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$(\bar{L}_n, g)$-SPACES. VARIATION OPERATOR

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1 Introduction

A Lagrangian theory of tensor fields over spaces with contravariant and covariant affine connections and metrics [$(\overline{L}_n, g)$-spaces] [1] has three essential structures [2], [3]: the *Lagrangian density*, the *Euler-Lagrange equations* and their corresponding *energy-momentum tensors*.

The Lagrangian density can be considered as a tensor density of rank 0 with the weight $q = \frac{1}{2}$, depending on tensor fields' components and their first and second partial (or covariant) derivatives.

The Euler-Lagrange equations can be obtained by means of the functional variation of a Lagrangian density and of these of its field variables considered as *dynamic field variables* (in contrast to the non-varied field variables considered as fixed and *non-dynamic field variables*).

The corresponding energy-momentum tensors can be found by means of the Lie variations (Lie derivatives) of a Lagrangian density and all of its field variables (dynamic and non-dynamic field variables). By means of Lie variations (change of the field variables by draggings-along of the tensor fields and their covariant derivatives) the corresponding energy-momentum tensors can be found.

1.1 Lagrangian densities of type 1 and type 2

In accordance to the two different considerations of a Lagrangian density one can introduce the following definitions:

Lagrangian density of type 1. Tensor density with weight $q = \frac{1}{2}$ and rank 0, depending on components of tensor fields (with finite rank) and their first (and second) *partial* derivatives with respect to the co-ordinates as well as on components of affine connections and their partial derivatives

$$\mathbf{L} = \sqrt{-d_g}.L(g_{ij},\, g_{ij,k},\, g_{ij,k,l},\, V^A{}_B,\, V^A{}_{B,i},\, V^A{}_{B,i,j}) \ , \tag{1}$$

where $L(x^k) = L'(x^{k'})$ is a Lagrangian invariant, g_{ij} are the components of the covariant metric tensor field $g = g_{ij}.dx^i.dx^j = g_{\alpha\beta}.e^\alpha.e^\beta$, $dx^i.dx^j = \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)$, $g_{ij} = g_{ji}$, $V^A{}_B$ are components of tensor fields V or components of an affine connection Γ or P, $d_g = \det(g_{ij}) < 0$,

$$V^A{}_{B,i} = \frac{\partial V^A{}_B}{\partial x^i} \ , \qquad V^A{}_{B,i,j} = \frac{\partial^2 V^A{}_B}{\partial x^j \partial x^i} \ . \tag{2}$$

Lagrangian density of type 2. Tensor density with weight $q = \frac{1}{2}$ and rank 0, depending on components of tensor fields (with finite rank) and their first (and second) *covariant* derivatives with respect to basic vector fields and to the corresponding affine connections

$$\mathbf{L} = \sqrt{-d_g}.L(g_{ij},\, g_{ij;k},\, g_{ij;k;l},\, V^A{}_B,\, V^A{}_{B;i},\, V^A{}_{B;i;j}) \ , \tag{3}$$

where $L(x^k) = L'(x^{k'})$ is a Lagrangian invariant, g_{ij} are the components of the covariant metric tensor field g, $V^A{}_B$ are components of tensor fields $V = V^A{}_B.e_A \otimes e^B = V^A{}_B.\partial_A \otimes dx^B$ with finite rank, $A = i_1...i_k$, $B = j_1...j_l$, $k, l \in \mathbf{N}$, $V^A{}_{B;i}$ are the covariant derivatives of $V^A{}_B$ with respect to the basic vector fields ∂_i (or e_i) (the Greek indices α, β,... are related to a non-coordinata basic vector field $e_i \equiv e_\alpha$).

From the properties of the tensor densities and the invariant volume element $d\omega$ [4] the properties of the product $L.d\omega$ determining the action S of a Lagrangian system,

$$S = \int\limits_{V_n} L.d\omega \ , \tag{4}$$

follow. $L.d\omega$ can be represented by means of the Lagrangian density $\mathbf{L}$

$$\begin{gathered} L.d\omega = \sqrt{-d_g}.L.\tfrac{1}{n!}.\varepsilon_A.\omega^A = \mathbf{L}.\tfrac{1}{n!}.\varepsilon_A.\omega^A = \\ = \sqrt{-d_g}.L.d^{(n)}x = \mathbf{L}.d^{(n)}x = \mathbf{L}.dV_n \ , \\ d^{(n)}x = dx^1 \wedge dx^2 \wedge ... \wedge dx^n \ , \ \dim M = n \ , \\ \varepsilon_A = \varepsilon_{i_1...i_n} = \varepsilon_{e_1...e_n} \ , \ \delta\varepsilon_{i_1...i_n} = 0 \ , \\ \omega^A = dx^{i_1} \wedge ... \wedge dx^{i_n} = e^{i_1} \wedge ... \wedge e^{i_n} \ . \end{gathered} \tag{5}$$

$\varepsilon_{i_1...i_n}$ is the Levi-Chivita symbol [4]. From

$$\nabla_\xi(L.d\omega) = (\xi L + \frac{1}{2}.L.\overline{g}[\nabla_\xi g]).d\omega = \sqrt{-d_g}(\xi L + \frac{1}{2}.L.\overline{g}[\nabla_\xi g]).d^{(n)}x \ , \tag{6}$$

where $\xi L = \xi^i.L_{,i} = L_{;i}.\xi^i$, the relation follows between the action of ∇_ξ and $\overline{\nabla}_\xi$ on $L.d\omega$

$$\nabla_\xi(\mathbf{L}.\frac{1}{n!}.\varepsilon_A.\omega^A) = (\overline{\nabla}_\xi\mathbf{L}).\frac{1}{n!}.\varepsilon_A.\omega^A \ , \tag{7}$$

$$\nabla_\xi(\mathbf{L}.d^{(n)}x) = (\overline{\nabla}_\xi\mathbf{L}).d^{(n)}x \ , \tag{8}$$

$$\begin{gathered} \overline{\nabla}_\xi\mathbf{L} = \sqrt{-d_g}\{\xi L + \tfrac{1}{2}.L.\overline{g}[\nabla_\xi g]\} = \\ = \sqrt{-d_g}(L_{;k} + \tfrac{1}{2}.L.g^{\overline{ij}}.g_{ij;k}).\xi^k \ . \end{gathered} \tag{9}$$

Two different definitions $\overline{\nabla}_u$ and $\widetilde{\nabla}_u$ for a covariant operator , acting on tensor densities (relative tensor fields) [4] as an automorphism in the relative tensors algebra, could be introduced and applied in the lagrangian formalism [1] for tensor fields. The covariant operator $\overline{\nabla}_u$ along a contravariant vector field $u = u^i.e_i = u^k.\partial_k$, acting on a tensor density $\overline{Q}$ of the type

$$\begin{gathered} \overline{Q} = (d_{\underline{K}})^q.Q \ , \ d_{\underline{K}} = \det(K_{ij}) \neq 0 \ , \ K_{ij} = K_{ji} \ , \qquad q \in \mathbf{R} \ , \\ Q = \overline{Q}^A{}_B.e_A \otimes e^B = Q^C{}_D.\partial_C \otimes dx^D \ , \\ A = i_1...i_j \ , \quad B = j_1...j_l \ , \quad C = l_1...l_j \ , \quad D = m_1...m_l \ , \end{gathered}$$

is determined as

$$\overline{\nabla}_u = \nabla_u + 2q.P^i_{ik}.u^k \ .$$

∇_u is the covariant differential operator [1], [5] along u, acting on tensor fields [for example, on a contravariant vector field $v = v^k.e_k = v^l.\partial_l$,

$$\nabla_u v = \nabla_{u^i.e_i} v = u^i.\nabla_{e_i} v = (e_i v^k + \Gamma^k_{li}.v^l).u^i.e_k = (v^k{}_{,i} + \Gamma^k_{li}.v^l).u^i.\partial_k \ ,$$

or on a covariant vector field $p = p_k.e^k = p_l.dx^l$,

$$\nabla_u p = \nabla_{u^i.e_i} p = u^i.\nabla_{e_i} p = (e_i p_k + P^l_{ki}.p_l).u^i.e^k = p_{k,i} + P^l_{ki}.p_l).u^i.dx^k \ .]$$

The covariant derivative $\overline{\nabla}_u\overline{Q}$ can be written in the forms

$$\begin{gathered} \overline{\nabla}_u\overline{Q} = (d_{\underline{K}})^q.\nabla_u Q + [\nabla_u(d_{\underline{K}})^q].Q = \\ = \nabla_u\overline{Q} + 2q.P^i_{ik}.u^k.\overline{Q} = \overline{Q}^A{}_{B;k}.u^k.e_A \otimes e^B = \\ = [\overline{Q}^A{}_{B,k} + \Gamma^A_{Ck}.\overline{Q}^C{}_B + P^D_{Bk}.\overline{Q}^A{}_D + 2q.P^i_{ik}.\overline{Q}^A{}_B].u^k.e_A \otimes e^B \ . \end{gathered} \tag{10}$$

The "Leibniz rule" for differentiation $\overline{\nabla}_u[(d_{\underline{K}})^q.Q] = (d_{\underline{K}})^q.\nabla_u Q + [\nabla_u(d_{\underline{K}})^q].Q$ is fulfilled for $\overline{\nabla}_u\overline{Q}$ from (10).

