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THREE-DIMENSIONAL FORMULATION OF THE RELATIVISTIC TWO-BODY PROBLEM IN TERMS OF RAPIDITIES

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§ 1. Introduction

The quasipotential approach (QPA)[1-3] is a very important tool for solving a number of physical problems[4]. The geometrical properties of the quasipotential equations (QPE) deduced in [2], made it possible to apply to them the technique of the relativistic Fourier transformation and to introduce the notion of the relativistic configurational space[5]. Here a scheme has been developed in many respects analogous to quantum mechanics[5-11].

The aim of this paper is to give a detailed analysis of a version of QPA in which the main role is played by the rapidity, the dynamical variable, canonically conjugated to the relativistic relative distance.

Let us now consider in more detail the geometrical properties of QPE deduced in[2]. The equations for the relativistic two-particle amplitude $A(\vec{p},\vec{q})$ and the wave function $\psi_q(\vec{p})$ have the form:

$$A(\vec{p},\vec{q})=-\frac{m}{4\pi}V(\vec{p},\vec{q};E_q)+\frac{1}{(2\pi)^3}\int V(\vec{p},\vec{\kappa};E_q)G_q(\kappa)A(\vec{\kappa},\vec{q})\,d\Omega_\kappa, \qquad (1.1)$$

$$\psi_q(\vec{p})=(2\pi)^3\delta^{(3)}(\vec{p}(-)\vec{q})+\frac{1}{(2\pi)^3}G_q(p)\int V(\vec{p},\vec{\kappa};E_q)\psi_q(\kappa)\,d\Omega_\kappa, \qquad (1.2)$$

where

$$G_q(\kappa)=\frac{1}{2E_q-2E_\kappa+i\varepsilon}\,. \qquad (1.3)$$

These equations have "absolute character" in respect to the geometry of the momentum space*), i.e., by form they do not differ from the nonrelativistic Lippmann-Schwinger equations.

Equations (1.1) and (1.2) can be gotten from the corresponding nonrelativistic equations replacing in the latter nonrelativistic (Euclidean) expressions for energy, volume element, etc., by their relativistic (noneuclidean) analogs[8]:

$$E_q = \frac{\vec{q}^{\,2}}{2} \longrightarrow E_q = \sqrt{\vec{q}^{\,2}+1}\,,$$

$$d\Omega_K = d\vec{K} \longrightarrow d\Omega_K = \frac{d\vec{K}}{\sqrt{1+\vec{K}^2}}\,,$$

$$\delta(\vec{p}-\vec{q}) \longrightarrow \delta(\vec{p}(-)\vec{q}) = \sqrt{1+\vec{p}^{\,2}}\,\delta(\vec{p}-\vec{q}). \qquad (1.5)$$

The connection between the quasipotential amplitude $A(\vec{p},\vec{q})$ and a differential cross-section of the elastic scattering has the form[5]:

$$\frac{d\sigma}{d\omega} = |A(\vec{p},\vec{q})|^2, \qquad (1.6)$$

which coincides with the normalization condition of the nonrelativistic amplitude of the elastic scattering. In case of a real quasipotential from (1.1) the unitarity condition follows:

$$Im\, A(\vec{p},\vec{q}) = \frac{|\vec{q}|}{4\pi}\int A(\vec{p},\vec{K})A^*(\vec{K},\vec{q})\,d\omega_K\,. \qquad (1.7)$$

The transition to the relativistic r -space is performed by means of the expansion in the matrix elements of Lorentz group representations[5] (relativistic Fourier transformation)

$$\xi(\vec{q},\vec{r}) = (ch\,\chi_q - (\vec{n}\vec{n}_q)\,sh\,\chi_q)^{-1-ir}, \qquad (1.8)$$

$$\vec{r} = r\vec{n},\quad ch\,\chi_q = E_q\,,\quad \vec{n}\,sh\,\chi_q = \vec{q}. \qquad (1.9)$$

