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EXTERNAL CURVATURE AND GLOBAL EXCITATIONS OF THE EINSTEIN EQUATIONS

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1 Introduction

This work is devoted to the investigation of Einstein equations from the kinemetric invariants point of view. The role of kinemetric quantities in General Relativity is known, and they are one to one related with ADM-parameters in Hamiltonian formulation of gravity [1]. Recently the Gaugeless Hamiltonian Reduction procedure for the gravity was proposed [2, 3]. This method is based on the explicit resolution of the first-order constraint under a canonical momentum, which is connected with the trace of the external curvature. In this context there is a need to investigate dynamical and geometrical properties of such a kinemetric invariant as a trace of external curvature. As the basis for our consideration we shall take the trace of the external curvature of the family of hypersurfaces $t = const$ and spatial averaging of dynamical equations.

In Section 2, usual Einstein equations are written in terms of ADM-parameters. Here we also show that the trace of external curvature cannot be globally converted to constant by any coordinate transformations and discuss the existence of global excitation.

In Section 3, we extract the global dynamics from Einstein equations and represent the approximations in which this dynamics repeats the Standard Cosmological Model. We discuss the relation of this dynamics and global variable with time-surface term and Hamiltonian reduction method.

All denotations and useful formulas are given in the Appendix.

2 External curvature and existence of global excitation

2.1 Einstein equations in terms of ADM-parameters

We can express the Einstein equations $R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \frac{8\pi\kappa}{c^4}T_{\alpha\beta}$ in terms of ADM-parameters

$$ds^2 = N^2dt^2 - h_{ij}\check{dx}^i\check{dx}^j, \quad \check{dx}^i = dx^i + N^i dt \tag{1}$$

by direct substitution. It is well known that such a substitution can be easily obtained by using the Gauss relations and Peterson-Kodaci equations of the theory of hypersurfaces in Riemannian manifolds [4, 5] (see the Appendix, B):

$$\kappa_0 \equiv \frac{1}{2}({}^3R + b^2 - b^i_k b^k_i) - \frac{8\pi\kappa}{c^4}T_{\alpha\beta}\nu^\alpha\nu^\beta = 0, \tag{2}$$

$$\kappa_k \equiv \nabla_k b - \nabla_i b^i_k + \frac{8\pi\kappa}{c^4}T_{k\alpha}\nu^\alpha = 0, \tag{3}$$

$$\frac{\overset{\odot}{b}_{ij}}{N} + {}^3R_{ij} - \frac{1}{N}\nabla_i\nabla_j N + bb_{ij} + \frac{h_{ij}}{2}(-{}^3R - b^l_k b^k_l - b^2 - \frac{2\overset{\odot}{b}}{N} + \frac{2\Delta N}{N}) = \frac{8\pi\kappa}{c^4}T_{ij}, \tag{4}$$

where

$$\overset{\odot}{b}_{ij} = \dot{b}_{ij} - (b_{ik}\nabla_j N^k + b_{jk}\nabla_i N^k + N^k\nabla_k b_{ij}) - 2Nb_{ik}b^k_j\ ,$$

$$\overset{\odot}{b}{}^i_j = \dot{b}^i_j - N^k \nabla_k b^i_j,$$

$$\overset{\odot}{b} = \dot{b} - N^k \nabla_k b,$$

$b = b_{ij} h^{ij}$, $b^i_j = h^{ik} b_{kj}$, ${}^3R$ is the scalar curvature of the inner metric h_{ij}.