For the introduced in (L_n, g)-spaces by other authors [4] covariant operator $\widetilde{\nabla}_u$ by definition in a co-ordinate basis $\{\partial_i\}$ as

$$\widetilde{\nabla}_u = \nabla_u + 2q.P^i_{ki}.u^k$$

with

$$\begin{gathered} \widetilde{\nabla}_u\overline{Q} = (d_{\underline{K}})^q.\nabla_u Q + [\nabla_u(d_{\underline{K}})^q].Q + 2q.U_{ki}{}^i.u^k.\overline{Q} = \\ = \nabla_u\overline{Q} + 2q.P^i_{ki}.u^k.\overline{Q} = \overline{Q}^A{}_{B;k}.u^k.\partial_A \otimes dx^B = \\ = [\overline{Q}^A{}_{B,k} + \Gamma^A_{Ck}.\overline{Q}^C{}_B + P^D_{Bk}.\overline{Q}^A{}_D + 2q.P^i_{ki}.\overline{Q}^A{}_B].u^k.\partial_A \otimes dx^B \ , \\ U_{ki}{}^i = P^i_{ki} - P^i_{ik} \ , \end{gathered} \tag{11}$$

the first relation of (10) is not fulfilled. For tensor fields $\overline{\nabla}_\xi = \nabla_\xi$, $\widetilde{\nabla}_\xi = \nabla_\xi$.

If the components $K^A{}_B$ of a tensor field $K = K^A{}_B.e_A \otimes e^B = K^A{}_B.\partial_A \otimes dx^B$ depend on components of other tensor fields and their covariant derivatives, i. e. if

$$K^A{}_B = K^A{}_B(V^C{}_D, ...) \ ,$$

where $A = i_1...i_l$, $B = j_1...j_m$, $C = k_1...k_r$, $D = l_1...l_s$, then the following relations will be fulfilled after a change of the tensor basis to another tensor basis:

$$\begin{gathered} K^{A'}{}_{B'} = T_A{}^{A'}.T_{B'}{}^B.K^A{}_B \ , \\ T_A{}^{A'} = A_{i_1}{}^{i'_1}...A_{i_l}{}^{i'_l} \ , \qquad T_{B'}{}^B = A_{j'_1}{}^{j_1}...A_{j'_m}{}^{j_m} \ , \\ T_C{}^{C'} = A_{k_1}{}^{k_1'}...A_{k_r}{}^{k'_r} \ , \quad T_{D'}{}^D = A_{l'_1}{}^{l_1}...A_{l'_s}{}^{l_s} \ , \\ K^{A'}{}_{B'} = K^{A'}{}_{B'}(V^{C'}{}_{D'}, ...) \ , \qquad V^C{}_D = T_{C'}{}^C.T_D{}^{D'}.V^{C'}{}_{D'} \ . \\ T_{C'}{}^C = A_{k'_1}{}^{k_1}...A_{k'_r}{}^{k_r} \ , \end{gathered}$$

$$\frac{\partial V^C{}_D}{\partial V^{C'}{}_{D'}} = T_{C'}{}^C.T_D{}^{D'} ,$$

$$\frac{\partial K^{A'}{}_{B'}}{\partial V^{C'}{}_{D'}} = \frac{\partial}{\partial V^{C'}{}_{D'}}(T_A{}^{A'}.T_{B'}{}^B.K^A{}_B) = T_A{}^{A'}.T_{B'}{}^B.\frac{\partial K^A{}_B}{\partial V^{C'}{}_{D'}} =$$
$$= T_A{}^{A'}.T_{B'}{}^B.\frac{\partial K^A{}_B}{\partial V^C{}_D}.\frac{\partial V^C{}_D}{\partial V^{C'}{}_{D'}} = T_A{}^{A'}.T_{B'}{}^B.T_{C'}{}^C.T_D{}^{D'}.\frac{\partial K^A{}_B}{\partial V^C{}_D} .$$

It follows from the last expression that *the partial derivatives of the components of a tensor field with respect to components of another tensor field on which the first components are depending are again components of a tensor field.* This statement is proved by Schmutzer [6] (pp. 51-52) and is generalized for spinor and bispinor fields [6] (pp. 36-37).

2 Variation operator

Variation (variational) operator. Operator, acting on the components of tensor fields in a given basis and mapping these tensor fields in tensor fields with the same rank, with the following properties:

1. Action on a tensor field K:

$$\delta : K \to \delta K , \qquad K, \ \delta K \in \otimes^k{}_l(M),$$
$$\delta K = \delta K^A{}_B.e_A \otimes e^B = \delta K^C{}_D.\partial_C \otimes dx^D, \ K = K^A{}_B.e_A \otimes e^B = K^C{}_D.\partial_C \otimes dx^D,$$
$$K^A{}_B \in C^r(M), \quad \delta K^A{}_B \in C^r(M), \ x \in M .$$

2. Action on a function f:

$$\delta : f \to \delta f , \qquad f, \delta f \in C^r(M) .$$

3. Linear operator with respect to tensor fields:

$$\delta(\alpha.K_1 + \beta.K_2) = \alpha.\delta K_1 + \beta.\delta K_2 ,$$
$$\alpha, \beta \in R \text{ (or } C), \quad K_i \in \otimes^k{}_l(M) , \ i = 1,2 .$$

4. Differential operator acting on tensor fields and obeying the Leibnitz rule:

$$\delta(f.g) = \delta f.g + f.\delta g , \quad f, g \in C^r(M) , \quad f, g \in \otimes^0{}_0(M),$$
$$\delta(Q \otimes S) = \delta Q \otimes S + Q \otimes \delta S , \quad Q \in \otimes^k{}_l(M) , \ S \in \otimes^m{}_r(M) .$$

5a. (Possible) commutation relations *(commutation relations of type A)* with the Lie-differential operator:

$$\delta \circ \pounds_{\partial_j} = \pounds_{\partial_j} \circ \delta , \quad \delta \circ \pounds_{e_\alpha} = \pounds_{e_\alpha} \circ \delta ,$$
$$\delta \circ \pounds_\xi - \pounds_{\delta\xi} = \pounds_\xi \circ \delta .$$

5b. (Possible) commutation relations (*commutation relations of type B*) with the covariant differential operator:

$$\delta \circ \nabla_{\partial_j} = \nabla_{\partial_j} \circ \delta , \quad \delta \circ \nabla_{e_\alpha} = \nabla_{e_\alpha} \circ \delta , \quad \delta \circ \nabla_\xi - \nabla_{\delta\xi} = \nabla_\xi \circ \delta .$$

5c. (Possible) commutation relations *(commutation relations of type C)* with the contraction operator S:

$$\delta \circ S = S \circ \delta .$$

From the properties 2. and 4. it follows that $\delta 1 = 0 , \ 1 \in \mathbf{N} \subset C^r(M)$.

Proof. $\delta(1.1) = (\delta 1).1 + 1.(\delta 1) = 2.(\delta 1) = \delta 1 : \delta 1 = 0$.

From the properties 2., 3. and 4. it follows that $\delta\alpha = 0, \alpha =$ const. $\in R$ (or C) $\subset C^r(M)$.

Proof. $\delta(\alpha.g) = \alpha.\delta g = \delta\alpha.g + \alpha.\delta g : \delta\alpha = 0, \forall g \in C^r(M)$.

3 Consequences from the commutation relations of the variation operator with the Lie differential operator

3.1 Consequences from $\delta \circ \pounds_{\partial_j} = \pounds_{\partial_j} \circ \delta$ and $\delta \circ \pounds_{e_\alpha} = \pounds_{e_\alpha} \circ \delta$

1. Action of δ and $\pounds_{\partial_j}$ on a function. From $\pounds_{\partial_j} f = \partial_j f = f_{,j}$, $f \in C^r(M)$, $\delta(\pounds_{\partial_j} f) = \delta(f_{,j})$, $\pounds_{\partial_j}(\delta f) = (\delta f)_{,j}$, and $\delta \circ \pounds_{\partial_j} = \pounds_{\partial_j} \circ \delta$ the commutation between the partial derivatives along the co-ordinates and the functional variation of a function f follows in the form

$$\delta(f_{,j}) = (\delta f)_{,j} . \tag{12}$$

2. Action of δ and $\pounds_{\partial_j}$ on a contravariant co-ordinate basic vector field. From $\pounds_{\partial_j} \partial_i = [\partial_j, \partial_i] = \mathbf{0} \in T(M)$, it follows that $\delta(\pounds_{\partial_j} \partial_i) = \delta\mathbf{0} = \mathbf{0}$. On the other side $\delta(\partial_i) = \delta(1.\partial_i) = (\delta 1).\partial_i = 0.\partial_i = \mathbf{0} \in T(M)$. Therefore, $\delta(\pounds_{\partial_j} \partial_i) = \pounds_{\partial_j}(\delta\partial_i) = \delta\mathbf{0} = \mathbf{0}$.

