm}$ and $\alpha_{(3)}^{\pm}$, respectively, simple inspection shows that they do not allow for constant Grassmann-odd solutions. Due to this reason ref. [1] concluded that the recursive procedure cannot be interrupted at an odd step, in distinction to the case of the bosonic Toda lattice [2]. As a crucial consequence of this conclusion there is no place for fermionic flows, at least not in the framework of this iteration procedure.

However, there is a subtle point in this argument, and we are going to revise the conclusion. The argument overlooks the possibility of solutions which are *superfield-independent* lattice functions. Indeed, we find the following solutions of eqs. (10) and (11):

$$\alpha_{(2p-1)}^{\pm} = \mathcal{I} \epsilon, \quad [\alpha_{(2p-1)}^{\pm}] = 0, \quad (20)$$

where ϵ is a dimensionless fermionic constant and \mathcal{I} is the simple dimensionless bosonic lattice function

$$\mathcal{I} \equiv (-1)^j \quad (21)$$

which takes only two values, +1 or -1, and possesses the following obvious properties:

$$\overleftarrow{\mathcal{I}} = \overrightarrow{\mathcal{I}} = -\mathcal{I} \quad \text{and} \quad \mathcal{I}^2 = 1 \quad (22)$$

Therefore, as in the bosonic case, the recursive procedure can be interrupted here as well at every odd step. It remains to show how fermionic flows originate from this background.

This goal in mind, let us represent the bosonic time derivative entering eq. (8) in the following form:

$$\frac{\partial}{\partial t_p} v = \epsilon \frac{\partial}{\partial \vartheta_p} \quad (23)$$

defining a fermionic time-derivative $\frac{\partial}{\partial \vartheta_p}$. Then, eq. (8) becomes

$$\begin{aligned} \epsilon \frac{\partial}{\partial \vartheta_p^\pm} v = -v \alpha_{(0)}^\pm, \quad \epsilon \frac{\partial}{\partial \vartheta_p^\pm} u = u \overrightarrow{\alpha}_{(0)}^\pm \quad \Rightarrow \\ [\vartheta_p^\pm] \equiv -[\alpha_{(0)}^\pm] = p - \frac{1}{2}, \end{aligned} \quad (24)$$

where $\alpha_{(0)}^\pm$ should be expressed in terms of $\alpha_{(2p-1)}^\pm$ (20) via relations (11), and eqs. (9), (12) and (20) have been exploited to arrive at eqs. (24). The fermionic constant ϵ enters linearly on both sides of eqs. (24), hence the fermionic flows $\frac{\partial}{\partial \vartheta_p}$ actually do not depend on ϵ . In this context we remark that ϵ is an artificial parameter, which need not be introduced at all. However, without ϵ it is necessary to consider the quantities $t_p, V_p, U_p, \alpha_{(2n)}^\pm$ ($\alpha_{(2n+1)}^\pm$) entering eqs. (8), (11) as fermionic (bosonic) ones from the beginning. Of course, at the end of the analysis one arrives at the same result (24).

Using eqs. (24), (20) and (11) for the fermionic flows, we elaborate the first two of them,

$$\mathcal{I} \frac{\partial}{\partial \vartheta_1^+} v = -D_+ v + 2v D_-^{-1}(uv), \quad \mathcal{I} \frac{\partial}{\partial \vartheta_1^+} u = -D_+ u - 2u D_-^{-1}(uv), \quad (25)$$

$$\begin{aligned} \mathcal{I} \frac{\partial}{\partial \vartheta_2^+} v = -D_+ \partial_+ v + 2(\partial_+ v) D_-^{-1}(uv) + \\ (D_+ v) D_-^{-1} D_+(uv) + v D_-^{-1} [u \partial_+ v + (D_+ v) D_+ v], \end{aligned}$$

$$\begin{aligned} \mathcal{I} \frac{\partial}{\partial \vartheta_2^+} u = +D_+ \partial_+ u + 2(\partial_+ u) D_-^{-1}(uv) + \\ (D_+ u) D_-^{-1} D_+(uv) + u D_-^{-1} [v \partial_+ u + (D_+ u) D_+ v]. \end{aligned} \quad (26)$$

Let us note that the two differential hierarchies arising for the two different values of \mathcal{I} (+1 or -1) are actually isomorphic. Indeed, one can easily see that they are related by the standard automorphism which changes the sign of all Grassmann numbers. Thus, in distinction to the bosonic Toda lattice, where the Darboux transformation does not change the direction of evolution times in the differential hierarchy (8), its supersymmetric counterpart (1) reverses the sign of fermionic times in the differential hierarchy. This supersymmetric peculiarity has no effect on the property that the supersymmetric *discrete* hierarchy is a collection of isomorphic *differential* hierarchies like in the bosonic case³.