For the completeness of the system of the equations (2)-(4) the definition of the external curvature must be added:

$$b_{ij} = \frac{1}{2N}(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i). \tag{5}$$

Taking into account that

$$(-{}^3R - b^l_k b^k_l - b^2 - \frac{2\overset{\odot}{b}}{N} + \frac{2\Delta N}{N}) \equiv R(g) = -\frac{8\pi\kappa}{c^4} T,$$

we can rewrite the equation (4) in the form

$$\frac{\overset{\odot}{b}_{ij}}{N} + {}^3R_{ij} - \frac{1}{N}\nabla_i \nabla_j N + b b_{ij} - \frac{8\pi\kappa}{c^4}\left(T_{ij} + \frac{1}{2} h_{ij} T\right) = 0. \tag{6}$$

Let us write the Einstein equations in the schematic form:
the constraints

$$\kappa_0 = 0, \quad \kappa_k = 0 \tag{7}$$

and the dynamic equations

$$\dot{b} = f_{(1)}(b, h, N, N^i, T_{\alpha\beta}), \tag{8}$$

$$\dot{h} = f_{(2)}(b, h, N, N^i). \tag{9}$$

From the structure of these equations we can see that the lapse function N and the shift vector N^k are not included in the constraints and there are no dynamical equations for these quantities. They are contained in the right-hand side of Eqs. (8), (9) and provide the invariance under group of diffeomorphism of the coordinate transformations

$$t' = t'(t, x), \quad x' = x'(t, x).$$

Fixing of these variables has a clear geometrical and physical sense. Indeed, from the lapse function and shift vector we can construct the space-time unit vector $\nu^\alpha = (\frac{1}{N}, -\frac{N^k}{N})$, which is normal to the family of hypersurfaces $t = const$, and the fixing of ν^α corresponds to the setting of continuum of the observers with synchronized clocks [6].

So we have a dynamical system with diffeomorphic group of invariance (four arbitrary functions), and the presence of four non-dynamical degrees of freedom provides the invariance. It is necessary to add new non-dynamical variables for extension of the invariant group. For example, the triadic extension of ADM-parameters involves a non-dynamical antisymmetric tensor t_{ij} which provides the $SO(3)$ local invariance (see the Appendix, B).

2.2 Existence of the global excitation

It has been shown [7] that any one of kinemetric invariants can be locally [1] converted to zero by fixing a family of hypersurfaces $t = const$.

Can it be done globally? That is the question which needs to be investigated in more details. First of all let us restrict the Einstein manifold by some topological conditions. Here and further we assume the existence of global foliation of the whole manifold on the family of space-like hypersurfaces $t = const$. We suggest that the following condition be fulfilled:

$$\int \Delta N \sqrt{h} d^3x = 0. \tag{10}$$

The condition (10) is trivial if the hypersurfaces $t = const$ are closed, for the asymptotically flat metrics of the hypersurfaces [2] the following behavior of the lapse function at infinity is sufficient:

$$N \to 1 + \frac{const}{r^{1+\epsilon}}, \quad \epsilon > 0. \tag{11}$$

All these suggestions are valid, for example, for homogeneous Friedmann metric. If we allow these assumptions then it is impossible to convert the trace of the external curvature to zero (or to constant) for nonstatic metric if the matter stress energy tensor satisfies the strong energy condition $T_{00} - \frac{1}{2}T \geq 0$.

Indeed, the trace of equation (6) with respect to h^{ij} gives

$$\frac{\dot{b}}{N} - \frac{N^k}{N}\nabla_k b + b^2 + {}^3R - \frac{1}{N}\Delta N = \frac{8\pi\kappa}{c^4}(T_{ij}h^{ij} + \frac{3}{2}T). \tag{12}$$

Taking (2) into account, we have

$$\frac{\dot{b}}{N} - \frac{N^k}{N}\nabla_k b - \frac{1}{N}\Delta N + b^i_j b^j_i + \frac{8\pi\kappa}{c^4}(T_{\alpha\beta}\nu^\alpha\nu^\beta - \frac{1}{2}T) = 0. \tag{13}$$

The spatial integral of the last equation gives

$$\int (\dot{b} - N^k \nabla_k b)\sqrt{h} d^3x + \int \{b_{ij} b^{ij} + \frac{8\pi\kappa}{c^4}(T_{\alpha\beta}\nu^\alpha\nu^\beta - \frac{1}{2}T)\} N \sqrt{h} d^3x = 0. \tag{14}$$

If the first integral is equal to zero ($\dot{b} = 0$, $b = const$, ...) then $b_{ij} = 0$ (static metric) and $T_{\alpha\beta}\nu^\alpha\nu^\beta - \frac{1}{2}T = 0$.