3. Action of δ and $\pounds_{e_\alpha}$ on a contravariant non-co-ordinate basic vector field. From the relations $\pounds_{e_\alpha} e_\beta = C_{\alpha\beta}{}^\gamma.e_\gamma$, $\delta e_\beta = (\delta 1).e_\beta = 0.e_\beta = \mathbf{0} \in T(M)$, $\delta(\pounds_{e_\alpha} e_\beta) = (\delta C_{\alpha\beta}{}^\gamma).e_\gamma$, $\pounds_{e_\alpha}(\delta e_\beta) = \pounds_{e_\alpha}\mathbf{0} = \mathbf{0}$, and $\delta(\pounds_{e_\alpha} e_\beta) = \pounds_{e_\alpha}(\delta e_\beta)$, it follows that

$$\delta C_{\alpha\beta}{}^\gamma = 0. \tag{13}$$

4. Action of δ and $\pounds_{\partial_j}$ on a covariant co-ordinate basic vector field. From the relations $\pounds_{\partial_j} dx^i = (P^i_{kj} + \Gamma^{\bar{i}}_{\bar{k}j}).dx^k$, $\delta dx^i = (\delta 1).dx^i = 0.dx^i = \mathbf{0}^* \in$

$T^*(M)$, $\delta(\pounds_{\partial_j} dx^i) = [\delta(P^i_{kj} + \Gamma^{\bar{i}}_{\underline{k}j})].dx^k$, $\pounds_{\partial_j}(\delta dx^i) = \mathbf{0}^*$, and $\delta(\pounds_{\partial_j} dx^i) = \pounds_{\partial_j}(\delta dx^i)$, it follows that

$$\delta(P^i_{kj} + \Gamma^{\bar{i}}_{\underline{k}j}) = 0 , \quad \delta P^i_{kj} = -\delta(\Gamma^{\bar{i}}_{\underline{k}j}) . \tag{14}$$

5. Action of δ and $\pounds_{e_\alpha}$ on a covariant non-co-ordinate basic vector field. From the relations $\pounds_{e_\alpha} e^\beta = (P^\beta_{\gamma\alpha} + \Gamma^{\bar{\beta}}_{\underline{\gamma}\alpha} + C_{\underline{\gamma}\alpha}{}^{\bar{\beta}}).e^\gamma$, $\delta(\pounds_{e_\alpha} e^\beta) = [\delta(P^\beta_{\gamma\alpha} + \Gamma^{\bar{\beta}}_{\underline{\gamma}\alpha} + C_{\underline{\gamma}\alpha}{}^{\bar{\beta}})].e^\gamma$, $\delta e^\beta = \mathbf{0}^*$, $\pounds_{e_\alpha}(\delta e^\beta) = \mathbf{0}^*$, and $\delta(\pounds_{e_\alpha} e^\beta) = \pounds_{e_\alpha}(\delta e^\beta)$, it follows that

$$\delta(P^\beta_{\gamma\alpha} + \Gamma^{\bar{\beta}}_{\underline{\gamma}\alpha} + C_{\gamma\alpha}{}^\beta) = 0 . \tag{15}$$

6. Action of δ and $\pounds_{\partial_j}$ on a mixed tensor field. From the relations

$$\begin{aligned}
\pounds_{\partial_j} K &= K^A{}_{B,j}.\partial_A \otimes dx^B + K^A{}_B.\pounds_{\partial_j}(\partial_A \otimes dx^B), \\
\pounds_{\partial_j}\partial_A &= 0, \ (\pounds_{\partial_j}\partial_i = 0), \quad \pounds_{\partial_j} dx^B = (P^B_{Dj} + \tilde{\Gamma}^B_{Dj}).dx^D, \\
\tilde{\Gamma}^B_{Dj} &= -S_{Di}{}^{Bk}.\Gamma^{\bar{i}}_{\underline{k}j}, \quad P^B_{Dj} = -S_{Di}{}^{Bk}.P^i_{kj},
\end{aligned}$$

$$\begin{aligned}
\pounds_{\partial_j} K &= [K^A{}_{B,j} + (P^D_{Bj} + \tilde{\Gamma}^D_{Bj}).K^A{}_D].\partial_A \otimes dx^B = \\
&= (\pounds_{\partial_j} K^A{}_B).\partial_A \otimes dx^B,
\end{aligned}$$

$$\begin{aligned}
\delta(\pounds_{\partial_j} K) &= \{\delta(K^A{}_{B,j}) + [\delta(P^D_{Bj} + \tilde{\Gamma}^D_{Bj})].K^A{}_D + (P^D_{Bj} + \tilde{\Gamma}^D_{Bj}).\delta K^A{}_D\}.\partial_A \otimes dx^B = \\
&= [\delta(\pounds_{\partial_j} K^A{}_B)].\partial_A \otimes dx^B, \ \delta S_{Ai}{}^{Bj} = 0 \ (\text{because of } \delta g^i_j = 0), \\
\pounds_{\partial_j}(\delta K) &= [(\delta K^A{}_B)_{,j} + (P^D_{Bj} + \tilde{\Gamma}^D_{Bj}).\delta K^A{}_D].\partial_A \otimes dx^B = \\
&= [\pounds_{\partial_j}(\delta K^A{}_B)].\partial_A \otimes dx^B , \\
P^D_{Bj} + \tilde{\Gamma}^D_{Bj} &= -S_{Bi}{}^{Dk}(P^i_{kj} + \Gamma^{\bar{i}}_{\underline{k}j}) ,
\end{aligned} \tag{16}$$

and p. 4., it follows that $\delta(P^D_{Bj} + \tilde{\Gamma}^D_{Bj}) = 0$, $\delta P^D_{Bj} = -\delta\tilde{\Gamma}^D_{Bj}$.

From the commutation relation $\delta(\pounds_{\partial_j} K) = \pounds_{\partial_j}(\delta K) \simeq \delta(\pounds_{\partial_j} K^A{}_B) = \pounds_{\partial_j}(\delta K^A{}_B)$ the commutation between the partial derivative and the functional variation follows

$$\delta(K^A{}_{B,j}) = (\delta K^A{}_B)_{,j} . \tag{17}$$

7. Action of δ and $\pounds_{\partial_j}$ on a contravariant affine connection Γ. From the relations

$$\begin{aligned}
&\nabla_{\partial_j}\partial_i = \Gamma^k_{ij}.\partial_k, \ \delta(\nabla_{\partial_j}\partial_i) = \delta\Gamma^k_{ij}.\partial_k, \\
&\pounds_{\partial_l}[\delta(\nabla_{\partial_j}\partial_i)] = \pounds_{\partial_l}(\delta\Gamma^k_{ij}.\partial_k) = (\delta\Gamma^k_{ij})_{,l}.\partial_k + \delta\Gamma^k_{ij}.\pounds_{\partial_l}\partial_k = (\delta\Gamma^k_{ij})_{,l}.\partial_k ,
\end{aligned}$$

$$\begin{aligned}
\pounds_{\partial_l}(\nabla_{\partial_j}\partial_i) &= \pounds_{\partial_l}(\Gamma^k_{ij}.\partial_k) = \Gamma^k_{ij,l}.\partial_k + \Gamma^k_{ij}.\pounds_{\partial_l}\partial_k = \Gamma^k_{ij,l}.\partial_k , \\
\delta[\pounds_{\partial_l}(\nabla_{\partial_j}\partial_i)] &= \delta(\Gamma^k_{ij,l}).\partial_k ,
\end{aligned}$$

the commutation between the partial derivative and the functional variation of a contravariant affine connection follows

$$(\delta\Gamma^i_{jk})_{,l} = \delta(\Gamma^i_{jk,l}) . \tag{18}$$

8. Action of δ and $\pounds_{\partial_j}$ on a covariant affine connection P. From the relations

$$\begin{aligned}
\pounds_{\partial_l}[\delta(\nabla_{\partial_j} dx^i)] &= \pounds_{\partial_l}(\delta P^i_{kj}.dx^k) = (\delta P^i_{kj})_{,l}.dx^k + \delta P^i_{kj}.\pounds_{\partial_l} dx^k , \\
\pounds_{\partial_l} dx^k &= (P^k_{ml} + \Gamma^{\bar{k}}_{\underline{m}l}).dx^m , \\
\pounds_{\partial_l}[\delta(\nabla_{\partial_j} dx^i)] &= [(\delta P^i_{kj})_{,l} + (P^m_{kl} + \Gamma^{\bar{m}}_{\underline{k}l}).\delta P^i_{mj}].dx^k ,
\end{aligned}$$

$$\begin{aligned}
\pounds_{\partial_l}(\nabla_{\partial_j} dx^i) &= \pounds_{\partial_l}(P^i_{kj}.dx^k) = P^i_{kj,l}.dx^k + P^i_{kj}.\pounds_{\partial_l} dx^k = \\
&= [P^i_{kj,l} + (P^m_{kl} + \Gamma^{\bar{m}}_{\underline{k}l}).\delta P^i_{mj}].dx^k , \\
&\delta(P^i_{kj} + \Gamma^{\bar{i}}_{\underline{k}j}) = 0 , \\
\delta[\pounds_{\partial_l}(\nabla_{\partial_j} dx^i)] &= [\delta(P^i_{kj,l}) + (P^m_{kl} + \Gamma^{\bar{m}}_{\underline{k}l}).\delta P^i_{mj}].dx^k ,
\end{aligned}$$

