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OPEN RIGID STRING
WITH THE GAUSS—BONNET TERM IN ACTION

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1 Introduction

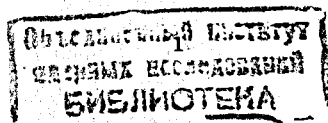
The difficulties in the Nambu-Goto string quantum theory, such as the nonphysical space-time dimension and the tachyonic state in the string spectrum, are well known. The need for an adequate description of the quark interaction in hadrons initiated the appearance of the rigid string model. This model was suggested by A. M. Polyakov and independently by H. Kleinert [1]. Due to its finite thickness, the rigid string is characterized not only by its tension but also by its resistance to transverse bending (rigidity). This is in complete analogy with classical dynamics of rods and beams. However the energy of Polyakov-Kleinert rigid string proves to be unbounded from below [2] because of the second derivatives in the string action. Therefore only Euclidean rigid string model is well defined.

In the applications to hadronic physics open strings are to be considered [3]. Here an important role is played by boundary conditions on the string dynamical variables (string coordinates). For example, when the boundary terms describing point-like masses on the string ends are added to the Nambu-Goto, the interquark potential is considerably modified. In the case of extremely asymmetric quark mass configuration this results in the removal of the tachyonic state contribution to the string potential [4].

It was specifically supposed that the boundary conditions in the effective rigid string model following from QCD enable one to suppress the oscillation modes giving negative contribution to the energy [5]. Unfortunately, this idea has not been implemented yet.

Another approach to the problem of energy unboundness from below was considered in [6], where the rigid string model with non-local action was put forward.

A consistent way to introduce the boundary conditions into the string dynamics is to add the corresponding terms (geometrical invariants) to the initial string action [7, 8, 9]. In this case the boundary conditions are consistent with the dynamical equations for sure.



The first candidate to modify the boundary conditions in string models is obviously the Gaussian curvature of string world surface. This geometrical invariant depends on the second derivatives of the metric induced on the string world sheet. As the rigid string action contains the second derivatives, it is natural to modify it with the Gaussian curvature term. According to the Gauss-Bonnet theorem [11] the surface integral of the gaussian curvature can be reduced to a contour integral along the boundary of the surface. As a result, for closed strings this term in the action gives the Eulerian characteristics of the string world sheet. For open strings it was shown that, when the action is modified in such a way, the string ends can not move with the velocity of light, as they do in the Nambu-Goto model with free ends [12].

A general mathematical analysis of the boundary conditions in the rigid string model was performed in [8]. However, the influence of these conditions on the concrete physical predictions is not properly studied. Only particular results were obtained here [9, 10].

The aim of the present note is to investigate the effect of the Gaussian curvature in the rigid string action on the interquark potential generated by the string. The layout of the paper is the following. In Section 2 the linearized equations of motion and boundary conditions in the rigid string model with the action modified by the Gaussian curvature are derived. Further the equation defining the eigenfrequencies of the string oscillations is obtained. Proceeding from this equation, in Section 3 the one-loop interquark potential generated by the string is found. By making use of the numerical calculations, it is shown that the modification of the string action by the topological term considerably changes the interquark potential at the distances comparable with the size of hadrons or less. Section 4 is devoted to the discussion of the results obtained.

2 Modification of the boundary conditions by the Gaussian curvature in the rigid string action

The action of the relativistic string with rigidity is the following [1]

$$S = -M_0^2 \int \int d^2\xi \sqrt{-g} \left[1 - \frac{\alpha}{2M_0^2} \Delta x^\mu \Delta x_\mu \right]. \quad (1)$$

Here M_0^2 is the string tension, α is a dimensionless parameter characterizing the string rigidity, $x^\mu(\xi_0, \xi_1)$ are the string coordinates in D -dimensional space-time, $\mu = 0, 1, \dots, D-1$. The curvilinear coordinates ξ_0, ξ_1 are introduced on the string world-sheet. The inbedding of the string world surface into the enveloping space-time induces a metric on this surface $g_{ij}(\xi) = \partial_i x^\mu \partial_j x_\mu$, $i, j = 0, 1$; $g_{ij}g^{jk} = \delta_i^k$; $g = \det g_{ij}$. The Laplace-Beltrami operator related to the induced metric is defined by

$$\Delta = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial \xi^i} \left(\sqrt{-g} g^{ij} \frac{\partial}{\partial \xi^i} \right). \quad (2)$$