For example, the Green function G_q in the configurational r -representation has the form

$$G_q(\vec{r},\vec{r}\,') = \frac{1}{(2\pi)^3}\int \xi(\vec{K},\vec{r})G_q(K)\xi^*(\vec{K},\vec{r})\,d\Omega_K \qquad (1.10)$$

and satisfies the equation

$$(H_0 - 2ch\,\chi_q)G_q(\vec{r},\vec{r}\,') = -\delta(\vec{r}-\vec{r}\,'), \qquad (1.11)$$

where H_0 is the differential-difference operator (free Hamiltonian) with the step, equal to the Compton wave length of the particle $\hbar/mc$:

$$H_0 = 2ch\,i\frac{\partial}{\partial r} + 2\frac{i}{r}sh\,i\frac{\partial}{\partial r} - \frac{\Delta_{\theta,\varphi}}{r^2}\exp(i\frac{\partial}{\partial r}). \qquad (1.12)$$

The equation for the wave function in the relativistic r -space[5]

$$[H_0 - 2E_q + V(r,E_q)]\,\Psi_q(r) = 0 \qquad (1.13)$$

as well as (1.1) and (1.2), do not differ by form from the Schrödinger equation.

It was shown (see review[6]) that relativistic differential-difference equations in the important cases of the Coulomb and oscillator potentials, as well as for the square well, were exactly solved in full analogy with quantum mechanics. The scattering theory in the relativistic configurational space is also in full analogy with the quantum-mechanical scattering theory.

The only serious difficulty in this scheme is connected with the peculiar formulation of the boundary conditions in the differential-difference case. In particular, it very complicates the investigation of analytical properties of the wave function and

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*) In equations (1.1) and (1.2) the integration is carried out over the Lobachevsky p -space, realised on the upper sheet of the hyperboloid

$$p_0^2 - \vec{p}^{\,2} = m^2c^2 \qquad (1.4)$$

(the mass shell of a particle with mass m). In the following we use the system of units, in which $\hbar = c = m = 1$.

scattering amplitude in the complex plane of angular momentum ℓ and energy E_q , although the dependence of the operator H_0 on these variables is the same as in the nonrelativistic case.

We will demonstrate that there exists a version of QPA which also has absolute character in respect to the geometry of the momentum space, but is free from the above mentioned difficulties (analysis of the boundary conditions is given in §3).

Let us write nonrelativistic operator of the kinetic energy in the form:

$$E_q = \frac{\vec{q}^{\,2}}{2} = \frac{S^2(q,0)}{2} , \tag{1.14}$$

where $S(q,0)$ is the distance between the point with coordinates $\vec{q}$ and the origin in nonrelativistic three-dimensional Euclidean momentum space.

Using eq.(1.14) the Lippmann-Schwinger equation has the form

$$A(\vec{p},\vec{q}) = -\frac{1}{4\pi} V(\vec{p},\vec{q}) + \frac{1}{(2\pi)^3} \int \frac{V(\vec{p},\vec{k}) A(\vec{k},\vec{q})\, d\Omega_k}{S^2(q,0) - S^2(k,0) + i\varepsilon} . \tag{1.15}$$

Now we pass to the relativistic equation (see (1.5)) replacing the nonrelativistic (Euclidean) quantities by the relativistic (non-euclidean) ones. However, when transforming Green's function, we do not now replace the expression for the energy, but pass from formula (1.14) for the distance in the Euclidean p -space to the expression for the distance in Lobachevsky p -space:

$$S(q,0) = \sqrt{\vec{q}^{\,2}} \longrightarrow \chi_q = \ln\left(E_q + \sqrt{E_q^2 - 1}\right). \tag{1.16}$$

The quantity χ_q , taken now instead of the energy E_q , is also called "rapidity". The relativistic equation for $A(\vec{p},\vec{q})$ has now the form*):

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*) This equation was first postulated in/5/. About the quasipotential $V(\vec{p},\vec{q};E_q)$ see §2.