the commutation between the partial derivative and the functional variation of a covariant affine connection follows

$$(\delta P^i_{jk})_{,l} = \delta(P^i_{jk,l}) . \tag{19}$$

9. From the relations

$$\begin{aligned}
&\pounds_{e_\alpha} e_\beta = \nabla_{e_\alpha} e_\beta - \nabla_{e_\beta} e_\alpha - T(e_\alpha, e_\beta) , \\
&\delta(\pounds_{e_\alpha} e_\beta) = \delta(\nabla_{e_\alpha} e_\beta) - \delta(\nabla_{e_\beta} e_\alpha) - \delta[T(e_\alpha, e_\beta)] , \\
&\delta C_{\alpha\beta}{}^\gamma.e_\gamma = (\delta\Gamma^\gamma_{\beta\alpha} - \delta\Gamma^\gamma_{\alpha\beta} - \delta T_{\alpha\beta}{}^\gamma).e_\gamma , \qquad \delta C_{\alpha\beta}{}^\gamma = 0 ,
\end{aligned}$$

it follows the condition for $\delta T_{\alpha\beta}{}^\gamma$

$$\delta T_{\alpha\beta}{}^\gamma = \delta\Gamma^\gamma_{\beta\alpha} - \delta\Gamma^\gamma_{\alpha\beta} . \tag{20}$$

3.2 Consequences from $\delta \circ \pounds_\xi - \pounds_{\delta\xi} = \pounds_\xi \circ \delta$

1. Action of δ and $\pounds_\xi$ on a function. From the relations $\pounds_\xi f = \xi f = \xi^j.f_{,j}$, $f \in C^r(M)$, $\pounds_{\delta\xi} f = (\delta\xi) f = \delta\xi^j.f_{,j}$, $\pounds_\xi(\delta f) = \xi(\delta f) = (\delta f)_{,j}.\xi^j$, $(\delta \circ \pounds_\xi - \pounds_{\delta\xi}) f = (\pounds_\xi \circ \delta)$, it follows that $\delta(f_{,j}) = (\delta f)_{,j}$ for $\forall \xi \in T(M)$ [s. (12)].

2. Action of δ and $\pounds_\xi$ on a contravariant co-ordinate basic vector field ∂_i. From the relations $\pounds_\xi \partial_i = [\xi, \partial_i] = -\xi^j{}_{,i}.\partial_j$, $\pounds_{\delta\xi}\partial_i = -(\delta\xi^j)_{,i}.\partial_j$, $\delta(\pounds_\xi \partial_i) - \pounds_{\delta\xi}\partial_i = [-\delta(\xi^j{}_{,i}) + (\delta\xi^j)_{,i}].\partial_j$, $\pounds_\xi(\delta\partial_i) = \mathbf{0}$, $(\delta \circ \pounds_\xi - \pounds_{\delta\xi})\partial_i = (\pounds_\xi \circ \delta)\partial_i = \mathbf{0}$ for $\forall \xi \in T(M)$, it follows the relation

$$\delta(\xi^j{}_{,i}) = (\delta\xi^j)_{,i} . \tag{21}$$

3. Action of δ and $\pounds_\xi$ on a covariant co-ordinate basic vector field dx^i. From the relations

$$\begin{aligned}
&\pounds_\xi dx^i = [\xi^{\bar{i}}{}_{,\underline{j}} + (P^i_{jk} + \Gamma^{\bar{i}}_{\underline{j}k}).\xi^k].dx^j , \\
&\pounds_{\delta\xi} dx^i = [(\delta\xi^{\bar{i}})_{,\underline{j}} + (P^i_{jk} + \Gamma^{\bar{i}}_{\underline{j}k}).\delta\xi^k].dx^j , \ (\delta\xi^{\bar{i}})_{,\underline{j}} = f^i{}_k.(\delta\xi^k)_{,l}.f_j{}^l , \\
&\delta(\pounds_\xi dx^i) = [\delta(\xi^{\bar{i}}{}_{,\underline{j}}) + \delta(P^i_{jk} + \Gamma^{\bar{i}}_{\underline{j}k}).\xi^k + (P^i_{jk} + \Gamma^{\bar{i}}_{\underline{j}k}).\delta\xi^k].dx^j .
\end{aligned}$$

$$
\begin{gathered}
\pounds_\xi(\delta dx^i) = \pounds_\xi(\mathbf{0}^*) = \mathbf{0}^* \ , \\
\delta(\pounds_\xi dx^i) - \pounds_{\delta\xi} dx^i = \pounds_\xi(\delta dx^i) = \mathbf{0}^* \ , \\
\delta(\xi^{\overline{i}}{}_{,\underline{j}}) + \delta(P^i_{jk} + \Gamma^{\overline{i}}_{\underline{j}k}).\xi^k + (P^i_{jk} + \Gamma^{\overline{i}}_{\underline{j}k}).\delta\xi^k - (\delta\xi^{\overline{i}})_{,\underline{j}} - (P^i_{jk} + \Gamma^{\overline{i}}_{\underline{j}k}).\delta\xi^k = \\
= \delta(\xi^{\overline{i}}{}_{,\underline{j}}) - (\delta\xi^{\overline{i}})_{,\underline{j}} \ , \quad \delta(P^i_{jk} + \Gamma^{\overline{i}}_{\underline{j}k}) = 0 \ ,
\end{gathered}
$$

a commutation relation follows in the form

$$
\delta(\xi^{\overline{i}}{}_{,\underline{j}}) = (\delta\xi^{\overline{i}})_{,\underline{j}} \ , \qquad \text{or} \qquad \delta(f^i{}_k.\xi^k{}_{,l}.f_j{}^l) = f^i{}_k.(\delta\xi^k)_{,l}.f_j{}^l \ . \tag{22}
$$

The last relation after using (21) leads to $\xi^k{}_{,l}.\delta(f^i{}_k.f_j{}^l) = 0$ for $\forall \xi^k{}_{,l} \in C^r(M)$ and therefore, to the relation for $f^i{}_k$ and $f_j{}^l$

$$
\delta(f^i{}_k.f_j{}^l) = 0 \ . \tag{23}
$$

From $\delta(P^i_{jk} + \Gamma^{\overline{i}}_{\underline{j}k}) = 0$ and $\delta(f^i{}_k.f_j{}^l) = 0$, it follows that

$$
\delta(\Gamma^{\overline{i}}_{\underline{j}k}) = \delta(f^i{}_l.\Gamma^l_{mk}.f_j{}^m) = \delta\Gamma^{\overline{i}}_{\underline{j}k} = f^i{}_l.\delta\Gamma^l_{mk}.f_j{}^m
$$

and

$$
\delta P^i_{jk} = - \ \delta\Gamma^{\overline{i}}_{\underline{j}k} \ . \tag{24}
$$

4. Action of δ and $\pounds_\xi$ on a contravariant non-co-ordinate basic vector field e_β. From the relations

$$
\begin{gathered}
\pounds_\xi e_\beta = -(e_\beta\xi^\alpha + C_{\beta\gamma}{}^\alpha.\xi^\gamma).e_\alpha \ , \\
\pounds_{\delta\xi} e_\beta = -[e_\beta(\delta\xi^\alpha) + C_{\beta\gamma}{}^\alpha.\delta\xi^\gamma].e_\alpha \ , \\
\delta(\pounds_\xi e_\beta) = -[\delta(e_\beta\xi^\alpha) + C_{\beta\gamma}{}^\alpha.\delta\xi^\gamma + \xi^\gamma.\delta C_{\beta\gamma}{}^\alpha].e_\alpha \ , \\
\delta(\pounds_\xi e_\beta) - \pounds_{\delta\xi} e_\beta = \pounds_\xi(\delta e_\beta) = \mathbf{0} \ ,
\end{gathered}
$$