Let us add to the action (1) a topological term, proportional to the integral Gaussian (intrinsic) curvature of the string world surface

$$- \beta \int \int d^2\xi \sqrt{-g} K. \quad (3)$$

By the Gauss-Bonnet theorem this term can be transformed into the integral along a closed contour $\partial\Omega$ bounding the string world surface

$$\int_{\Omega} d^2\xi \sqrt{-g} K = - \oint_{\partial\Omega} k_g ds + \text{const}, \quad (4)$$

where k_g is the geodesic curvature. For a curve lying on a surface and defined by natural parametrization $\mathbf{r}(s)$, $(d\mathbf{r}/ds)^2 = 1$, the geodesic curvature, k_g , is given by the formula [11]

$$k_g^2 = (\mathbf{k}_{\parallel})^2, \quad (5)$$

where k_{\parallel} is a tangential component of the curvature vector

$$\mathbf{k} \equiv \frac{d^2 \mathbf{r}}{ds^2} = \mathbf{k}_{\perp} + \mathbf{k}_{\parallel}. \quad (6)$$

Now we choose the coordinate set on the string world surface in a way that the trajectories of the string ends were defined by a condition $\xi_1 = \text{const.}$ then the geodesic curvature of these trajectories can be expressed through the components of the metric tensor g_{ij}

$$k_g = -\frac{1}{2} \frac{g_{00}\dot{g}_{00} - 2g_{00}\dot{g}_{01} + g_{01}g_{00}}{\sqrt{-g}(g_{00})^{3/2}}. \quad (7)$$

Further we shall use the nonparametric (Gaussian) definition of the string world sheet

$$x^{\mu}(\xi_i) = (\xi_0, \xi_1, x_2, \dots, x_{D-1}) = (\xi_i, \mathbf{u}(\xi_k)), \quad (8)$$

$$i = 0, 1, \quad \xi_0 = t, \quad \xi_1 = r, \quad 0 < r < R.$$

In this parametrization the geodesic curvature is given by

$$k_g = -\frac{\ddot{\mathbf{u}} \dot{\mathbf{u}}}{\sqrt{1 + \dot{\mathbf{u}}^2 - \ddot{\mathbf{u}}^2} (1 - \dot{\mathbf{u}}^2)^{3/2}}, \quad (9)$$

where $\dot{\mathbf{u}} = \partial \mathbf{u} / \partial t$, $\dot{\mathbf{u}} = \partial \mathbf{u} / \partial r$.

In what follows we shall treat $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ as small quantities

$$\begin{aligned} \sqrt{-g} &= \sqrt{1 - \dot{\mathbf{u}}^2 + \ddot{\mathbf{u}}^2} \simeq 1 - \frac{1}{2}\dot{\mathbf{u}}^2 + \frac{1}{2}\ddot{\mathbf{u}}^2, \\ \frac{1}{\sqrt{-g}} &\simeq 1 + \frac{1}{2}\dot{\mathbf{u}}^2 - \frac{1}{2}\ddot{\mathbf{u}}^2. \end{aligned} \quad (10)$$

Taking into account (10) the string action in harmonic approximation acquires the form

$$S = -M_0^2 \int_0^R dr \int_{t_1}^{t_2} dt \left[1 + \frac{1}{2}(\dot{\mathbf{u}}^2 - \ddot{\mathbf{u}}^2) + \frac{\alpha}{2M_0^2} (\square \mathbf{u})^2 \right] - \beta \int_{t_1}^{t_2} dt \ddot{\mathbf{u}} \dot{\mathbf{u}}, \quad (11)$$

where $\square = \partial^2 / \partial t^2 - \partial^2 / \partial r^2$ is the two dimensional d'Alembert operator. The action (11) gives rise to the linear equations of motion

$$\square(M_0^2 + \alpha \square) \mathbf{u} = 0 \quad (12)$$

and the boundary conditions

$$\frac{\partial}{\partial r} (M_0^2 \mathbf{u} + \beta \ddot{\mathbf{u}} + \alpha \square \mathbf{u}) = 0, \quad r = 0, R, \quad (13)$$

$$\beta \ddot{\mathbf{u}} - \alpha \square \mathbf{u} = 0, \quad r = 0, R. \quad (14)$$

Due to the second derivatives in the rigid string action the number of obtained boundary conditions is twice compared with the Nambu-Goto case.