$$A(\vec{p},\vec{q}) = -\frac{1}{4\pi} \hat{V}(\vec{p},\vec{q};E_q) + \frac{1}{(2\pi)^3} \frac{\chi_q}{\operatorname{sh}\chi_q} \int \frac{\hat{V}(\vec{p},\vec{k};E_q) A(\vec{k},\vec{q})\, d\Omega_k}{\chi_q^2 - \chi_k^2 + i\varepsilon} . \tag{1.17}$$

The new quasipotential Green function

$$g_q(k) = \frac{\chi_q}{\operatorname{sh}\chi_q} \frac{1}{\chi_q^2 - \chi_k^2 + i\varepsilon} \tag{1.18}$$

has a pole on the energy shell $E_k = E_q$ as other quasipotential Green functions, and its discontinuity ensures the fulfilment of the two-particle unitarity condition (1.7).

Since in the following rapidities χ_q will play the key role, let us pay attention to the following facts. The relativistic relative distance r and rapidity χ are canonically conjugated variables in the sense of Fourier analysis on the Lorentz group/5/ (see also/12,13/). The analysis of many properties of QPE is essentially simplified, if it is based on the properties of physical quantities, as functions of the rapidity/5-9/. Invariant inclusive cross-sections of many-particle processes at high energies are simple functions of the rapidity. Recently an interesting attempt was made in ref./14/ to analyse this fact on the basis of the relativistic Fourier analysis. In papers/15/, the apparatus developed in/5-9/ was succesfully applied in the study of the properties of the hadron wave function in the parton model.

Therefore, there is a number of evidences that the rapidity is an adequate dynamical variable in the relativistic region.

§ 2. Connection with Other Relativistic Equations

Let us consider the complex plane of the variable χ . Energy and momentum (1.9) are periodic functions (with the period $2\pi i$) of the repidity along the pure imaginary direction in χ -plane. The kernel of the relativistic Fourier transformation, however, does not possess this property of periodicity. Dealing with this transformation, one should consider different quantities as functions of the rapidity in the whole complex χ -plane. For instance, for calculating the integral (1.10) one has to use the Jordan's lemma, closing contour of the infinite radius in the upper, or lower, half-plane χ , depending on the sign of the difference $\tau - \tau'$.

The mapping (1.16) is the infinite-sheet one. Only two sheets of the infinite-sheet Riemann surface, corresponding to the strips

$$0 < Im\,\chi < \pi, \qquad -\pi < Im\,\chi < 0 \tag{2.1}$$

have nonrelativistic analogs: the physical $\Im m\sqrt{E} > 0$ and the nonphysical $\Im m\sqrt{E} < 0$ sheets.

The Bethe-Salpeter equation and infinite set of quasipotential equations/16/ have the dependence only on $ch\,\chi$ and $sh\,\chi$. From the point of view of complex rapidities, for its studying the principal branch of logarithm is sufficient. We have no reason, however, to require QPE Green function to be $2\pi i$ -periodic function of the rapidity. To satisfy the main requirement of the two-particle relativistic unitarity, it is sufficient for the Green function to have only one pole in the upper χ semiplane. An infinite chain of poles, corresponding to the periodic dependence, does not contribute to the unitarity condition.

Let us study the connection between equations (1.17) and (1.1). We use the relation

$$G_q(\kappa) = \frac{1}{2(ch\chi_q - ch\chi_\kappa + i\varepsilon)} = -\frac{1}{sh\chi_q}\Big[cth\,\frac{\chi_\kappa - \chi_q - i\varepsilon}{2} - cth\,\frac{\chi_\kappa + \chi_q + i\varepsilon}{2}\Big] = g_q(\kappa) + \Omega_q(\kappa), \tag{2.2}$$

where

$$\Omega_q(\kappa) = -\frac{1}{sh\chi_q}\sum_{n=1}^{\infty}\Big[\frac{\chi_\kappa - \chi_q}{(\chi_\kappa - \chi_q)^2 + (2\pi n)^2} - \frac{\chi_\kappa + \chi_q}{(\chi_\kappa + \chi_q)^2 + (2\pi n)^2}\Big] \tag{2.3}$$

Thus, we have represented the Green function $G_q(\kappa)$ of equation (1.1) as the sum of two terms, the Green function $g_q(\kappa)$ of equation (1.17) and the quantity $\Omega_q(\kappa)$. The latter contains the whole infinite chain of "extra",(i.e.,not contributing to the elastic unitarity), poles in the complex χ -plane.