the commutation relation follows in the form

$$
\delta(e_\beta\xi^\alpha) = e_\beta(\delta\xi^\alpha) \ . \tag{25}
$$

5. Action of δ and $\pounds_\xi$ on a covariant non-co-ordinate basic vector field. From the relations

$$
\begin{gathered}
\pounds_\xi e^\beta = [e_{\underline{\alpha}}\xi^{\overline{\beta}} + (P^\beta_{\alpha\gamma} + \Gamma^{\overline{\beta}}_{\underline{\alpha}\gamma} + C_{\underline{\alpha}\gamma}{}^{\overline{\beta}}).\xi^\gamma].e^\alpha \ , \\
\delta(\pounds_\xi e^\beta) = [\delta(e_{\underline{\alpha}}\xi^{\overline{\beta}}) + \delta(P^\beta_{\alpha\gamma} + \Gamma^{\overline{\beta}}_{\underline{\alpha}\gamma} + C_{\underline{\alpha}\gamma}{}^{\overline{\beta}}).\xi^\gamma + (P^\beta_{\alpha\gamma} + \Gamma^{\overline{\beta}}_{\underline{\alpha}\gamma} + C_{\underline{\alpha}\gamma}{}^{\overline{\beta}}).\delta\xi^\gamma].e^\alpha \ , \\
\delta(P^\beta_{\alpha\gamma} + \Gamma^{\overline{\beta}}_{\underline{\alpha}\gamma} + C_{\underline{\alpha}\gamma}{}^{\overline{\beta}}) = 0 \ , \\
\pounds_{\delta\xi} e^\beta = [e_{\underline{\alpha}}(\delta\xi^{\overline{\beta}}) + (P^\beta_{\alpha\gamma} + \Gamma^{\overline{\beta}}_{\underline{\alpha}\gamma} + C_{\underline{\alpha}\gamma}{}^{\overline{\beta}}).\delta\xi^\gamma].e^\alpha \ , \\
e_{\underline{\alpha}}(\delta\xi^{\overline{\beta}}) = f_\alpha{}^\gamma.[e_\gamma(\delta\xi^\rho)].f^\beta{}_\rho \ , \qquad \pounds_\xi(\delta e^\beta) = \mathbf{0}^* \ , \\
\delta(\pounds_\xi e^\beta) - \pounds_{\delta\xi} e^\beta = \pounds_\xi(\delta e^\beta) = \mathbf{0}^* \ ,
\end{gathered}
$$

the relations follow in the form

$$
\begin{gathered}
\delta(e_{\underline{\alpha}}\xi^{\overline{\beta}}) = e_{\underline{\alpha}}(\delta\xi^{\overline{\beta}}) \ , \quad \delta(f^\alpha{}_\rho.f_\beta{}^\gamma) = 0 : \delta f^\alpha{}_\beta = \delta f^{\underline{\alpha}}{}_{\overline{\beta}} \ , \\
\delta f^{\underline{\alpha}}{}_{\overline{\beta}} = f^\rho{}_\beta.\delta f^\sigma{}_\rho.f_\sigma{}^\alpha \ .
\end{gathered} \tag{26}
$$

6. Action on δ and $\pounds_\xi$ on a mixed tensor field. From the relations

$$
\begin{gathered}
\pounds_\xi K = K^A{}_{B,j}.\xi^j.\partial_A \otimes dx^B + K^A{}_B.\pounds_\xi(\partial_A \otimes dx^B) = \pounds_\xi K^A{}_B.\partial_A \otimes dx^B \ , \\
\pounds_\xi \partial_A = S_{Am}{}^{Cn}.\xi^m{}_{,n}.\partial_C \ , \\
\pounds_\xi dx^B = -[\xi^{\overline{m}}{}_{,\underline{n}} + (P^m_{nk} + \Gamma^{\overline{m}}_{\underline{n}k}).\xi^k].dx^A \ , \\
\pounds_\xi K^A{}_B = K^A{}_{B,j}.\xi^j + S_{Cm}{}^{An}.K^C{}_B.\xi^m{}_{,n} - \\
- S_{Bm}{}^{Dn}.K^A{}_D.[\xi^{\overline{m}}{}_{,\underline{n}} + (P^m_{nk} + \Gamma^{\overline{m}}_{\underline{n}k}).\xi^k] \ ,
\end{gathered}
$$

the cornmutation relations follow in the form

$$
\delta(K^A{}_{B,j}) = (\delta K^A{}_B)_{,j} \ . \tag{27}
$$

7. Action of δ and $\pounds_\xi$ on a contravariant affine connection Γ. From the relations

$$
\begin{gathered}
\nabla_{e_\beta} e_\alpha = \Gamma^\gamma_{\alpha\beta}.e_\gamma \ , \ \delta(\nabla_{e_\beta} e_\alpha) = (\delta\Gamma^\gamma_{\alpha\beta}).e_\gamma \ , \ \delta(e_\gamma) = \mathbf{0} \ , \\
\pounds_\xi[\delta(\nabla_{e_\beta} e_\alpha)] = [\xi^\delta.e_\delta(\delta\Gamma^\gamma_{\alpha\beta}) - (e_\delta\xi^\gamma + C_{\delta\rho}{}^\gamma.\xi^\rho).\delta\Gamma^\delta_{\alpha\beta}].e_\gamma \ , \\
\pounds_{\delta\xi}(\nabla_{e_\beta} e_\alpha) = \{\delta\xi^\delta.e_\delta\Gamma^\gamma_{\alpha\beta} - [e_\delta(\delta\xi^\gamma) + C_{\delta\rho}{}^\gamma.\delta\xi^\rho].\Gamma^\delta_{\alpha\beta}\}.e_\gamma \ , \\
\delta C_{\alpha\beta}{}^\gamma = 0 \ , \ \delta(e_\beta\xi^\alpha) = e_\beta(\delta\xi^\alpha) \ ,
\end{gathered}
$$

the commutation relations in a non-co-ordinate basis follow in the form

$$
\delta(e_\delta\Gamma^\gamma_{\alpha\beta}) = e_\delta(\delta\Gamma^\gamma_{\alpha\beta}) \ . \tag{28}
$$

8. Action of δ and $\pounds_\xi$ on a covariant affine connection P. From the relations

$$
\begin{gathered}
\nabla_{e_\gamma} e^\alpha = P^\alpha_{\beta\gamma}.e^\beta \ , \qquad \delta(\nabla_{e_\gamma} e^\alpha) = \delta P^\alpha_{\beta\gamma}.e^\beta \ , \\
\pounds_\xi(\nabla_{e_\gamma} e^\alpha) = (\xi^\sigma.e_\sigma P^\alpha_{\beta\gamma}).e^\beta + P^\alpha_{\kappa\gamma}.[e_{\underline{\beta}}\xi^{\overline{\kappa}} + (P^\kappa_{\beta\rho} + \Gamma^{\overline{\kappa}}_{\underline{\beta}\rho} + C_{\underline{\beta}\rho}{}^{\overline{\kappa}}).\xi^\rho].e^\beta \ , \\
\delta[\pounds_\xi(\nabla_{e_\gamma} e^\alpha)] = [\delta\xi^\sigma.e_\sigma P^\alpha_{\beta\gamma} + \xi^\sigma.\delta(e_\sigma P^\alpha_{\beta\gamma})].e^\beta + \\
+ \delta P^\alpha_{\kappa\gamma}.[e_{\underline{\beta}}\xi^{\overline{\kappa}} + (P^\kappa_{\beta\rho} + \Gamma^{\overline{\kappa}}_{\underline{\beta}\rho} + C_{\underline{\beta}\rho}{}^{\overline{\kappa}}).\xi^\rho].e^\beta + \\
+ P^\alpha_{\kappa\gamma}.[\delta(e_{\underline{\beta}}\xi^{\overline{\kappa}}) + \delta\xi^\rho.(P^\kappa_{\beta\rho} + \Gamma^{\overline{\kappa}}_{\underline{\beta}\rho} + C_{\underline{\beta}\rho}{}^{\overline{\kappa}}) + \xi^\rho.\delta(P^\kappa_{\beta\rho} + \Gamma^{\overline{\kappa}}_{\underline{\beta}\rho} + C_{\underline{\beta}\rho}{}^{\overline{\kappa}})].e^\beta \ , \\
\delta(P^\kappa_{\beta\rho} + \Gamma^{\overline{\kappa}}_{\underline{\beta}\rho} + C_{\underline{\beta}\rho}{}^{\overline{\kappa}}) = 0 \ , \quad \delta(e_{\underline{\beta}}\xi^{\overline{\kappa}}) = e_{\underline{\beta}}(\delta\xi^{\overline{\kappa}}) \ ,
\end{gathered}
$$

the commutation relations in a non-co-ordinate basis follow in the form

$$
\delta(e_\sigma P^\alpha_{\beta\gamma}) = e_\sigma(\delta P^\alpha_{\beta\gamma}) \ . \tag{29}
$$

4 Consequences from the commutation relations of the variation operator with the covariant differential operator

4.1 Consequences from $\delta \circ \nabla_{\partial_j} = \nabla_{\partial_j} \circ \delta$ and $\delta \circ \nabla_{e_\alpha} = \nabla_{e_\alpha} \circ \delta$

