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8D OSCILLATOR AS A HIDDEN SU(2)-MONOPOLE

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1 Introduction

This paper deals with the problem of monopole generation from oscillator–like systems, i.e. systems with a potential chosen as "oscillator + anything". In turn, the mentioned problem is connected with the search for the electromagnetic duality (ED) in the structure of Quantum Mechanics (QM). The existence of QM–duality seems important for two reasons. First, QM is a mathematically more simple theory than the gauge theories, so we have an excellent polygon for experience in ED. Second, there appears a wide range of applications because of ED pretentions to realize accurate calculations outside perturbation theory: according to ED, strongly coupled gauge theories can be formulated in the form of weakly coupled magnetic monopoles [1].

During the last years, the following machinery has been developed for a monopole generation: Hurwitz–like transformations applied to 2D, 4D and 8D quantum oscillators transfer them into the charge–monopole bound systems in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^5$, respectively [2-4]. In two space dimensions the oscillator model was also constructed which can be transformed into a charge–monopole bound system with fractional statistics, interpolating the bosonic and fermionic limits [5]. Thus, the important extension of ED to the world of anyons is achieved.

Recently, the algebraic approach has been developed to clarify the relation between the 8D quantum oscillator and the charge–dyon bound system with the $SU(2)$–monopole [6][4]. This approach is exhaustive but rather abstract. We make here an attempt to fulfill this gap[5] by presenting the analytical approach that is more explicit and hence more acceptable for understanding. Special attention is given to the space–gauge coupling and to the spectroscopy of the charge–dyon bound system.

2 SU(2)–Monopole

Let us recall the way used for passage from the 8D oscillator to the 5D $SU(2)$–monopole. The initial system is governed by the equation

$$\frac{\partial^2 \psi}{\partial u_\mu^2} + \frac{2m}{\hbar^2}\left(E - \frac{m\omega^2 u^2}{2}\right)\psi = 0, \tag{1}$$

where $u_\mu \in \mathbb{R}^8$, $\mu = 0, 1, ..., 7$; $u^2 = u_\mu u_\mu$.

With the help of the special transformation[6]

$$\begin{aligned} x_0 &= u_0^2 + u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2 - u_6^2 - u_7^2 \\ x_1 &= 2(u_0u_4 + u_1u_5 - u_2u_6 - u_3u_7) \end{aligned}$$

[4] The particular version of the problem was previously considered in [7].

[5] See also [2] where the analytical approach to the same problem was presented more concisely.

[6] Formula (2) is known as the Hurwitz transformation [8]; (3) is copied from [9].

$$x_2 = 2(u_0u_5 - u_1u_4 + u_2u_7 - u_3u_6) \quad (2)$$
$$x_3 = 2(u_0u_6 + u_1u_7 + u_2u_4 + u_3u_5)$$
$$x_4 = 2(u_0u_7 - u_1u_6 - u_2u_5 + u_3u_4)$$

$$\alpha_T = \frac{i}{2}\ln\frac{(u_0+iu_1)(u_2-iu_3)}{(u_0-iu_1)(u_2+iu_3)} \in [0,2\pi)$$
$$\beta_T = 2\arctan\left(\frac{u_2^2+u_3^2}{u_0^2+u_1^2}\right)^{1/2} \in [0,\pi] \quad (3)$$
$$\gamma_T = \frac{i}{2}\ln\frac{(u_0+iu_1)(u_2+iu_3)}{(u_0-iu_1)(u_2-iu_3)} \in [0,4\pi)$$

we present $\mathbb{R}^8$ as a direct product $\mathbb{R}^5 \otimes S^3$ of the new configuration space $\mathbb{R}^5$ with the Cartesian coordinates $x_j \in (-\infty,\infty)$ and the intrinsic space S^3 with the coordinates α_T, β_T and γ_T. In the new coordinates, Eq.(1) can be led to the form

$$\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x_j} - \hbar A_j^a\hat{T}_a\right)^2\psi + \frac{\hbar^2}{2mr^2}\hat{T}^2\psi - \frac{e^2}{r}\psi = \epsilon\psi\,. \quad (4)$$