The flows $\frac{\partial}{\partial \vartheta_k^-}$ and $\frac{\partial}{\partial t_k^-}$ can easily be derived by applying the invariance transformations

$$\partial_\pm \rightarrow \partial_\mp, \quad D_\pm \rightarrow \pm D_\mp \quad (27)$$

of the $N=(1|1)$ supersymmetry algebra (2) and eqs. (1), (6) and (10) to the flows $\frac{\partial}{\partial \vartheta_k^\pm}$ (25)-(26) and $\frac{\partial}{\partial t_k^\pm}$ (18)-(19), respectively, but we do not write them down here.

Using the explicit expressions for the bosonic and fermionic flows constructed here, one can calculate their algebra

$$\begin{aligned} \left\{ \frac{\partial}{\partial \vartheta_k^\pm}, \frac{\partial}{\partial \vartheta_l^\pm} \right\} = -2 \frac{\partial}{\partial t_{k+l-1}^\pm}, \quad \left\{ \frac{\partial}{\partial \vartheta_k^\pm}, \frac{\partial}{\partial \vartheta_l^\mp} \right\} = 0, \\ \left[\frac{\partial}{\partial t_k^\pm}, \frac{\partial}{\partial t_l^\pm} \right] = \left[\frac{\partial}{\partial t_k^\mp}, \frac{\partial}{\partial t_l^\mp} \right] = \left[\frac{\partial}{\partial t_k^\pm}, \frac{\partial}{\partial \vartheta_l^\pm} \right] = \left[\frac{\partial}{\partial t_k^\mp}, \frac{\partial}{\partial \vartheta_l^\mp} \right] = 0. \end{aligned} \quad (28)$$

³For the one-dimensional bosonic Toda lattice hierarchy the isomorphism which relates the differential hierarchies is trivial because they are identical copies of the single Nonlinear Schrödinger hierarchy [7].

This algebra coincides with the one used in [9, 10], where the super Toda lattice (STL) hierarchy has been expressed as a system of infinitely many equations for *infinitely many* superfields. Our formulation involves only *two independent* superfields (v and u). From the point of view of the former approach this corresponds to extracting those STL hierarchy equations which can be realized in terms of the superfields v and u alone after excluding all other superfields of the STL hierarchy. Keeping in mind this correspondence it is quite natural to suppose that the algebra (28) is not only valid for the flows (18), (19), (25)–(27) for which it was in fact calculated, but for all the other flows as well. If this is the case, eqs. (28) may be realized in the superspace $\{t_k^+, \theta_k^+, t_k^-, \theta_k^-\}$,

$$\frac{\partial}{\partial \theta_k^\mp} = \frac{\partial}{\partial \theta_k^\pm} - \sum_{l=1}^{\infty} \theta_l^\pm \frac{\partial}{\partial t_{k+l-1}^\pm}, \quad (29)$$

which is used in what follows. Here, θ_k^+ and θ_k^- are abelian fermionic evolution times with the dimensions

$$[\theta_k^\pm] = k - \frac{1}{2}. \quad (30)$$

In closing this section we only mention that the flows and their algebras (2) and (28) admit a consistent reduction to a one-dimensional subspace by setting

$$\partial_+ = \partial_- \equiv \partial \quad \Leftrightarrow \quad \frac{\partial}{\partial t_k^+} = \frac{\partial}{\partial t_k^-} \equiv \frac{\partial}{\partial t_k}. \quad (31)$$

As a result, the $N=(1|1)$ supersymmetric Toda chain hierarchy arises, but its detailed description is beyond the scope of the present work.