1. Action of δ and ∇_{∂_j} on a function. From the relations $\nabla_{\partial_j} f = \partial_j f = f_{,j}$, $f \in C^r(M)$, $\delta(\nabla_{\partial_j} f) = \delta(f_{,j})$, $\nabla_{\partial_j}(\delta f) = (\delta f)_{,j}$, and $(\nabla_{\partial_j} \circ \delta) f = (\delta \circ \nabla_{\partial_j}) f$, it follows that

$$
(\delta f)_{,j} = \delta(f_{,j}) .
$$

2. Action of δ and ∇_{∂_j} on a contravariant co-ordinate basic vector field ∂_i. From the relations

$$\nabla_{\partial_j}\partial_i = \Gamma^k_{ij}.\partial_k \ , \ \delta(\nabla_{\partial_j}\partial_i) = \delta\Gamma^k_{ij}.\partial_k \ , \ \delta\partial_i = \mathbf{0} \ ,$$
$$\nabla_{\partial_j}(\delta\partial_i) = \nabla_{\partial_j}\mathbf{0} = \mathbf{0} \ ,$$
$$(\delta \circ \nabla_{\partial_j})\partial_i = (\nabla_{\partial_j} \circ \delta)\partial_i : \delta(\nabla_{\partial_j}\partial_i) = \nabla_{\partial_j}(\delta\partial_i) = \mathbf{0} \ ,$$

it follows that

$$\delta\Gamma^k_{ij} = 0 \ . \qquad (30)$$

3. Action of δ and ∇_{e_β} on a contravariant non-co-ordinate basic vector field. From the relations

$$\nabla_{e_\beta}e_\alpha = \Gamma^\gamma_{\alpha\beta}.e_\gamma \ , \ \delta(\nabla_{e_\beta}e_\alpha) = \delta\Gamma^\gamma_{\alpha\beta}.e_\gamma \ , \ \delta e_\alpha = \mathbf{0} \ ,$$
$$\nabla_{e_\beta}(\delta e_\alpha) = \nabla_{e_\beta}\mathbf{0} = \mathbf{0} \ ,$$
$$(\delta \circ \nabla_{e_\beta})e_\alpha = (\nabla_{e_\beta} \circ \delta)e_\alpha : \delta(\nabla_{e_\beta}e_\alpha) = \nabla_{e_\beta}(\delta e_\alpha) = \mathbf{0} \ ,$$

it follows that

$$\delta\Gamma^\alpha_{\beta\gamma} = 0 \ . \qquad (31)$$

4. Action of δ and ∇_{∂_j} on a covariant co-ordinate basic vector field dx^i. From the relations

$$\nabla_{\partial_j}dx^i = P^i_{kj}.dx^k \ , \ \delta(\nabla_{\partial_j}dx^i) = \delta P^i_{kj}.dx^k \ , \ \delta(dx^i) = \delta dx^i = \mathbf{0}^* \ ,$$
$$\nabla_{\partial_j}(\delta dx^i) = \nabla_{\partial_j}\mathbf{0}^* = \mathbf{0}^* \ ,$$
$$(\delta \circ \nabla_{\partial_j})dx^i = (\nabla_{\partial_j} \circ \delta)dx^i : \delta(\nabla_{\partial_j}dx^i) = \nabla_{\partial_j}(\delta dx^i) = \mathbf{0}^* \ ,$$

it follows that

$$\delta P^i_{jk} = 0 \ . \qquad (32)$$

5. Action of δ and ∇_{e_β} on a covariant non-co-ordinate basic vector field e^α. From the relations

$$\nabla_{e_\beta}e^\alpha = P^\alpha_{\gamma\beta}.e^\gamma \ , \ \delta(\nabla_{e_\beta}e^\alpha) = \delta P^\alpha_{\gamma\beta}.e^\gamma \ , \ \delta(e^\alpha) = \delta e^\alpha = \mathbf{0}^* ,$$
$$\nabla_{e_\beta}(\delta e^\alpha) = \nabla_{e_\beta}\mathbf{0}^* = \mathbf{0}^* \ ,$$
$$(\delta \circ \nabla_{e_\beta})e^\alpha = (\nabla_{e_\beta} \circ \delta)e^\alpha : \delta(\nabla_{e_\beta}e^\alpha) = \nabla_{e_\beta}(\delta e^\alpha) = \mathbf{0}^* \ ,$$

it follows that

$$\delta P^\alpha_{\beta\gamma} = 0 \ . \qquad (33)$$

6. Action of δ and ∇_{∂_j} on a mixed tensor field. From the relations

$$\nabla_{\partial_j}K = K^A{}_{B;j}.\partial_A \otimes dx^B \ ,$$
$$\delta(\nabla_{\partial_j}K) = \delta(K^A{}_{B;j}).\partial_A \otimes dx^B \ ,$$
$$\nabla_{\partial_j}(\delta K) = (\delta K^A{}_B)_{;j}.\partial_A \otimes dx^B \ ,$$
$$(\delta \circ \nabla_{\partial_j})K = (\nabla_{\partial_j} \circ \delta)K : \delta(\nabla_{\partial_j}K) = \nabla_{\partial_j}(\delta K) \ ,$$

the commutation relations between the covariant derivative and the functional variation follow in the form

$$\delta(K^A{}_{B;j}) = (\delta K^A{}_B)_{;j} \ . \qquad (34)$$

From the last expression, after writing the explicit form of $\delta(K^A{}_{B;j})$ and $(\delta K^A{}_B)_{;j}$ using the expressions

$$K^A{}_{B;j} = K^A{}_{B,j} + \Gamma^A_{Cj}.K^C{}_B + P^D_{Bj}.K^A{}_D \ ,$$
$$\Gamma^A_{Cj} = -S_{Ci}{}^{Ak}.\Gamma^i_{kj} \ , \ P^D_{Bj} = -S_{Bi}{}^{Dk}.P^i_{kj} \ , \ \delta S_{Ci}{}^{Ak} = 0 \ ,$$
$$\delta\Gamma^A_{Cj} = 0 \ , \ \delta P^D_{Bj} = 0 \ , \delta\Gamma^i_{kj} = 0 \ , \ \delta P^i_{kj} = 0 \ ,$$

$$\delta(K^A{}_{B;j}) = \delta(K^A{}_{B,j}) + \Gamma^A_{Cj}.\delta K^C{}_B + P^D_{Bj}.\delta K^A{}_D \ ,$$
$$(\delta K^A{}_B)_{;j} = (\delta K^A{}_B)_{,j} + \Gamma^A_{Cj}.\delta K^C{}_B + P^D_{Bj}.\delta K^A{}_D \ ,$$
$$\delta(K^A{}_{B;j}) = (\delta K^A{}_B)_{;j} \ ,$$

the commutation relations between the partial derivative and the functional variation follow

$$\delta(K^A{}_{B,j}) = (\delta K^A{}_B)_{,j} \ . \qquad (35)$$

7. Action of δ and ∇_{∂_j} on a contravariant affine connection Γ. From the relations

$$\nabla_{\partial_j}\partial_i = \Gamma^k_{ij}.\partial_k \ , \ \delta(\nabla_{\partial_j}\partial_i) = \delta\Gamma^k_{ij}.\partial_k \ , \ \delta\Gamma^i_{kj} = 0 \ ,$$
$$\nabla_{\partial_l}(\nabla_{\partial_j}\partial_i) = \nabla_{\partial_l}(\Gamma^k_{ij}.\partial_k) = (\Gamma^k_{ij,l} + \Gamma^m_{ij}.\Gamma^k_{ml}).\partial_k \ ,$$
$$\delta[\nabla_{\partial_l}(\nabla_{\partial_j}\partial_i)] = [\delta(\Gamma^k_{ij,l})].\partial_k \ ,$$
$$\nabla_{\partial_l}[\delta(\nabla_{\partial_j}\partial_i)] = \nabla_{\partial_l}[\delta\Gamma^k_{ij}.\partial_k] = (\delta\Gamma^k_{ij})_{,l}.\partial_k + \delta\Gamma^k_{ij}.\nabla_{\partial_l}\partial_k =$$
$$= (\delta\Gamma^k_{ij})_{,l}.\partial_k = 0 \ ,$$
$$\delta[\nabla_{\partial_l}(\nabla_{\partial_j}\partial_i)] = \nabla_{\partial_l}[\delta(\nabla_{\partial_j}\partial_i)] \ ,$$

the commutation relations follow in the form

$$\delta(\Gamma^k_{ij,l}) = (\delta\Gamma^k_{ij})_{,l} = 0. \qquad (36)$$

From $\delta\Gamma^i_{jk} = 0$ and $\delta(\Gamma^i_{jk,l}) = (\delta\Gamma^i_{jk})_{,l} = 0$, it follows that the functional variation of the components of the contravariant curvature tensor is equal to zero, i. e.

$$\delta R^i{}_{jkl} = 0 \ . \qquad (37)$$

8. Action of δ and ∇_{∂_j} on a covariant affine connection P. From the relations

$$\nabla_{\partial_j}dx^i = P^i_{kj}.dx^k \ , \ \delta(\nabla_{\partial_j}dx^i) = \delta P^i_{kj}.dx^k \ , \ \delta P^i_{kj} = 0 \ ,$$
$$\nabla_{\partial_l}[\delta(\nabla_{\partial_j}dx^i)] = (\delta P^i_{kj})_{,l}.dx^k + \delta P^i_{kj}.\nabla_{\partial_l}dx^i = (\delta P^i_{kj})_{,l}.dx^k = 0 \ ,$$
$$\delta[\nabla_{\partial_l}(\nabla_{\partial_j}dx^i)] = [\delta(P^i_{kj,l})].dx^k \ ,$$
$$\delta[\nabla_{\partial_l}(\nabla_{\partial_j}dx^i)] = \nabla_{\partial_l}[\delta(\nabla_{\partial_j}dx^i)] = 0 \ .$$

the commutation relations follow in the form

$$\delta(P^i_{kj,l}) = (\delta P^i_{kj})_{,l} = 0. \tag{38}$$

From $\delta P^i_{jk} = 0$ and $\delta(P^i_{jk,l}) = (\delta P^i_{jk})_{,l} = 0$, it follows that the functional variation of the components of the covariant curvature tensor is equal to zero, i. e.