Let us clarify the connection between quasipotentials V and $\hat{V}$ in equations (1.1) and (1.17). We write both equations in the symbolic form

$$A = -\frac{1}{4\pi}V + \frac{1}{(2\pi)^3}VGA, \tag{2.4a}$$

$$A = -\frac{1}{4\pi}\hat{V} + \frac{1}{(2\pi)^3}\hat{V}gA. \tag{2.4b}$$

Transforming eq.(2.4a)

$$A = -\frac{1}{4\pi}V + \frac{1}{(2\pi)^3}VgA + \frac{1}{(2\pi)^3}V\Omega A, \tag{2.5}$$

we come to the conclusion, that eq. (2.4a) is equivalent to (2.4b), if V and $\hat{V}$ are connected as follows

$$\hat{V} = V + \frac{1}{(2\pi)^3}V\Omega\hat{V}. \tag{2.6}$$

It is evident, that the Born approximations in (2.4a) and (2.4b) coincide with each other.

Let us pass to the connection of equation (1.17) with the Bethe-Salpeter equation:

$$T(p_1,p_2;p_1',p_2') = K(p_1,p_2;p_1',p_2') + \int K(p_1,p_2;\tfrac{P}{2}+\kappa,\tfrac{P}{2}-\kappa)G(P,\kappa)T(\tfrac{P}{2}+\kappa,\tfrac{P}{2}-\kappa;p_1',p_2')d^4\kappa, \tag{2.7}$$

where $P = p_1 + p_2$. In the c.m. system $P = (2ch\chi_q, \vec{0})$.

The two-particle propagator $G(P,\kappa)$ has the form

$$G(P,\kappa) = \frac{i}{(2\pi)^4}\Big\{\frac{\;}{[(\frac{P}{2}+\kappa)^2 - m^2 + i\varepsilon][(\frac{P}{2}-\kappa)^2 - m^2 + i\varepsilon]}\Big\}. \tag{2.8}$$

One of the ways of transition from the four-dimensional equation (2.7) to the three-dimensional QPE is to replace the Green function $G(P,\kappa)$ by "dispersion type" integral $E(P,\kappa)$ providing two-particle unitarity

$$G(P,\kappa)\to E(P,\kappa) = -\frac{1}{(2\pi)^3}\int_{4m^2}^{\infty}\frac{ds'}{s'-s}\, f(s',s)\times$$
$$\times\, \mathcal{D}^{(+)}\left(\frac{P'}{2}+\kappa\right)\mathcal{D}^{(+)}\left(\frac{P'}{2}-\kappa\right), \qquad (2.9)$$

where

$$\mathcal{D}^{(+)}(\kappa)=\theta(\kappa^0)\,\delta(\kappa^2-m^2), \qquad (2.10)$$
$$s=P^2,\; s'=P'^2,\quad P'=\frac{\sqrt{s'}}{2ch\chi_q}P,$$

and the quantity $f(s',s)$ satisfies the only condition

$$f(s,s)=1. \qquad (2.11)$$

If instead of (2.9) one uses expression*)

$$\hat{g}(P,\kappa)=-\frac{1}{(2\pi)^3}\int_0^{\infty}\frac{d\chi'}{\chi'^2-\chi_q^2-i\varepsilon}\,\mathcal{D}^{(+)}\left(\frac{P'}{2}+\kappa\right)\mathcal{D}^{(+)}\left(\frac{P'}{2}-\kappa\right), \qquad (2.12)$$

where $\chi'=\ln\left(\frac{\sqrt{s'}}{2}+\sqrt{\frac{s'}{4}-1}\right)$, then in the c.m. system

$$\hat{g}(P,\kappa)=\frac{1}{4(2\pi)^3}\frac{\delta(\kappa^0)}{ch^2\chi_q}\,g_q(\kappa). \qquad (2.13)$$