3. The tau function of the semi-infinite $N=(1|1)$ superconformal Toda lattice.

For the case of the semi-infinite hierarchy, i.e. for the hierarchy interrupted from the left by the boundary condition

$$u_{-1} = 0, \quad (32)$$

the bosonic and fermionic flows for the remaining boundary superfield v_{-1} have the extremely simple form⁴,

$$\frac{\partial}{\partial \theta_k^\pm} v_{-1} = -D_\pm \partial_\pm^{k-1} v_{-1} \quad \text{and} \quad \frac{\partial}{\partial t_k^\pm} v_{-1} = \partial_\pm^k v_{-1}, \quad (33)$$

and can easily be solved. These equations are consistent with the algebra (28), and in its realization (29) their general solution is

$$v_{-1} = \int \left(\prod_{\alpha=\pm} d\lambda_\alpha d\eta_\alpha \right) \varphi(\lambda_+, \lambda_-, \eta_+ - \theta_+, \eta_- - \theta_-) \times \exp \sum_{\alpha=\pm} \left[(x^\alpha - \eta_\alpha \theta^\alpha) \lambda_\alpha + \sum_{k=1}^{\infty} (t_k^\alpha + \eta_\alpha \theta_k^\alpha) \lambda_\alpha^k \right] \quad (34)$$

where φ is an arbitrary function of bosonic (λ_\pm) and fermionic (η_\pm) spectral parameters with dimensions

$$[\lambda_\pm] = -1, \quad [\eta_\pm] = \frac{1}{2}. \quad (35)$$

Let us construct the general solution for the superfields \overleftarrow{v}^s and \overleftarrow{u}^s at $s \geq 0$. This can be done by expressing them in terms of the boundary superfield v_{-1} (34) via eqs. (1) through an obvious iterative procedure,

$$\overleftarrow{u}^{(s+1)} = \frac{1}{\overleftarrow{v}^s}, \quad \overleftarrow{v}^{(s+1)} = \overleftarrow{v}^s (D_- D_+ \ln \overleftarrow{v}^s - \overleftarrow{u}^s \overleftarrow{v}^s), \quad s \geq -1. \quad (36)$$

We have explicitly checked for the next few values of s that the resulting expressions, obtained by iteration of eq. (36), convert to the following nice form:

$$\begin{aligned} v_{2s} &= +(-1)^s \frac{\tau_{2s}}{\tau_{2s+1}}, & v_{2s+1} &= +(-1)^s \frac{\tau_{2(s+1)}}{\tau_{2s+1}}, \\ u_{2s} &= -(-1)^s \frac{\tau_{2s-1}}{\tau_{2(s-1)}}, & u_{2s+1} &= -(-1)^s \frac{\tau_{2s-1}}{\tau_{2s}}. \end{aligned} \quad (37)$$

⁴To derive these equations it is only necessary to take into account the $U(1)$ invariance of the flows (consequently, only linear equations for v_{-1} are admissible at $u_{-1} = 0$), the dimensions (17) and (30) of bosonic and fermionic times and the algebra (28).

where the τ_s are⁵

$$\begin{aligned} \tau_0 &\equiv -v_{-1} \quad , \quad \tau_{2s} = \text{sdet} \left(\begin{array}{cc} \partial_+^i \partial_-^j \tau_0 & \partial_+^i \partial_-^m D_- \tau_0 \\ \partial_+^k \partial_-^j D_+ \tau_0 & \partial_+^k \partial_-^m D_+ D_- \tau_0 \end{array} \right)_{\substack{0 \leq i, j \leq s \\ 0 \leq k, m \leq s-1}} \quad , \\ \tau_{-1} &= 1 \quad , \quad \tau_{2s+1} = \text{sdet} \left(\begin{array}{cc} \partial_+^i \partial_-^j \tau_0 & \partial_+^i \partial_-^m D_- \tau_0 \\ \partial_+^k \partial_-^j D_+ \tau_0 & \partial_+^k \partial_-^m D_+ D_- \tau_0 \end{array} \right)_{\substack{0 \leq i, j \leq s \\ 0 \leq k, m \leq s}} \quad (38) \end{aligned}$$

The supermatrices in eqs. (38) can be embedded into a single supermatrix

$$(D_+^p D_-^q \tau_0)_{0 \leq p, q \leq N} \quad (39)$$

with the obvious correspondence. These formulae are plausibly valid for any value of s .