$$\delta P^i{}_{jkl} = 0 \ . \tag{39}$$

9. From the relations

$$\delta\Gamma^\gamma_{\alpha\beta} = 0 \ , \ \delta T_{\alpha\beta}{}^\gamma = \delta\Gamma^\gamma_{\beta\alpha} - \delta\Gamma^\gamma_{\alpha\beta} - \delta C_{\alpha\beta}{}^\gamma \ ,$$

it follows that

$$\delta T_{\alpha\beta}{}^\gamma = - \ \delta C_{\alpha\beta}{}^\gamma \ . \tag{40}$$

4.2 Consequences from $\delta \circ \nabla_\xi - \nabla_{\delta\xi} = \nabla_\xi \circ \delta$

1. Action of δ and ∇_ξ on a function. From the relations

$$\begin{gathered}
\nabla_\xi f = \xi f = \xi^j.f_{,j} \ , \quad f \in C^r(M) \ , \qquad \nabla_{\delta\xi} f = (\delta\xi) f = \delta\xi^j.f_{,j} \ , \\
\delta(\nabla_\xi f) = \delta(\xi f) = \delta(\xi^j.f_{,j}) = \delta\xi^j.f_{,j} + \xi^j.\delta(f_{,j}) \ , \\
\delta(\nabla_\xi f) - \ \nabla_{\delta\xi} f = \xi^j.\delta(f_{,j}) \ , \qquad \nabla_\xi \delta f = \xi(\delta f) = \xi^j.(\delta f)_{,j} \ ,
\end{gathered}$$

the commutation relations follow in the form (12)

$$\begin{gathered}
\delta(f_{,j}) = (\delta f)_{,j} \text{ (in a co-ordinate basis),} \\
\delta(e_\alpha f) = e_\alpha(\delta f) \text{ (in a non-co-ordinate basis).}
\end{gathered}$$

2. Action of δ and ∇_ξ on a contravariant basic vector field (e_α or ∂_i). From the relations for e_α (analogous relations are valid for ∂_i)

$$\begin{gathered}
\nabla_\xi e_\alpha = \xi^\beta.\nabla_{e_\beta} e_\alpha = \xi^\beta.\Gamma^\gamma_{\alpha\beta}.e_\gamma \ , \\
\delta(\nabla_\xi e_\alpha) = \delta(\xi^\beta.\nabla_{e_\beta} e_\alpha) = (\delta\xi^\beta.\Gamma^\gamma_{\alpha\beta} + \xi^\beta.\delta\Gamma^\gamma_{\alpha\beta}).e_\gamma \ , \\
\nabla_{\delta\xi} e_\alpha = \delta\xi^\beta.\nabla_{e_\beta} e_\alpha = \delta\xi^\beta.\Gamma^\gamma_{\alpha\beta}.e_\gamma \ , \\
\delta(\nabla_\xi e_\alpha) - \nabla_{\delta\xi} e_\alpha = \xi^\beta.\delta\Gamma^\gamma_{\alpha\beta}.e_\gamma \ , \\
\nabla_\xi(\delta e_\alpha) = \nabla_\xi \mathbf{0} = \mathbf{0} \ , \\
\delta(\nabla_\xi e_\alpha) - \nabla_{\delta\xi} e_\alpha = \xi^\beta.\delta\Gamma^\gamma_{\alpha\beta}.e_\gamma = \nabla_\xi(\delta e_\alpha) = \mathbf{0} \ , \ \forall \xi \in T(M) \ , \\
\xi^\beta.\delta\Gamma^\gamma_{\alpha\beta} = 0 \ ,
\end{gathered}$$

the following relations for the variation of $\Gamma^\gamma_{\alpha\beta}$ (or Γ^i_{jk}) are fulfilled [s. (31), (30)]

$\delta\Gamma^\gamma_{\alpha\beta} = 0$ (in a non-co-ordinate basis), $\delta\Gamma^i_{jk} = 0$ (in a co-ordinate basis).

3. Action of δ and ∇_ξ on a covariant basic vector field (e^α or dx^i). From the relations for e^α (analogous relations are valid for dx^i)

$$\begin{gathered}
\nabla_\xi e^\alpha = \xi^\gamma.\nabla_{e_\gamma} e^\alpha = \xi^\gamma.P^\alpha_{\beta\gamma}.e^\beta \ , \\
\delta(\nabla_\xi e^\alpha) = \delta(\xi^\gamma.\nabla_{e_\gamma} e^\alpha) = (\delta\xi^\gamma.P^\alpha_{\beta\gamma} + \xi^\gamma.\delta P^\alpha_{\beta\gamma}).e^\beta \ , \\
\nabla_{\delta\xi} e^\alpha = \delta\xi^\gamma.\nabla_{e_\gamma} e^\alpha = P^\alpha_{\beta\gamma}.\delta\xi^\gamma.e^\beta \ , \\
\delta(\nabla_\xi e^\alpha) - \nabla_{\delta\xi} e^\alpha = \xi^\gamma.\delta P^\alpha_{\beta\gamma}.e^\beta \ , \\
\nabla_\xi(\delta e^\alpha) = \nabla_\xi \mathbf{0}^* = \mathbf{0}^* \ , \\
\delta(\nabla_\xi e^\alpha) - \nabla_{\delta\xi} e^\alpha = \xi^\gamma.\delta P^\alpha_{\beta\gamma}.e^\beta = \nabla_\xi(\delta e^\alpha) = \mathbf{0}^* \ , \\
\xi^\gamma.\delta P^\alpha_{\beta\gamma} = 0 \text{ for } \forall \xi \in T(M) \ ,
\end{gathered}$$

the following relations for the variation of $P^\gamma_{\alpha\beta}$ (or P^i_{jk}) are fulfilled [s. (33), (32)]

$\delta P^\gamma_{\alpha\beta} = 0$ (in a non-co-ordinate basis), $\delta P^i_{jk} = 0$ (in a co-ordinate basis).

4. Action of δ and ∇_ξ on a mixed tensor field. From the relations

$$\begin{gathered}
\nabla_\xi K = K^A{}_{B/\alpha}.\xi^\alpha.e_A \otimes e^B \ , \\
\delta(\nabla_\xi K) = [\delta(K^A{}_{B/\alpha}.\xi^\alpha)].e_A \otimes e^B = \\
= [\xi^\alpha.\delta(K^A{}_{B/\alpha}) + \delta\xi^\alpha.K^A{}_{B/\alpha}].e_A \otimes e^B \ , \\
\nabla_{\delta\xi} K = K^A{}_{B/\alpha}.\delta\xi^\alpha.e_A \otimes e^B \ , \\
\delta(\nabla_\xi K) - \nabla_{\delta\xi} K = \xi^\alpha.\delta(K^A{}_{B/\alpha}).e_A \otimes e^B \ , \\
\nabla_\xi \delta K = (\delta K^A{}_B)_{/\alpha}.\xi^\alpha.e_A \otimes e^B \ , \\
\delta(\nabla_\xi K) - \nabla_{\delta\xi} K = \nabla_\xi \delta K \ , \\
\xi^\alpha.\delta(K^A{}_{B/\alpha}) = (\delta K^A{}_B)_{/\alpha}.\xi^\alpha \ , \ \forall \xi \in T(M) \ ,
\end{gathered}$$

the commutation relations follow in the form [s. (34)]

$$\begin{gathered}
\delta(K^A{}_{B/\alpha}) = (\delta K^A{}_B)_{/\alpha} \text{ (in a non-co-ordinate basis),} \\
\delta(K^A{}_{B;i}) = (\delta K^A{}_B)_{;i} \text{ (in a co-ordinate basis).}
\end{gathered}$$

By means of the relations

$$\begin{gathered}
\delta(K^A{}_{B/\alpha}) = \delta(e_\alpha K^A{}_B) + \Gamma^A_{C\alpha}.\delta K^C{}_B + P^D_{B\alpha}.\delta K^A{}_D \ , \\
(\delta K^A{}_B)_{/\alpha} = e_\alpha(\delta K^A{}_B) + \Gamma^A_{C\alpha}.\delta K^C{}_B + P^D_{B\alpha}.\delta K^A{}_D \ ; \\
\text{(because of } \delta\Gamma^A_{C\alpha} = - \ \delta(S_{C\gamma}{}^{A\beta}.\Gamma^\gamma_{\beta\alpha}) = 0 \ , \\
\delta P^D_{B\alpha} = - \ \delta(S_{B\gamma}{}^{D\beta}.P^\gamma_{\beta\alpha}) = 0) \ ,
\end{gathered}$$

commutation relations follow in the form [s. (35)]

$$\begin{gathered}
\delta(e_\alpha K^A{}_B) = e_\alpha(\delta K^A{}_B) \text{ (in a non-co-ordinate basis),} \\
\delta(K^A{}_{B,i}) = (\delta K^A{}_B)_{,i} \text{ (in a co-ordinate basis).}
\end{gathered}$$