Substituting (2.13) into equation (2.7) and passing to the amplitude A /5/, we derive eq.(1.17). Connection between $\hat{V}$ and the Bethe-Salpeter kernel K has the form:

$$\hat{V}=K+K(G-\hat{g})\hat{V}. \qquad (2.14)$$

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*) It can be shown, that dispersion integrals in the χ - plane are just of type (2.12). Analytic properties of the solutions of equation (1.17) in χ - and ℓ -planes will be described elsewhere. We note that analytical properties of the amplitude in the variable $W=\frac{1}{2}+\frac{i}{\pi}\ln(\nu+\sqrt{\nu^2-1})$, $\nu=E_{lab}$, similar to χ, have been studied in ref./17/.

§ 3. The Equation for the Wave Function in the Configurational Space

We shall introduce the wave function $\psi_q(\vec{p})$ in the continuous spectrum:

$$\psi_q(\vec{p})=(2\pi)^3\delta(\vec{p}(-)\vec{q})-4\pi\frac{\chi_q}{sh\chi_q}\frac{A(\vec{p},\vec{q})}{\chi_q^2-\chi_p^2+i\varepsilon} \qquad (3.1)$$

Equation for $\psi_q(\vec{p})$ has the form

$$\psi_q(\vec{p})=(2\pi)^3\delta(\vec{p}(-)\vec{q})+\frac{1}{(2\pi)^3}\frac{\chi_q}{sh\chi_q}\frac{\int\hat{V}(\vec{p},\vec{k};E_q)\psi_q(\vec{k})\,d\Omega_k}{\chi_q^2-\chi_p^2+i\varepsilon} \qquad (3.2)$$

Making in (3.2) relativistic Fourier transformation, we arrive at the Schrödinger equation for the wave function $\psi_q(\vec{r})$

$$\psi_q(\vec{r})=\frac{1}{(2\pi)^3}\int\xi(\vec{p},\vec{r})\,\psi_q(\vec{p})\,d\Omega_p \qquad (3.3)$$

in the relativistic configurational $\vec{r}$ -representation:

$$\psi_q(\vec{r})=\xi(\vec{q},\vec{r})+\int g_q(\vec{r},\vec{r}')V(\vec{r}')\psi_q(\vec{r}')\,d\vec{r}' \qquad (3.4)$$

The Green function $g_q(\vec{r},\vec{r}')$ is given by the expressions

$$g_q(\vec{r},\vec{r}')=\frac{1}{(2\pi)^3}\frac{\chi_q}{sh\chi_q}\int\frac{\xi(\vec{k},\vec{r})\,\xi^*(\vec{k},\vec{r}')}{\chi_q^2-\chi_k^2+i\varepsilon}\,d\Omega_k=$$
$$=\frac{1}{4\pi r r'}\sum_{\ell=0}^{\infty}(2\ell+1)\,g_{q\ell}(r,r')\,P_\ell(\vec{n}\vec{n}'), \qquad (3.5)$$

where $g_{q\ell}(r,r')$ is the partial Green function (see Appendix A)

$$g_{q\ell}(r,r')=\frac{2}{\pi}\frac{\chi_q}{sh\chi_q}\int_0^{\infty}\frac{s_\ell(r,\chi_k)\,s_\ell^*(r',\chi_k)}{\chi_q^2-\chi_k^2+i\varepsilon}\,d\chi_k=$$
$$=-\frac{\nu_\ell(r')}{sh\chi_q}\left\{\theta(r-r')\,e_\ell^{(1)}(r,\chi_q)\,s_\ell(r',\chi_q)+\theta(r'-r)\times\right. \qquad (3.6)$$
$$\left.\times\, e_\ell^{(1)}(r',\chi_q)\,s_\ell(r,\chi_q)\right\},\quad \nu_\ell(r)=(-1)^{\ell+1}\frac{r^{(\ell+1)}}{(-r)^{(\ell+1)}},$$

and $r^{(\lambda)}$ is the so-called generalized power.