If $\dot{b} \neq 0$ then in the rest frame of reference where $N^k = 0$ (line of time envelops normal unit vector ν^α) the $\sqrt{h}$ depends on time and, of course, cannot be a constant. The next step of our consideration is connected with extraction of the factor $a_0(t)$ from metric by conformal transformation. It can be called global excitation as it depends only on time, and, as will be shown below, the dynamics of this quantity involves some spatial integral characteristics of the remaining local variables.

[1] "Locally" means in the frame of some coordinate map.

[2] It means the following behavior at infinity: $h_{ij}|_\infty \to f(t)\delta_{ij}$.

3 Dynamics of global excitation

3.1 Global excitation

The extraction of the factor $a(t)$ from metric by conformal transformation means the following parametrization with respect to the $(1+3)$ decomposition:

$$ds^2 = a(t)^2(N_c^2 dt^2 - h_{(c)ij}\check{dx}^i\check{dx}^j), \quad \check{dx}^i = dx^i + N_c^i dt, \tag{15}$$

or

$$ds^2 = a(t)^2 ds_c^2 \quad or \quad g_{\mu\nu} = a(t) g_{(c)\mu\nu}. \tag{16}$$

The conformal transformation has one-to-one correspondence between the hypersurfaces $t = const$ of the manifolds $g_{\mu\nu}$ and $g_{(c)\mu\nu}$. The conformal transformation of the external curvature is

$$b_{ij} = a(t) b_{(c)ij} + h_{(c)ij}\dot{a}(t), \tag{17}$$

$$b = \frac{b_{(c)}}{a} + 3\frac{\dot{a}}{N_c a^2}. \tag{18}$$

Some useful formulas of conformal transformation are presented in the Appendix, C. Here we write the constraint (2)

$$\frac{{}^3R_{(c)}}{a^2} + \frac{1}{a^2}(b_{(c)}^2 - b_{(c)k}{}^i b_{(c)i}{}^k) + \frac{4\dot{a}b_{(c)}}{N_c a^3} + \frac{6\dot{a}^2}{N_c^2 a^4} - \frac{8\pi\kappa}{a^2 c^4} 2T_{\alpha\beta}\nu_c^\alpha\nu_c^\beta = 0. \tag{19}$$

There is one additional degree of freedom $a(t)$ in the parametrization (15). In order to save the whole amount of independent variables, it is necessary to involve an additional global condition. We choose the following condition:

$$\int b_{(c)}\sqrt{h_c}d^3x = 0. \tag{20}$$

In other words, it means that "spatial averaging" of the divergence of the unit vector ν_c^α is equal to zero:

$$\int \nabla_\alpha \nu_c^\alpha \sqrt{h_c} d^3x = 0. \tag{21}$$

Now from (19), using the space integration with the volume element $\sqrt{-g} = N_c a^4\sqrt{h_c}$, we obtain

$$\frac{\dot{a}^2}{a^2} + \frac{< R_c >}{6} = \frac{< T_{00} >}{3} + \frac{1}{6} < (b_{(c)k}{}^i b_{(c)i}{}^k - b_{(c)}^2) >; \tag{22}$$

in the same way from the trace of the equation (4) we obtain

$$-\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - 2\frac{\dot{a}}{a}\frac{\dot{V}_0}{V_0} - \frac{< R_c >}{6} = \frac{1}{3} < T_{ii} > + \frac{1}{2} < (b_{(c)k}^i b_{(c)i}^k - b_c^2) >, \tag{23}$$