5. Action of δ and ∇_ξ on a contravariant affine connection Γ. From the relation

$$\nabla_\xi(\nabla_{e_\beta} e_\alpha) = \nabla_\xi(\Gamma^\gamma_{\alpha\beta}.e_\gamma) = [e_\sigma \Gamma^\gamma_{\alpha\beta} + \Gamma^\rho_{\alpha\beta}.\Gamma^\gamma_{\rho\sigma}].\xi^\sigma.e_\gamma \ ,$$

commutation relations follow in the form [s. (36)]

$$\delta(e_\sigma \Gamma^\gamma_{\alpha\beta}) = e_\sigma(\delta\Gamma^\gamma_{\alpha\beta}) = 0 \text{ (in a non-co-ordinate basis)},$$
$$\delta(\Gamma^i_{jk,l}) = (\delta\Gamma^i_{jk})_{,l} = 0 \text{ (in a co-ordinate basis)}.$$

6. Action of δ and ∇_ξ on a covariant affine connection P. From the relation

$$\nabla_\xi(\nabla_{e_\gamma} e^\alpha) = \nabla_\xi(P^\alpha_{\beta\gamma}.e^\beta) = (e_\sigma P^\alpha_{\beta\gamma} + P^\alpha_{\rho\gamma}.P^\rho_{\beta\sigma}).\xi^\sigma.e^\beta ,$$

commutation relations follow in the form [s. (40)]

$$\delta(e_\sigma P^\alpha_{\beta\gamma}) = e_\sigma(\delta P^\alpha_{\beta\gamma}) = 0 \text{ (in a non-co-ordinate basis)},$$
$$\delta(P^i_{jk,l}) = (\delta P^i_{jk})_{,l} = 0 \text{ (in a co-ordinate basis)}.$$

5 Consequences from the commutation relations of the variation operator with the contraction operator

If the variation operator commutes with the contraction operator, i.e. if the commutation relation

$$\delta \circ S = S \circ \delta , \tag{41}$$

is valid, then by means of the relations in a co-ordinate basis or in a non-co-ordinate basis

$$S(dx^i \otimes \partial_j) = f^i{}_j \, , \, S(e^\alpha \otimes e_\beta) = f^\alpha{}_\beta \, ,$$
$$(\delta \circ S)(dx^i \otimes \partial_j) = \delta f^i{}_j \, , \, (\delta \circ S)(e^\alpha \otimes e_\beta) = \delta f^\alpha{}_\beta \, ,$$
$$\delta(dx^i \otimes \partial_j) = \overline{0} \in \otimes^1{}_1(M) \, , \, \delta(e^\alpha \otimes e_\beta) = \overline{0} \in \otimes^1{}_1(M) \, ,$$
$$\delta(dx^i) = \mathbf{0}^* \, , \, \delta e^\alpha = \mathbf{0}^* \, , \, \delta\partial_j = \mathbf{0} \, , \, \delta e_\beta = \mathbf{0} \, ,$$
$$(\delta \circ S)(dx^i \otimes \partial_j) = (S \circ \delta)(dx^i \otimes \partial_j) = 0 \, ,$$

the conditions for $\delta f^i{}_j$ and $\delta f^\alpha{}_\beta$

$$\delta f^i{}_j = 0 \, , \qquad \delta f^\alpha{}_\beta = 0 \tag{42}$$

follow.

Since the contraction operator commutes with the covariant differential operator and with the Lie-differential operator the following commutation relations can be used if the variation operator commutes

(a) with the covariant differential operator:

$$\begin{aligned} \nabla_{\partial_j} \circ (\delta \circ S) &= (\delta \circ S) \circ \nabla_{\partial_j} \, , \\ (S \circ \delta) \circ \nabla_{\partial_j} &= \nabla_{\partial_j} \circ (S \circ \delta) \, , \end{aligned} \tag{43}$$

(b) with the Lie differential operator:

$$\begin{aligned} \pounds_{\partial_j} \circ (\delta \circ S) &= (\delta \circ S) \circ \pounds_{\partial_j} \, , \\ (S \circ \delta) \circ \pounds_{\partial_j} &= \pounds_{\partial_j} \circ (S \circ \delta) \, , \\ \pounds_{\partial_j} \circ (\delta \circ S) &= (S \circ \delta) \circ \pounds_{\partial_j} \, . \end{aligned} \tag{44}$$

From the relations (42) and

$$(\pounds_{\partial_j} \circ \delta \circ S)(dx^i \otimes \partial_k) = (\pounds_{\partial_j} \circ \delta) f^i{}_k = (\delta f^i{}_k)_{,j} = 0 \, ,$$
$$(S \circ \delta \circ \pounds_{\partial_j})(dx^i \otimes \partial_k) = (\delta \circ \pounds_{\partial_j}) f^i{}_k = \delta(f^i{}_{k,j}),$$

it follows that

$$\begin{aligned} \delta(f^i{}_{k,j}) &= (\delta f^i{}_k)_{,j} = 0 \, , \\ \delta(P^i_{mj} + \Gamma^{\overline{i}}_{\underline{mj}}) &= 0 \text{ (because of } \pounds_{\partial_j} \circ \delta = \delta \circ \pounds_{\partial_j}), \\ \delta[(P^i_{mj} + \Gamma^{\overline{i}}_{\underline{mj}}).f^m{}_k] &= 0 \, . \end{aligned} \tag{45}$$

From the commutation relations $\nabla_{\partial_j} \circ \delta \circ S = S \circ \delta \circ \nabla_{\partial_j}$, $\nabla_{\partial_j} \circ \delta = \delta \circ \nabla_{\partial_j}$ and $\delta \circ S = S \circ \delta$ it also follows that

$$\delta(f^i{}_{k,j}) = (\delta f^i{}_k)_{,j} = 0 \, .$$

The method using commutation relations of type A is the common (conventional) method used in the classical field theories and could be called *method of Lagrangians with partial derivatives* (MLPD). The method using commutation relations of type B is called *method of Lagrangians with covariant derivatives* (MLCD). In this case the affine connections appear as non-dynamic fields variables and the variation commutes simultaneously with the partial and the covariant derivatives of the tensor fields components. The commutation relations of type C could be used when the contraction tensor field $Sr = f^i{}_j.\partial_i \otimes dx^j = f^\alpha{}_\beta.e_\alpha \otimes e^\beta$ is considered as a (fixed) non-dynamical tensor field or when $Sr = Kr = g^i_j.\partial_i \otimes dx^j = g^\alpha_\beta.e_\alpha \otimes e^\beta$, i.e. when the contraction tensor field Sr is equal to the Kronecker tensor field Kr. In both cases $\delta \circ S = S \circ \delta$ appears as a sufficient condition for $\delta f^i{}_j = 0$.

The MLCD has been used in the Einstein theory of gravitation [7] for finding out in a trivial manner the Einstein equations and the corresponding energy-momentum tensors. It has also been used for constructing the Einstein theory of gravitation in $(\overline{V}_n, g)$-spaces [8].

6 Conclusions

In this paper the variation operator is introduced and its commutation relations with the covariant differential operator, with the Lie differential operator and with the contraction operator acting on tensor fields are considered. It is shown that the commutation relations of the variational operator with the covariant differential operator lead to necessary conditions for the application of the method of Lagrangians with covariant derivatives in $(\overline{L}_n, g)$-spaces. In this method the affine connections appear as non-dynamic field variables and only the tensor fields and their covariant derivatives as constructive elements in a Lagrangian density take the role of dynamic field variables. This fact distinguish the MLCD from the MLPD and could be used in entirely covariant Lagrangian formalism for describing tensor field theories over differentiable manifolds with affine connections and metric.

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Манов С., Димитров Б. E2-98-205
$(\overline{L}_n, g)$-пространства. Вариационный оператор

Определен вариационный оператор как линейный дифференциальный оператор, действующий на тензорное поле в заданном базисе. Введены коммутационные соотношения этого оператора с дифференциальным оператором Ли, с ковариантным дифференциальным оператором и с оператором свертки (контракции). Получены следствия приложений этих коммутационных соотношений вариационного оператора к разным дифференциально-геометрическим объектам (функциям, связностям, тензорным полям). На этой основе определены два типа лагранжевых метода: обычный (канонический) метод лагранжианов с частными производными (МЛЧП) и метод лагранжианов с ковариантными производными (МЛКП).

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Manoff S., Dimitrov B. E2-98-205
$(\overline{L}_n, g)$-Spaces. Variation Operator

A variation operator is determined over $(\overline{L}_n, g)$-spaces as a linear differential operator, acting on tensor fields in a given basis. Its commutation relations with the Lie differential operator, with the covariant differential operator and with the contraction operator are imposed. The corollaries from using the different commutation relations in a Lagrangian formalism are found and two types of variation methods are distinguished: the common (canonical) method of Lagrangians with partial derivatives (MLPD) and the method of Lagrangians with covariant derivatives (MLCD).

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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