Here $r = (x_jx_j)^{1/2}$ and $j = 0,1,..,4$

$$\epsilon = -m\omega^2/8, \qquad e^2 = E/4. \quad (5)$$

The operators $\hat{T}_a$ are the generators of the $SU(2)$ group and have the form

$$\hat{T}_1 = i\left(\cos\alpha_T\cot\beta_T\frac{\partial}{\partial\alpha_T} + \sin\alpha_T\frac{\partial}{\partial\beta_T} - \frac{\cos\alpha_T}{\sin\beta_T}\frac{\partial}{\partial\gamma_T}\right)$$
$$\hat{T}_2 = i\left(\sin\alpha_T\cot\beta_T\frac{\partial}{\partial\alpha_T} - \cos\alpha_T\frac{\partial}{\partial\beta_T} - \frac{\sin\alpha_T}{\sin\beta_T}\frac{\partial}{\partial\gamma_T}\right) \quad (6)$$
$$\hat{T}_3 = -i\frac{\partial}{\partial\alpha_T}\,.$$

The 5D vectors $\vec{A}^a$ have the form

$$\vec{A}^1 = \frac{1}{r(r+x_0)}(0,-x_4,-x_3,x_2,x_1)$$
$$\vec{A}^2 = \frac{1}{r(r+x_0)}(0,x_3,-x_4,-x_1,x_2)$$
$$\vec{A}^3 = \frac{1}{r(r+x_0)}(0,x_2,-x_1,x_4,-x_3)\,.$$

Every term of the triplet A_j^a coincides with the vector potential of 5D Dirac monopole[7] with a unit topological charge and the line of singularity along the nonpositive part of the x_0-axis. The vectors A_j^a are orthogonal to each other,

$$A_j^aA_j^b = \frac{1}{r^2}\frac{(r-x_0)}{(r+x_0)}\delta_{ab}$$

and also to the vector $\vec{x} = (x_0,x_1,x_2,x_3,x_4)$.

We see, that Eq.(4) describes the charge–dyon system with the $SU(2)$–monopole.

By using the orthogonality condition for vectors A_j^a we can transform the Eq.(4) into

$$\left(\Delta_5 - 2iA_j^a\hat{T}_a\frac{\partial}{\partial x_j} - \frac{2}{r(r+x_0)}\hat{T}^2\right)\psi + \frac{2m}{\hbar^2}\left(\epsilon + \frac{e^2}{r}\right)\psi = 0\,. \quad (7)$$

3 LT–Coupling

Let us note that

$$iA_j^a\frac{\partial}{\partial x_j} = \frac{2}{r(r+x_0)}\hat{L}_a,$$

where

$$\hat{L}_1 = \frac{i}{2}[D_{41}(x) + D_{32}(x)]$$
$$\hat{L}_2 = \frac{i}{2}[D_{13}(x) + D_{42}(x)]$$
$$\hat{L}_3 = \frac{i}{2}[D_{12}(x) + D_{34}(x)]$$

and

$$D_{ij}(x) = -x_i\frac{\partial}{\partial x_j} + x_j\frac{\partial}{\partial x_i}\,.$$

Using these formulae we can transform Eq.(7) into

$$\left(\Delta_5 - \frac{4}{r(r+x_0)}\hat{\vec{L}}\hat{\vec{T}} - \frac{2}{r(r+x_0)}\hat{T}^2\right)\psi + \frac{2m}{\hbar^2}\left(\epsilon + \frac{e^2}{r}\right)\psi = 0\,. \quad (8)$$

We see that Eq.(8) contains the *LT-coupling* term demonstrating that we have no way to separate the wave function dependence on $\mathbb{R}^5$ and S^3.

[7]The SU(2)–monopole theory in $\mathbb{R}^5$ was constructed by Yang [10].

Let us introduce in $\mathbb{R}^5$ the hyperspherical coordinates $r \in [0, \infty)$, $\theta \in [0, \pi]$, $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$, $\gamma \in [0, 4\pi)$ according to

$$\begin{aligned} x_0 &= r\cos\theta \\ x_1 + ix_2 &= r\sin\theta\cos\frac{\beta}{2}e^{i\frac{\alpha+\gamma}{2}} \\ x_3 + ix_4 &= r\sin\theta\sin\frac{\beta}{2}e^{i\frac{\alpha-\gamma}{2}} . \end{aligned}$$