The solutions of the boundary value problem (12-14) can be sought in the form

$$\mathbf{u}(r, t) \sim C e^{i\omega t + ikr}. \quad (15)$$

Substituting (15) into the equations of motion (12), one obtains the dispersive equation

$$(\omega^2 - k^2)[M_0^2 - \alpha(\omega^2 - k^2)] = 0 \quad (16)$$

with four branches of solutions

$$k_1 = \omega, \quad k_2 = -\omega, \quad (17)$$

$$k_3 = \Omega, \quad k_4 = -\Omega, \quad \Omega = \sqrt{\omega^2 - M_0^2/\alpha}.$$

Now the general solution to the string coordinates (15) can be rewritten as

$$\mathbf{u}(t, r) = \mathbf{u}_0 e^{i\omega t} \sum_{j=1}^4 C_j e^{ik_j r}, \quad (18)$$

where \mathbf{u}_0 is a constant vector, and C_j are amplitudes determined by initial conditions.

The solutions of Eq. (12) should satisfy the boundary conditions (13) and (14). The substitution of (18) into (13) and (14) results in a system of linear homogeneous equations for the amplitudes C_j :

$$\begin{aligned} qC_1 - qC_2 - \beta\omega\Omega C_3 + \beta\omega\Omega C_4 &= 0, \\ qe^{i\omega R}C_1 - qe^{-i\omega R}C_2 - \beta\omega\Omega e^{i\Omega R}C_3 + \beta\omega\Omega e^{-i\Omega R}C_4 &= 0, \\ \beta\omega^2 C_1 + \beta\omega^2 C_2 - qC_3 - qC_4 &= 0, \\ \beta\omega^2 e^{i\omega R}C_1 + \beta\omega^2 e^{-i\omega R}C_2 - qe^{i\omega R}C_3 - qe^{-i\omega R}C_4 &= 0, \end{aligned} \quad (19)$$

where $q = M_0^2 - \beta\omega^2$. The equation for the eigenfrequencies is obtained by setting the determinant of the system (19) equal to zero

$$\begin{aligned} f(\omega) \equiv \sin(\omega R) \sin(\Omega R) [(M_0^2 - \beta\omega^2)^4 + \beta^4 \omega^6 \Omega^2] \\ - 2(M_0^2 - \beta\omega^2)^2 \beta^2 \omega^3 \Omega [1 - \cos(\omega R) \cos(\Omega R)] = 0. \end{aligned} \quad (20)$$

This equation is rather complicated. Nevertheless it contains information that enables one to make concrete physical predictions in the framework of the string model in hand.

3 String potential

Making use of the string eigenfrequencies we are able to calculate the first quantum correction to the energy of the string ground state, i.e. the Casimir energy of the system. Considering this energy as a function of the string length R we arrive at the interquark potential generated by the string. The interquark potential $V(R)$ introduced in this way is defined by the formula [12]

$$\exp[-V(R)/T] = \int [D\mathbf{u}] \exp\{-S^T[\mathbf{u}]\}, \quad T \rightarrow 0, \quad (21)$$

where

$$S^T = M_0^2 \int_0^{1/T} dt \int_0^R dr \left[1 + \frac{1}{2} \mathbf{u} \left(1 - \frac{\alpha}{M_0^2} \Delta \right) (-\Delta) \mathbf{u} \right] - \beta \int_0^{1/T} dt \ddot{\mathbf{u}} \dot{\mathbf{u}} \quad (22)$$

is the Euclidean string action. $\Delta = \partial^2/\partial t^2 + \partial^2/\partial r^2$ is the two-dimensional Laplace operator, T is the temperature. The functional integration is carried out over the string coordinates obeying the periodicity condition $\mathbf{u}(t, r) = \mathbf{u}(t + 1/T, r)$. The Euclidean equations of motion and boundary conditions are obtained from (12), (13), and (14) by means of substitution $t \rightarrow it$. It is easy to demonstrate that the Euclidean eigenfrequencies are defined by the same equation (20).

After functional integration in (21) we derive the expression for the string potential

$$V(R) = M_0^2 R + \frac{D-2}{2\beta} \text{Tr} \ln \left[\left(1 - \frac{\alpha}{M_0^2} \Delta \right) (-\Delta) \right], \quad T \rightarrow 0. \quad (23)$$

The boundary term (4) in the action is taken into account when calculating the eigenfrequencies of the operator $(1 - \alpha\Delta/M_0^2)(-\Delta)$ which determine the functional trace in (23).