where

$$V_0 = \int \frac{1}{N_c}\sqrt{h_c}d^3x, \quad < R_c > = \frac{1}{V_0}\int R_c N_c \sqrt{h_c} d^3x, \tag{24}$$

$$< T_{00} > = \frac{8\pi\kappa}{c^4}\frac{1}{V_0}\int T_{\alpha\beta}\nu_c^\alpha\nu_c^\beta N_c\sqrt{h_c}d^3x, \tag{25}$$

$$< T_{ii} > = \frac{8\pi\kappa}{c^4}\frac{1}{V_0}\int T_{ij}h_c^{ij}N_c\sqrt{h_c}d^3x, \tag{26}$$

$$< (b_{(c)k}{}^i b_{(c)i}{}^k - b_{(c)}^2) > = \frac{1}{V_0}\int (b_{(c)k}{}^i b_{(c)i}{}^k - b_{(c)}^2) N_c\sqrt{h_c}d^3x. \tag{27}$$

In the case of open (and asymptotically flat) hypersurfaces $t = const$ we assume

$$< R_c > \neq \infty, \quad < T_{ii} > \neq \infty, \quad < T_{00} > \neq \infty, \tag{28}$$

$$< (b_{(c)k}{}^i b_{(c)i}{}^k - b_{(c)}^2) > \neq \infty. \tag{29}$$

3.2 Relation with dynamics of Standard Cosmological Model

If we neglect the kinetic inhomogeneous part of gravitation with respect to spatial averaging of stress energy tensor

$$< (b_{(c)k}{}^i b_{(c)i}{}^k - b_{(c)}^2) > = 0 \tag{30}$$

and assume

$$\frac{\dot{V}_0}{V_0} = 0, \tag{31}$$

then the equations (22) and (23) take the form

$$\frac{\dot{a}^2}{a^2} + \frac{< R_c >}{6} = \frac{< T_{00} >}{3}, \tag{32}$$

$$-\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{< R_c >}{6} = \frac{1}{3} < T_{ii} > . \tag{33}$$

Now we can compare the global dynamical equations (32),(33) with the Einstein equation of Friedmann-Robertson-Walker (FRW) homogeneous metric:

$$ds^2 = a_{FRW}^2 ds_c^2 = a_{FRW}^2[d^2t - d^2\chi - d\Omega], \quad k = 1, 0, -1, \tag{34}$$

$$3\left(\frac{\dot{a}_{FRW}^2}{a_{FRW}^2} + k\right) = T_{00}, \quad \left(-\frac{2\ddot{a}_{FRW}}{a_{FRW}} + \frac{\dot{a}_{FRW}^2}{a_{FRW}^2} - k\right) h_{(cFRW)ij} = T_{ij}. \tag{35}$$

The FRW dynamics of the global variable $a(t)$ can be obtained by specifying the properties of the quantities $< T_{ii} >$, $< T_{00} >$ and $< R_c >$. In such a way we obtain a mathematical equivalence between the dynamics of the global excitation and the dynamics of space factor in FRW cosmological model.

3.3 Time surface term

The Hilbert-Einstein action of pure gravity in the $(1+3)$ decomposition has the form

$$W \sim \int d^4x\sqrt{-g}R = \int dt d^3x N\sqrt{h}(-{}^3R - (b_j^i b_i^j - b^2)) + 2\Sigma. \tag{36}$$

where

$$\Sigma = \int dt d^3x\left(-\frac{\partial}{\partial t}(\sqrt{h}b) + \sqrt{h}(\Delta N + N^i b)\right)$$

is a surface term, containing time and space surface terms. If the condition (10) is valid and $N^k = 0$, then we have the time surface term only

$$\Sigma = \int dt d^3x \left(-\frac{\partial}{\partial t}(\sqrt{h}b)\right).$$

After the conformal transformation (15) and fulfilment of (20), the above expression takes the form

$$\Sigma = -3 \int dt \frac{\partial}{\partial t}(\dot{a}V_0), \qquad (37)$$

and we can conclude that the global variable a determines the time surface term.