In these coordinates

$$\Delta_5 = \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial}{\partial\theta}\right) - \frac{4}{r^2\sin^2\theta}\hat{L}^2$$

where

$$\begin{aligned} \hat{L}_1 &= i\left(\cos\alpha\cot\beta\frac{\partial}{\partial\alpha} + \sin\alpha\frac{\partial}{\partial\beta} - \frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right) \\ \hat{L}_2 &= i\left(\sin\alpha\cot\beta\frac{\partial}{\partial\alpha} - \cos\alpha\frac{\partial}{\partial\beta} - \frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right) \\ \hat{L}_3 &= -i\frac{\partial}{\partial\alpha} \end{aligned}$$

and

$$\hat{L}^2 = -\left[\frac{\partial^2}{\partial\beta^2} + \cot\beta\frac{\partial}{\partial\beta} + \frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\alpha^2} - 2\cos\beta\frac{\partial^2}{\partial\alpha\partial\gamma} + \frac{\partial^2}{\partial\gamma^2}\right)\right] .$$

Let us introduce $\hat{J}_a = \hat{L}_a + \hat{T}_a$. Since $\hat{J}^2 = \hat{L}^2 + \hat{T}^2 + 2\hat{L}_a\hat{T}_a$, Eq.(8) can be rewritten as

$$\left(\Delta_{r\theta} - \frac{\hat{L}^2}{r^2\sin^2\theta/2} - \frac{\hat{J}^2}{r^2\cos^2\theta/2}\right)\psi + \frac{2m}{\hbar^2}\left(\epsilon + \frac{e^2}{r}\right)\psi = 0 \qquad (9)$$

where

$$\Delta_{r\theta} = \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial}{\partial\theta}\right) .$$

Emphasize that

$$[\hat{L}_a, \hat{L}_b] = i\epsilon_{abc}\hat{L}_c, \quad [\hat{J}_a, \hat{J}_b] = i\epsilon_{abc}\hat{J}_c .$$

Introduce the separation ansatz

$$\psi = \Phi(r,\theta)G(\alpha,\beta,\gamma;\alpha_T,\beta_T,\gamma_T)$$

where G are the eigenfunctions of $\hat{L}^2$, $\hat{T}^2$ and $\hat{J}^2$ with the eigenvalues $L(L+1)$, $T(T+1)$ and $J(J+1)$. If this is substituted into Eq.(9), the differential equation for the function $\Phi(r,\theta)$ immediately follows

$$\left(\Delta_{r\theta} - \frac{L(L+1)}{r^2\sin^2\theta/2} - \frac{J(J+1)}{r^2\cos^2\theta/2}\right)\Phi + \frac{2m}{\hbar^2}\left(\epsilon + \frac{e^2}{r}\right)\Phi = 0 . \qquad (10)$$

Because of an LT-interaction, we look for the function G in the form

$$G = \sum_{M=m+t} (JM|L,m\,;T,t\,)\, D^L_{mm'}(\alpha,\beta,\gamma) D^T_{tt'}(\alpha_T,\beta_T,\gamma_T)$$

where $(JM|L,m\,;T,t\,)$ are the Clebsch–Gordan coefficients and $D^L_{mm'}$ and $D^T_{tt'}$ are the Wigner functions.

4 Hypermomentum

Pick up the function $\Phi(r,\theta)$ of the form

$$\Phi(r,\theta) = R(r)Z(\theta) .$$

Then, equation (10) is separated into

$$\frac{1}{\sin^3\theta}\frac{d}{d\theta}\left(\sin^3\theta\frac{dZ}{d\theta}\right) - \frac{2L(L+1)}{1-\cos\theta}Z - \frac{2J(J+1)}{1+\cos\theta}Z + \lambda(\lambda+3)Z = 0 \qquad (11)$$

and a purely radial equation

$$\frac{1}{r^4}\frac{d}{dr}\left(r^4\frac{dR}{dr}\right) - \frac{\lambda(\lambda+3)}{r^2}R + \frac{2m}{\hbar^2}\left(\epsilon + \frac{e^2}{r}\right)R = 0 \qquad (12)$$

with the separation constant $\lambda(\lambda+3)$ equal to the nonnegative eigenvalues of the hypermomentum operator.

It is convenient to make in Eq.(11) a change of variables, $y = (1-\cos\theta)/2$ and write

$$Z(y) = y^L(1-y)^J W(y) .$$

Substituting this into Eq.(11), we obtain the hypergeometric equation

$$y(1-y)\frac{d^2W}{dy^2} + [c - (a+b+1)y]\frac{dW}{dy} - abW = 0$$

with $a = -\lambda + L + J$, $b = \lambda + L + J + 3$, $c = 2L + 2$.