In the limit $T \rightarrow 0$ we have [3]

$$V(R) = M_0^2 R + \frac{D-2}{2} \sum_{n=1}^{\infty} \omega_n, \quad (24)$$

where ω_n are the roots of the equation (20). The first term in the formula (24) is the classical string energy proportional to its length (confining potential). The second term is the Casimir energy in the string model under consideration. The string potential given by Eq. (24) obviously requires renormalization because the sum of the eigenfrequencies diverges. The renormalized string potential at large distances should coincide with its classical value

$$V^{ren}(R)|_{R \rightarrow \infty} = M^2 R, \quad (25)$$

where M^2 is the renormalized string tension [13]. Starting with Eq. (24) and taking into account the necessity to regularize all the divergent expressions, one arrives at the following formula for the renormalized string potential

$$V^{ren}(R) = M_0^2 R + (D-2) E_{C'}^{reg}(R, \Lambda)|_{\Lambda \rightarrow \infty}$$

$$\begin{aligned}
&= M_0^2 R + (D-2) \{E_C^{reg}(R, \Lambda) - E_C^{reg}(R \rightarrow \infty, \Lambda)\}|_{\Lambda \rightarrow \infty} \\
&\quad + (D-2) E_C^{reg}(R \rightarrow \infty, \Lambda)|_{\Lambda \rightarrow \infty} \\
&= M^2 R + (D-2) E_C^{ren}, \tag{26}
\end{aligned}$$

where Λ is the regularization parameter, M^2 is the renormalized value of the string tension

$$M^2 = M_0^2 + \frac{D-2}{R} E_C^{reg}(R \rightarrow \infty, \Lambda)|_{\Lambda \rightarrow \infty}. \tag{27}$$

The renormalized Casimir energy is defined by

$$E_C^{ren}(R) = [E_C^{reg}(R, \Lambda) - E_C^{reg}(R \rightarrow \infty, \Lambda)]|_{\Lambda \rightarrow \infty}, \tag{28}$$

where M_0^2 should be substituted by M^2 according to (27). The sum (24) can be represented in terms of the contour integral by making use of the Cauchy theorem (argument principle) [14]

$$\frac{1}{2} \sum_{n=0}^{\infty} \omega_n = \frac{1}{4\pi i} \oint_C d\omega \ln[f(\omega)], \tag{29}$$

where the function $f(\omega)$ is given by (20), and the contour C encloses the zeros of $f(\omega)$ situated in the right half-plane of complex variable ω . The frequency equation (20) is real, therefore according to the Riemann-Schwarz theorem [14] its complex roots are lying symmetrically with respect to the real axis. Consequently under the summation their imaginary parts are mutually cancelled and the Casimir energy proves to be real.

Because of the square root, the function (20) is obviously two-valued. To select its single-valued branch, we make a cut connecting the branch points $\omega_0 = \pm M/\sqrt{\alpha}$ along the real axis. After that the contour can be chosen as it is shown in Fig. 1. The radius of the contour Λ stands for a renormalization parameter for the divergent sum $1/2 \sum_{n=1}^{\infty} \omega_n$. Integration along the semicircle in $\Lambda \rightarrow \infty$ limit contributes only to the counterterm [13]. The integrals along the edges of the cut are mutually cancelled, and the integration around

the branch point $\omega = \omega_0$ gives the R -independent constant. The integration along the interval $(-i\Lambda, i\Lambda)$ produces the Casimir energy of the rigid string with modified action

$$E_C^{reg}(M, \alpha, \beta, R) = \frac{1}{2\pi} \int_0^\Lambda dy \ln(f(iy)), \tag{30}$$

$$\begin{aligned}
f(iy) &= -\sinh(Ry) \sinh(R\Omega) [(M^2 + \beta y^2)^4 + \beta^4 y^6 \Omega^2] \\
&\quad - 2(M^2 + \beta y^2)^2 \beta^2 y^3 \Omega [1 - \cosh(Ry) \cosh(R\Omega)],
\end{aligned}$$

where $\Omega = \sqrt{y^2 + M^2/\alpha}$. For the renormalized value of the Casimir energy to be obtained, the asymptotics of (30) when $R \rightarrow \infty$ is needed

$$E_C^{reg}(R \rightarrow \infty) = \frac{1}{2\pi} \int_0^\Lambda dy \ln \left\{ -e^{Ry} e^{R\Omega} [(M^2 + \beta y^2)^2 - \beta^2 y^3 \Omega]^2 \right\}. \tag{31}$$