4 Discussion

The canonical quantization of gravity requires solving the problem of construction of the "physical Hamiltonian", which can be treated as the whole energy of considered system. Recently the method of Hamiltonian reduction was proposed in [2, 3] for isolating physical degrees of freedom and constructing a physical Hamiltonian and energy. Conceptually, this method relies on the canonical transformation $(P_a, a) \rightarrow (P_\eta, \eta)$, which absorbs the time surface term, and on explicitly solving a first-order constraint with respect to the new canonical momentum P_η. This allows us to treat the variable η as a new invariant [3] parameter of evolution and the momentum P_η as a physical Hamiltonian or energy. But the main peculiarity of this method is that the variables P_a and a must be global variables (dependent on time only) and, of course, it is tested only for FRW cosmological model.

The main result of the present work is the existence of global excitation dynamics and the possibility of extraction of this dynamics from Einstein equations. This dynamics can be made similar to the FRW metric scale factor dynamics by some common and quite wide assumptions. The Hamiltonian reduction method [3] can be directly applied to the global excitation variable and is valid for more common case then homogeneous FRW model.

[3] With respect to time reparametrization.

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Appendix

A Denotations

We use the following definitions of the Riemann and Ricci curvature:

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) A^\gamma = -A^\rho R^\gamma_{.\rho\beta\alpha}, \qquad (38)$$

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) A^\beta = -A^\rho R_{\rho\alpha}. \qquad (39)$$

Here ∇_α denotes the covariant derivative with respect to the space-time metric $g_{\alpha\beta}$ with a signature (1,-1,-1,-1). Greek indices denote the four-dimensional space-time components $\alpha, \beta, \gamma ... = 0, 1, 2, 3$; whereas the latin indices denote the space components $a, b, c ... = 1, 2, 3$. All the underlined indices correspond to the nonholonomic tetradic

$$g_{\alpha\beta} = h_{\alpha\underline{\rho}} h^{\underline{\rho}}_\beta, \ \ \underline{\rho} = 0, 1, 2, 3 \qquad (40)$$

or triadic components

$$h_{ij} = \omega_{i\underline{k}} \omega_{j\underline{k}}, \ \ \underline{k} = 1, 2, 3. \qquad (41)$$

Here h_{ij} denotes a space metric with a signature (1,1,1).

Note that the covariant derivative ∇_α does not act on underlined indices

$$\nabla_\alpha T^\beta_{\underline{\gamma}} = \partial_\alpha T^\beta_{\underline{\gamma}} + \Gamma^\beta_{\rho\alpha} T^\rho_{\underline{\gamma}}. \qquad (42)$$

We define the covariant derivative which acts on usual and underlined indices as

$$D_\alpha T^\beta_{\underline{\gamma}} = \partial_\alpha T^\beta_{\underline{\gamma}} + \Gamma^\beta_{\rho\alpha} T^\rho_{\underline{\gamma}} - \gamma^{\underline{\rho}}_{\underline{\gamma}\alpha} T^\beta_{\underline{\rho}}, \qquad (43)$$

where $\gamma^{\underline{\rho}}_{\underline{\gamma}\alpha}$ is the Ricci rotation coefficients, which are consistent with the tetrads $h_{\alpha\underline{\beta}}$

$$D_\alpha h^\gamma_{\underline{\beta}} = 0. \qquad (44)$$

The same is valid for the triadic derivative correspondingly:

$$D_i \omega^j_{\underline{k}} = 0. \qquad (45)$$

B The $(1+3)$ decomposition

B.1 Gause equation and Peterson-Codaci identity

In accordance with the general theory [5], we have the following Gause equation and Peterson-Codaci identity:

$$-R_{ij,k\rho}\nu^\rho = \pm(\nabla_i b_{jk} - \nabla_j b_{ik}), \qquad (46)$$

$$\overset{*}{R}_{ij,kl} = R_{ij,kl} \pm (b_{ik}b_{jl} - b_{jk}b_{il}), \tag{47}$$

where $\overset{*}{R}$ is the inner curvature of hypersurface. We take plus in formula if $\nu^\rho \nu_\rho = 1$ or minus if $\nu^\rho \nu_\rho = -1$. Here ν^ρ is the vector orthogonal to hypersurface, b_{ik} is the second curvature, ∇_i is the covariant derivative with respect to the inner metric of hypersurface.

Below you can find all the traces of (46),(47):

$$R_{i\rho}\nu^\rho = \pm(\nabla_i b - \nabla_j b^j_i), \tag{48}$$

$$-\overset{*}{R}_{ij} = -R_{ij} \mp R_{\alpha i,j\beta}\nu^\alpha\nu^\beta \pm (b^k_i b_{kj} - b b_{ij}), \tag{49}$$

$$-\overset{*}{R} = -R \pm 2R_{\alpha\beta}\nu^\alpha\nu^\beta \pm (b^i_j b^j_i - b^2), \tag{50}$$

$$R_{\alpha\beta}\nu^\alpha\nu^\beta = \pm\nabla_\alpha\tau^\alpha - b^i_j b^j_i + b^2\ ,\ \tau^\alpha = \{\nu^0 b, \partial^i \log\nu^0 + \nu^i b\}, \tag{51}$$

$$R = \overset{*}{R} + 2\nabla_\alpha\tau^\alpha \mp (b^i_j b^j_i + b^2). \tag{52}$$

B.2 ADM-Parametrization

We use the following ADM-parametrization of metric:

$$g_{\alpha\beta} = \begin{pmatrix} N^2 - N^k N_k & , & -N_i \\ -N_j & , & -h_{ij} \end{pmatrix};\ g^{\alpha\beta} = \begin{pmatrix} \frac{1}{N^2} & , & -\frac{N^i}{N^2} \\ -\frac{N^j}{N^2} & , & \frac{N^i N^j}{N^2} - h^{ij} \end{pmatrix}, \tag{53}$$

or in a simpler form:

$$ds^2 = N^2 dt^2 - h_{ij}\check{dx}^i\check{dx}^j,\ \ \check{dx}^i = dx^i + N^i dt. \tag{54}$$

The second curvature is defined as

$$b_{ij} = \frac{1}{2N}(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i), \tag{55}$$

and the unit time-like vector

$$\nu^\alpha = \left(\frac{1}{N}, -\frac{N^\alpha}{N}\right) \tag{56}$$

is normal to the family of hypersurfaces $t = const$. Gause equation and Peterson-Codaci identity in the ADM-parametrization look as

$$R_{i\rho}\nu^\rho = -(\nabla_i b - \nabla_j b^j_i), \tag{57}$$

$$R_{ij} = {}^3R_{ij} - R_{\alpha i,j\beta}\nu^\alpha\nu^\beta - (b^k_i b_{kj} - b b_{ij}), \tag{58}$$

$$R = -{}^3R + 2R_{\alpha\beta}\nu^\alpha\nu^\beta + (b^i_j b^j_i - b^2), \tag{59}$$

$$R_{\alpha\beta}\nu^\alpha\nu^\beta = \nabla_\alpha\tau^\alpha - (b^k_i b^i_k - b^2),\ \tau^\alpha = \{-\frac{b}{N}, \frac{\partial^i N}{N} + \frac{N^i}{N}b\}, \tag{60}$$

$$R = -{}^3R + 2\nabla_\alpha\tau^\alpha - (b^i_j b^j_i - b^2). \tag{61}$$

The last five formulas can be obtained from (48)-(52) by using the upper sign and taking into account signature of the metric:

$$b \to -b,\ \ b \to -b^i_j,\ \ \overset{*}{R} \to -{}^3R,\ \ \overset{*}{R}_{ij} \to {}^3R_{ij}.$$