Thus, we find that

$$Z(\theta) = (1-\cos\theta)^L(1+\cos\theta)^J$$

$${}_2F_1\left(-\lambda + J + L, \lambda + J + L + 3; 2L + 2; \frac{1-\cos\theta}{2}\right).$$

This solution is well behaved at $\theta = \pi$ if the series ${}_2F_1$ terminates, i.e.

$$-\lambda + J + L = -n_\theta$$

where $n_\theta = 0, 1, 2, \ldots$.

5 Energy Levels

Let us now turn to the radial equation and introduce the function

$$f(r) = e^{\kappa r} r^{-\lambda} R(r) .$$

It is easy to verify that the equation for $f(r)$ has the form of the confluent hypergeometric equation

$$z\frac{d^2 f}{dz^2} + (c - z)\frac{df}{dz} - af = 0$$

where $z = 2\kappa r$, $\kappa = \sqrt{-2m\epsilon/\hbar^2}$, $c = 2\lambda + 4$, $a = \lambda + 2 - 1/\kappa r_0$ and $r_0 = \hbar^2/me^2$. For the bound state solutions ($\epsilon < 0$)

$$\lambda + 2 - 1/\kappa r_0 = -n_r = 0, -1, -2, \ldots$$

and therefore

$$\epsilon_N^T = -\frac{me^4}{2\hbar^2(\frac{N}{2} + 2)^2}$$

where

$$N = 2(n_r + \lambda) = 2(n_r + n_\theta + J + L) .$$

6 Degeneracy

For fixed T, the energy levels ϵ_N^T do not depend on L, J and λ, i.e. are degenerate. The total degeneracy is

$$g_N^T = (2T+1)\sum_\lambda \sum_L (2L+1) \sum_J (2J+1) .$$

Since $\lambda = n_\theta + J + L$ and $N = 2(n_r + \lambda)$, it follows that (for fixed N and T) $\lambda = T, T+1, \ldots, N/2$. Then, $L_{max} = \lambda - J_{min}$ (L_{max} is fixed) and therefore $L_{max} = \lambda - (L_{max} - T)$ or $L_{max} = (\lambda + T)/2$. Thus,

$$g_N^T = (2T+1)\sum_{\lambda=T}^{N/2} \sum_{L=0,1/2}^{\frac{\lambda-T}{2}} (2L+1) \sum_J (2J+1) .$$

Now, comparing $|L-T| \le J \le L+T$ and $J \le \lambda - L$ we conclude that

(a) $J = |L-T|, |L-T|+1, \ldots, L+T$, for $L = 0, \frac{1}{2}, \ldots, \frac{\lambda-T}{2}$

(b) $J = |L-T|, |L-T|+1, \ldots, \lambda - L$, for $L = \frac{\lambda-T+1}{2}, \ldots, \frac{\lambda+T}{2}$

and rewrite the formula for g_N^T in a more explicit form

$$g_N^T = (2T+1)\sum_{\lambda=T}^{N/2}\Big\{ \sum_{L=0,1/2}^{\frac{\lambda-T}{2}} (2L+1) \sum_{J=|L-T|}^{L+T} (2J+1) + \sum_{L=\frac{\lambda-T+1}{2}}^{\frac{\lambda+T}{2}} (2L+1) \sum_{J=|L-T|}^{\lambda-L} (2J+1)\Big\} .$$

Finally, after some tedious calculations we obtain the following result:

$$g_N^T = \frac{1}{12}(2T+1)^2\left(\frac{N}{2} - T + 1\right)\left(\frac{N}{2} - T + 2\right)$$
$$\left\{\left(\frac{N}{2} - T + 2\right)\left(\frac{N}{2} - T + 3\right) + 2T(N+5)\right\} .$$

For $T = 0$ and $N = 2n$ (even) the r.h.s. of the last formula is equal to $(n+1)(n+2)^2(n+3)/12$, i.e. to the degeneracy of pure Coulomb levels. Further, $T = 0, 1, \ldots N/2$ and $T = 1/2, 3/2, ..N/2$ for even and odd N, respectively. Therefore,

$$g_N = \sum_{T=0,1/2}^{N/2} g_N^T = \frac{(N+7)!}{7!N!}$$

i.e. we obtain the degeneracy of the energy levels for the 8D isotropic quantum oscillator.