After the subtraction (28) we are left with the renormalized Casimir energy

$$\begin{aligned}
E_C^{ren}(M, \alpha, \beta, R) &= \frac{1}{2\pi} \int_0^\infty dy \ln \left\{ (1 - e^{-2Ry}) (1 - e^{-2R\Omega}) \right. \\
&\quad \left. - F(\alpha, \beta, M^2, y) (e^{-Ry} - e^{-R\Omega})^2 \right\}, \tag{32}
\end{aligned}$$

where

$$F(\alpha, \beta, M^2, y) = \frac{4\beta^2 y^3 \Omega (M^2 + \beta y^2)^2}{[(M^2 + \beta y^2)^2 - \beta^2 y^3 \Omega]^2}.$$

This expression determines in the one-loop approximation the first quantum correction to the classical linearly rising string potential $V(R) \sim M^2 R$. When $\beta = 0$ the function F vanishes, and Eq. (32) reduces to the one-loop Casimir energy of rigid string [3]. It is rather difficult to examine Eq. (32) analytically therefore we turn below to numerical calculations. The interquark potential $V(\rho)/M$, $\rho = MR$

for different values of the parameters α and β (formulae (26) and (32) with $D = 4$) is presented in Fig. 2. The potential generated by the Polyakov–Kleinert rigid string ($\beta = 0$) for $\alpha = 1, 10, 100$ is plotted in Fig. 2a. The values of the parameter α are chosen with allowance for the following considerations. In Abelian gauge model with simple Higgs potential (Nielsen–Olesen vortex model for relativistic string [15]) it was shown that the ratio $\alpha/(r_s^2 M^2)$, where r_s is the gluonic tube radius, approximately equals 20. Keeping in mind that the quantity M^{-1} gives the hadronic size in string models, one can put $r_s \sim (1/3)M^{-1}$. After that for the parameter α we obtain $\alpha \sim 2$.

Figures 2b, c, d show the impact of the Gaussian curvature term in the string action on the interquark potential. With increasing β the potential curves are shifted to the right with respect to the interquark potential generated by Polyakov–Kleinert string ($\beta = 0$). For small β it is easy to calculate the value of this shift

$$\delta V(R) = -\frac{2(D-2)\beta^2}{M^4} \int_0^\infty dy y^3 \bar{\Omega} \frac{(e^{-Ry} - e^{-R\bar{\Omega}})^2}{(1 - e^{-2Ry})(1 - e^{-2R\bar{\Omega}})} < 0. \quad (33)$$

The potential curves in Figs. 2b, c, d testify to obvious correlation between the values of the parameters α and β . To be exact, for fixed α one can obtain the same alteration of the potential by setting $\alpha \sim \beta$. Of course, this correlation appears only at the qualitative level. The curves in Fig. 2 convincingly demonstrate that the modification of the boundary conditions due to the Gaussian curvature in the string action leads to a considerable alteration of the interquark potential at the distances $\leq M^{-1}$. In this range the effect of the Gaussian curvature term turns out to be comparable with the transition from the Nambu–Goto model to the rigid string. At large distances $R \rightarrow \infty$ all the potential curves tend to the same asymptotics $\sim M^2 R$.

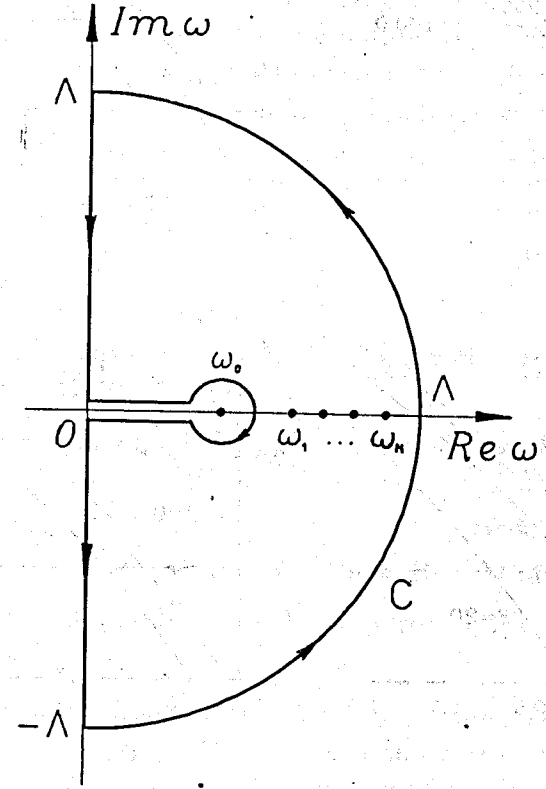


Figure 1: The integration contour used in Eq. (29) to sum the roots of frequency equation (20).