Substituting eqs. (37) in to eqs. (1) one obtains the following equation for τ_s :

$$D_- D_+ \ln \tau_s = - \left(\frac{\tau_{s-1}}{\tau_{s+1}} \right)^{(-1)^s} \quad (40)$$

Thus, we see that the general solution (37) of all equations belonging to the semi-infinite $N=(1|1)$ Toda lattice hierarchy can be expressed in terms of the single lattice function τ_s depending via τ_0 on all hierarchy times. In this respect we can treat τ_s as the tau function of the hierarchy. Moreover, this identification is supported by the fact that the τ_s (38) are in agreement with a more general expression for the tau function of the STL hierarchy discussed in [9].

It is an established fact by now that the tau function of the semi-infinite bosonic Toda chain hierarchy (restricted by the Virasoro constraints) reproduces the partition function of the one-matrix model, which defines two-dimensional minimal conformal matter interacting with two-dimensional quantum gravity (for review, see e.g. [4, 5] and references therein). In this context we are led to consider the reduction (31) of the tau function (38), (34), and obtain the tau function of the $N=(1|1)$ supersymmetric Toda chain hierarchy. The latter may be relevant for attacking the old yet unsolved problem of constructing non-trivial supersymmetric matrix and/or

⁵The superdeterminant is defined as $\text{sdet} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \det(A - BD^{-1}C)(\det D)^{-1}$.

eigenvalue models [11, 12]. The reduction can be done rather straightforwardly, and we present the resulting formulae without any comments,

$$\begin{aligned} \tau_{2s} &= \text{sdet} \left(\begin{array}{cc} \partial^{i+j} \tau_0 & \partial^{i+m} D_- \tau_0 \\ \partial^{k+j} D_+ \tau_0 & \partial^{k+m} D_+ D_- \tau_0 \end{array} \right)_{\substack{0 \leq i, j \leq s \\ 0 \leq k, m \leq s-1}} \quad , \\ \tau_{2s+1} &= \text{sdet} \left(\begin{array}{cc} \partial^{i+j} \tau_0 & \partial^{i+m} D_- \tau_0 \\ \partial^{k+j} D_+ \tau_0 & \partial^{k+m} D_+ D_- \tau_0 \end{array} \right)_{\substack{0 \leq i, j \leq s \\ 0 \leq k, m \leq s}} \quad , \quad (41) \end{aligned}$$

where

$$\begin{aligned} \tau_0 &= - \int d\lambda \, d\eta_+ \, d\eta_- \, \varphi(\lambda, \eta_+ - \theta_+, \eta_- - \theta_-) \times \\ &\quad \exp \left[(x - \sum_{\alpha=\pm} \eta_\alpha \theta^\alpha) \lambda + \sum_{k=1}^{\infty} (t_k + \sum_{\alpha=\pm} \eta_\alpha \theta_k^\alpha) \lambda^k \right] \quad . \quad (42) \end{aligned}$$

It is also instructive to discuss their bosonic limit which looks as

$$\tau_{2s} | = \frac{\tau_s^T[\rho_1(\lambda)]}{\tau_{s-1}^T[\rho_2(\lambda)]} \quad , \quad \tau_{2s+1} | = \frac{\tau_s^T[\rho_1(\lambda)]}{\tau_s^T[\rho_2(\lambda)]} \quad (43)$$

Here, the vertical line means that all fermionic quantities entering τ_s are put equal to zero,

$$\rho_1(\lambda) \equiv \left(\frac{\partial}{\partial \eta_-} \frac{\partial}{\partial \eta_+} \varphi \right) (\lambda, 0, 0) \quad , \quad \rho_2(\lambda) \equiv \lambda^2 \varphi(\lambda, 0, 0) \quad , \quad (44)$$

and $\tau_s^T[\rho(\lambda)]$ denotes the tau function of the bosonic semi-infinite Toda chain hierarchy with a spectral density $\rho(\lambda)$. The functions τ_s^T can be represented in a determinant, eigenvalue integral, or matrix integral form [4, 5],

$$\begin{aligned} \tau_s^T[\rho(\lambda)] &= \det \left(\partial^{i+j} \int d\lambda \, \rho(\lambda) \exp \left\{ x\lambda + \sum_{k=1}^{\infty} t_k \lambda^k \right\} \right)_{0 \leq i, j \leq s} \\ &\equiv \frac{1}{s!} \int \left(\prod_{i=1}^s d\lambda_i \right) \left(\prod_{i \geq j=1}^s (\lambda_i - \lambda_j)^2 \right) \exp \sum_{i=1}^s [x\lambda_i + \sum_{k=1}^{\infty} t_k \lambda_i^k + \ln \rho(\lambda_i)] \\ &\equiv \int dM \exp \text{Tr} \left[xM + \sum_{k=1}^{\infty} t_k M^k + \ln \rho(M) \right] \quad , \quad (45) \end{aligned}$$

where M is an $s \times s$ hermitean matrix. It would be very interesting to find similar representations (if they exist) for the supersymmetric tau function τ_s (41), but we postpone a discussion of this rather non-trivial problem for future publications.