It is necessary to specify one term in (58) more exactly:

$$\begin{aligned} R_{\alpha i,j\beta}\nu^\alpha\nu^\beta \ \ &= -\tfrac{1}{N}(\dot{b}_{ij} - N^p\nabla_p b_{ij} - b_{ik}\nabla_j N^k - b_{jk}\nabla_i N^k) + \\ &+ \tfrac{1}{N}\nabla_i\nabla_j N + b^k_i b_{kj}, \end{aligned} \tag{62}$$

where $\dot{b}_{ij}$ is the partial derivative with respect to time.

B.3 Triadic formulation and eigenvalue of external curvature

For diagonalization of the external curvature it is convenient to write down all Einstein equations in triadic form

$$h_{ij} = \omega_{i\underline{k}}\omega_{j\underline{k}}, \tag{63}$$

where under repeated indices we assume the summation and the underlined indices denote the triadic or the nonholonomic components.

Any local $SO(3)$ transformations do not produce changes in the metric h_{ij}. So triadic formalism extends the group of the symmetry of the original theory. The Einstein equations in triadic formalism take the form

$$\kappa_0 \ \equiv\ {}^3R + b^2 - b_{\underline{i}\,\underline{k}}b_{\underline{i}\,\underline{k}} - \frac{8\pi\kappa}{c^4}2T_{\alpha\beta}\nu^\alpha\nu^\beta = 0, \tag{64}$$

$$\kappa_{\underline{k}} \ \equiv\ \nabla_{\underline{k}}b - \nabla_{\underline{i}}b_{\underline{ik}} + \frac{8\pi\kappa}{c^4}T_{\underline{k}\alpha}\nu^\alpha = 0, \tag{65}$$

$$\frac{\overset{\odot}{b}_{\underline{i}\,\underline{j}}}{N} + {}^3R_{\underline{i}\,\underline{j}} - \frac{1}{N}\nabla_{\underline{i}}\nabla_{\underline{j}}N + b b_{\underline{i}\,\underline{j}} - \frac{8\pi\kappa}{c^4}\left(T_{\underline{i}\,\underline{j}} + \frac{1}{2}h_{\underline{i}\,\underline{j}}T\right) = 0, \tag{66}$$

$$\dot{\omega}^i_{\underline{j}} = -\omega^i_{\underline{k}}\left[N(b_{\underline{k}\,\underline{j}} + t_{\underline{k}\,\underline{j}}\nabla^n_{[\underline{i}}N_{i]} - N^m\omega^n_{\underline{j}}\nabla_m\omega_{\underline{k}n}\right]. \tag{67}$$

where

$$\overset{\odot}{b}_{\underline{i}\,\underline{j}} = \dot{b}_{\underline{i}\,\underline{j}} - N^k\nabla_k b_{\underline{i}\,\underline{j}} + N(t_{\underline{i}\,\underline{k}}b_{\underline{k}\,\underline{j}} - b_{\underline{i}\,\underline{k}}t_{\underline{k}\,\underline{j}}).$$

and $t_{\underline{ij}}$ is the antisymmetric tensor

$$t_{\underline{ij}} = \frac{1}{N}(\omega^n_{[\underline{i}}\dot{\omega}_{\underline{j}]n} + \nabla_{[\underline{j}}N_{\underline{i}]} + N^m\omega^n_{\underline{j}}\nabla_m\omega_{\underline{i}n}).$$

The system of Eqs. (64)-(67) can be also written in a schematic form, i.e. with the constraints

$$\kappa_0 = 0, \quad \kappa_k = 0 \tag{68}$$

and the dynamical equations

$$\dot{b} = f_{(1)}(b, h, N, N^i, T_{\alpha\beta}, t_{ij}), \tag{69}$$

$$\dot{h} = f_{(2)}(b, h, N, N^i, T_{\alpha\beta}, t_{ij}). \tag{70}$$

So, we can note the presence of the non-dynamical antisymmetric tensor $T_{\underline{ij}}$ that provides $SO(3)$ local invariance.