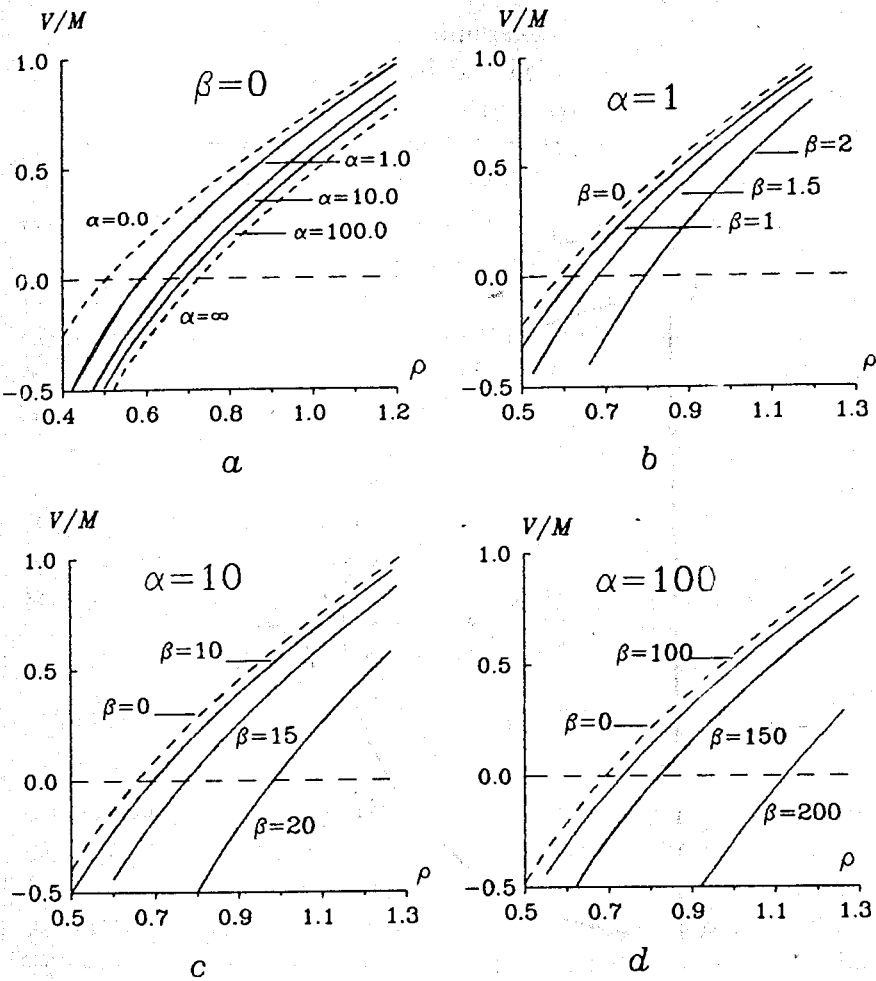


Figure 2: The interquark potential $V(\rho)/M$ generated by the modified rigid string with the Gaussian curvature in the action at different values of the parameters α and β . In Fig. 2a the dashed curves represent the limiting cases $\alpha = 0$ (the Nambu-Goto string) and $\alpha = \infty$. All the potential curves with finite values of α lie between them. In Fig. 2b, c, d the potential for $\beta = 0$ is plotted by dashed lines. With increasing β the potential curves are shifted to the right with respect to the curve corresponding to $\beta = 0$.

4 Conclusion

In view of an important role of the modification of the string action in hand it is natural to inquire oneself about the physical meaning of the parameter β . As this parameter enters only the boundary conditions for string variables, one can treat it as a coupling constant characterizing the residual quark interaction with gluonic field. This interaction has been ignored when the collective string variables were introduced. Obviously it is essential at small distances where localized gluonic tube (string) does not reproduce adequately the real physical picture.

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