The form of the bosonic limit (43) of the tau function is not unexpected because the $N=(1|1)$ Toda lattice equation (6) then reduces to the direct sum of two bosonic Toda lattice equations with opposite signatures of their kinetic terms. In eqs. (43) this property is in fact reflected by the appearance of two Toda tau functions raised to opposite powers. Furthermore, supersymmetry fixes the relative dimensions of their spectral densities,

$$[\rho_1(\lambda)] = [\rho_2(\lambda)] + 1, \quad (46)$$

as one can see from eqs. (44) and (35). If, in addition, we require scaling invariance (meaning that only dimensionless constants are allowed) the spectral densities are forced to obey

$$\rho_2(\lambda) = \lambda \rho_1(\lambda) \quad (47)$$

modulo an inessential dimensionless factor. It is a rather curious fact that the two choices

$$\left\{ \rho_1(\lambda) = 1, \quad \rho_2(\lambda) = \lambda \right\} \text{ or } \left\{ \rho_1(\lambda) = \frac{1}{\lambda}, \quad \rho_2(\lambda) = 1 \right\} \quad (48)$$

yield the partition functions of the one-matrix model ($\rho = 1$) and the one of the generalized Penner model ($\ln \rho = \pm \ln \lambda$) [13]. In closing we would like to refer also to recent interesting work [14], where a Berezinian construction and similar bosonic limits have been derived in the context of the reduced Manin-Radul $N=1$ supersymmetric KP hierarchy.

4. Conclusion. In this work we have derived an infinite class of fermionic flows for the $N=(1|1)$ superconformal Toda lattice hierarchy, which are given by eqs. (24), (20) and (11). Their algebraic structure (28) has been produced as well. Further, we have constructed the general solution of the semi-infinite $N=(1|1)$ Toda lattice hierarchy and proposed an explicit expression (38) for its tau function in a superdeterminant form. Finally we have obtained the reduced tau function (41) which corresponds to the

semi-infinite $N=(1|1)$ Toda chain hierarchy. It was seen to have the appropriate bosonic limit and may be relevant for discovering non-trivial supersymmetric matrix models.

Acknowledgments. We would like to thank H. Aratyn, L. Bonora, F. Delduc, V. Kazakov and A.N. Leznov for their interest in our activity and for useful discussions. This work was partially supported by the Russian Foundation for Basic Research, Grant No. 96-02-17634, RFBR-DFG Grant No. 96-02-00180 and by INTAS grants INTAS-93-127-ext. and INTAS-96-0538.

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Received by Publishing Department
on October 19, 1998.

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E2-98-285

Фермионные потоки и тау-функция $N = (1|1)$
суперконформной решеточной иерархии Тоды

Построен бесконечный класс фермионных потоков для $N = (1|1)$ суперконформной решеточной иерархии Тоды и изучена их алгебраическая структура. Мы полностью решили полуограниченную решеточную и цепочную тодовские $N = (1|1)$ иерархии, которые могут быть важны для построения суперсимметричных матричных моделей. Обсуждены также их бозонные пределы.

Работа выполнена в Лаборатории теоретической физики им. Н.Н.Боголюбова ОИЯИ и в Институте теоретической физики Ганноверского университета (Германия).

Препринт Объединенного института ядерных исследований. Дубна, 1998

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E2-98-285

Fermionic Flows and Tau Function of the $N = (1|1)$
Superconformal Toda Lattice Hierarchy

An infinite class of fermionic flows of the $N = (1|1)$ superconformal Toda lattice hierarchy is constructed and their algebraic structure is studied. We completely solve the semi-infinite $N = (1|1)$ Toda lattice and chain hierarchies and derive their tau functions, which may be relevant for building supersymmetric matrix models. Their bosonic limit is also discussed.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR and at the Institute für Theoretische Physik, Universität Hannover (Germany).

Preprint of the Joint Institute for Nuclear Research. Dubna, 1998