If we choose a system of the eigenvectors of external curvature as a triadic system $b_{\underline{ij}} = \delta_{\underline{ij}}\lambda_{\underline{i}}$, the equation (66) takes the form (here for simplicity $T_{\alpha\beta} = 0$)

$$\overset{\odot}{\lambda}_{\underline{i}} = {}^3R_{\underline{ii}} - b\lambda_{\underline{i}} + \frac{1}{N}\nabla_{\underline{i}}\nabla_{\underline{i}}N, \tag{71}$$

$$\left(t_{\underline{ki}} - \frac{N^p}{N}\gamma_{\underline{k}\,\underline{i}\,\underline{p}}\right)(\lambda_{\underline{i}} - \lambda_{\underline{k}}) = {}^3R_{\underline{i}\,\underline{k}} + \frac{1}{N}\nabla_{\underline{i}}\nabla_{\underline{k}}N, \quad \underline{i} \neq \underline{k}, \tag{72}$$

where $\overset{\odot}{\lambda}_{\underline{i}} \equiv \frac{1}{N}\dot{\lambda}_{\underline{i}} - \frac{N^k}{N}\partial_k\lambda_{\underline{i}}$ and $\gamma_{\underline{k}\,\underline{i}\,\underline{p}}$ is the Ricci coefficient. The constraint (65) can be represented as

$$\partial_{\underline{k}}\lambda_{\underline{k}} - \partial_{\underline{k}}b + \sum_{\underline{i}}(\lambda_{\underline{k}} - \lambda_{\underline{i}})\gamma_{\underline{iki}} = 0. \tag{73}$$

C Conformal transformation

The conformal transformation

$$ds^2 = a^2(t, x)ds_c^2 \tag{74}$$

produces the following transformation of kinemetric quantities:

$${}^3R_{ij} = {}^3R_{(c)ik} - \frac{1}{a}\nabla_i\nabla_k a + \frac{2}{a^2}\partial_i a\partial_k a - \frac{h_{(c)ik}}{a}\Delta a, \tag{75}$$

$${}^3R = \frac{{}^3R_{(c)}}{a^2} - \frac{4}{a}\Delta a + \frac{2}{a^2}\nabla^l a\nabla_l a, \tag{76}$$

$$\Gamma^k_{ij} = \Gamma^k_{(c)ij} + \frac{1}{a}(\delta^k_i\partial_j a + \delta^k_j\partial_i a - h_{(c)ij}\partial^k a), \tag{77}$$

$$b_{ij} = ab_{(c)ij} + h_{(c)ij}\nu^\alpha_c\partial_\alpha a\,, \quad b = \frac{b_{(c)}}{a} + \frac{3\nu^\alpha_c\partial_\alpha a}{a^2}, \quad \nu^\alpha = \frac{1}{a}\nu^\alpha_c, \tag{78}$$

$$b^2 = \frac{1}{a^2}\left(b^2_{(c)} + 6\frac{\overset{\odot}{a}}{a^2}b_{(c)} + 9(\frac{\overset{\odot}{a}}{a})^2\right), \tag{79}$$

$$b^{ij}b_{ij} = \frac{1}{a^2}\left(b^{ij}_{(c)}b_{(c)ij} + 2\frac{\overset{\odot}{a}}{a^2}b_{(c)} + \frac{3\,\overset{\odot}{a}{}^2}{a^2}\right), \tag{80}$$

where

$$\overset{\odot}{a} = \nu^\alpha_c\partial_\alpha a.$$

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