7 Conclusions

Formulae (2) and (3) together with the ansatz (5) form the duality transformation mapping of the 8D quantum oscillator into the charge–dyon system with the $SU(2)$–monopole. Let us stress the meaning we use for the term duality. Both Eq.(1) and Eq.(4) contain two quantities, ω and E. For Eq.(1) ω is the fixed parameter (coupling constant) and E is the quantity to be quantized (energy of the 8D oscillator). On the contrary, as it easy to see from (5), for Eq.(4) E is a fixed parameter (Coulomb coupling constant) and ω is the quantity to be quantized (ω^2–energy of the final system). Thus, the 8D quantum oscillator and the charge–dyon bound system with the $SU(2)$–monopole are not identical, but dual to each other.

This type duality is valid not only for the 8D, 4D and 2D oscillators, but also for oscillator–like systems with the potentials

$$V(u^2) = c_0 + c_1 u^2 + W(u^2)$$

where $W(u^2)$ has a polynomial form

$$W(u^2) = \sum_{n=2}^{\infty} c_n u^{2n}.$$

For such modified potentials, the ansatz (5) can be rewritten as

$$\epsilon = -\frac{c_1}{4}, \quad e^2 = \frac{E - c_0}{4} .$$

Thus, the value of the function $V(u^2)$ at $u^2 = 0$ contributes to the Coulomb coupling constant e^2. It is also easy to verify that the l.h.s. of Eq.(4) acquires the additional term $(-W(r)/4r)$.

8 Appendix

Consider the normalization of the wave function $\psi(\vec{x}, \alpha_T, \beta_T, \gamma_T)$. A standard calculation shows that the radial wave function $R(r)$ normalized by the condition

$$\int_0^\infty r^4 \left[R_{n_r\lambda}(r)\right]^2 dr = 1$$

has the form

$$R_{n_r\lambda}(r) = \frac{4}{r_0^{5/2}(n_r + \lambda + 2)^3} \frac{1}{(2\lambda + 3)!} \sqrt{\frac{(n_r + 2\lambda + 3)!}{(n_r)!}}$$
$$(2\kappa r)^\lambda e^{-\kappa r} F(-n_r; 2\lambda + 4; 2\kappa r) .$$

The full wave function

$$\psi = C^\lambda_{LTJ} R_{n_r\lambda}(r) Z_{\lambda LJ}(\theta) G^{JM}_{LTm't'}(\alpha, \beta, \gamma; \alpha_T, \beta_T, \gamma_T)$$

is normalized by the condition

$$\int |\psi|^2 dv = 1$$

where

$$dv = r^4 \sin^3\theta dr d\theta d\Omega d\Omega_T$$

and

$$d\Omega = \frac{1}{8}\sin\beta d\beta d\alpha d\gamma, \quad d\Omega_T = \frac{1}{8}\sin\beta_T d\beta_T d\alpha_T d\gamma_T .$$

Using the formula

$${}_2F_1\left(-n, n + a + b + 1; a + 1; \frac{1 - y}{2}\right) = \frac{n!\Gamma(a + 1)}{\Gamma(n + a + 1)} P_n^{(a,b)}(y)$$

where $P_n^{(a,b)}(y)$ are the Jacobi polynomials, and taking into account that

$$\int_{-1}^{1} (1 - y)^a (1 + y)^b \{P_n^{(a,b)}(y)\}^2 dy =$$
$$\frac{2^{a+b+1}}{2n + a + b + 1} \frac{\Gamma(n + a + 1)\Gamma(n + b + 1)}{n!\Gamma(n + a + b + 1)}$$

$$\int D^{j_2}_{m_2m_2'}(\alpha, \beta, \gamma) D^{j_1}_{m_1m_1'}(\alpha, \beta, \gamma) d\Omega = \frac{2\pi^2}{2j_1 + 1} \delta_{j_1j_2}\delta_{m_1m_2}\delta_{m_1'm_2'}$$

it is easy to obtain that

$$C^\lambda_{LJT} = \sqrt{\frac{(2L + 1)(2T + 1)(2\lambda + 3)(\lambda - J - L)!\Gamma(\lambda + J + L + 3)}{2^{2J+2L+5}\pi^4\Gamma(\lambda + J - L + 2)\Gamma(\lambda - J + L + 2)}} .$$

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