The definition of $r^{(\lambda)}$ as well as relativistic free wave functions by form from the corresponding Green function of the Schrödinger equation. The distinction is only in free solutions. Let us emphasize that eq.(3.6) comprises the usual step θ -function

$$\theta(r) = \begin{cases} 1 & r > 0 \\ 0 & r < 0, \end{cases} \tag{3.21}$$

while partial Green function of equation (1.1) in the relativistic r -space contains "smeared" $\hat{\theta}$ -function

$$\hat{\theta}(r) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{ixr}}{e^{x} - 1 - i\varepsilon}\, dx, \tag{3.22}$$

and that results in the difference character of the operator (1.12) and difficulties in the formulation of boundary conditions.

For equations (1.17) and (3.2), as well as equivalent equations (3.13) and (3.17), the situation is essentially different. Here we deal with differential equations of the second order. To determine its solutions it is necessary to put two boundary conditions (at $r \to 0$ and $r \to \infty$). We write down these conditions taking into account that $R_\ell(r,\chi_q) \underset{r\to\infty}{=} 0$ and assuming that potentials $V(r)$ are the short-range ones.

For the scattering $(\chi_q \geq 0)$

$$\varphi_{q\ell}(r) \underset{r\to\infty}{\sim} \sin\left(\chi_q r - \frac{\pi\ell}{2}\right) + A_\ell(\chi_q)\, e^{i(\chi_q r - \frac{\pi\ell}{2})} \tag{3.23}$$

For the bound states $\chi_q = i\zeta_q \ (0 < \zeta_q < \frac{\pi}{2})$

$$E_q = \cos\zeta_q, \tag{3.24}$$

$$\varphi_{q\ell}(r) \sim e^{-\zeta_q r} \tag{3.25}$$

The boundary condition at the origin in both cases is the same

$$\varphi_{q\ell}(0) = 0. \tag{3.26}$$

§ 4. Some Simple Potentials

For $\ell = 0$ equation (3.17) has the form

$$\frac{d^2\varphi_{q0}(r)}{dr^2} + \chi_q^2\, \varphi_{q0}(r) = \frac{\chi_q}{\operatorname{sh}\chi_q} V(r)\, \varphi_{q0}(r). \tag{4.1}$$

We consider the square well[6]

$$V(r) = -\theta(R - r)\, V_0 \tag{4.2}$$

For the bound states (see (3.24)) we obtain the discrete spectrum of the rapidities from the transcendental equation

$$\zeta_q + \sigma(\zeta_q)\, \operatorname{ctg}(R\sigma(\zeta_q)) = 0, \tag{4.3}$$

where

$$\sigma(\zeta_q) = \sqrt{-\zeta_q^2 + \zeta_q V_0/\sin\zeta_q} \tag{4.4}$$

Note, that the behaviour of the wave function $\varphi_{q0}(r) \sim \sin\sigma r$ inside well is different depending on the sign of the expression under square root in (4.4).

In the continuous spectrum we obtain the following expression for the scattering phase

$$\delta_0(\chi_q) = -\chi_q R + \operatorname{arctg}\left[\frac{\sigma(\chi_q)}{\chi_q} \operatorname{ctg}(R\sigma(\chi_q))\right]. \tag{4.5}$$

For the attractive Coulomb potential

$$V(r) = -\frac{\alpha}{r\, \operatorname{ch}\chi_q}, \quad \alpha > 0, \tag{4.6}$$

corresponding to the massless scalar particle exchange, the wave function of the discrete spectrum has the form

$$\varphi_n(r) = \alpha\left(\frac{2\zeta_n}{\alpha}\right)^{3/2} r\, e^{-r\zeta_n}\, \Phi(1-n, 2, 2r\zeta_n). \tag{4.7}$$

The energy spectrum is given by formula (cf.[7])

$$E_n = \sqrt{\frac{1 + \sqrt{1 - \alpha^2/n^2}}{2}}, \qquad n = 1, 2, \ldots \tag{4.8}$$

From the asymptotic behaviour of the wave function of the continuous spectrum

$$\varphi_{q0}(r)=\sqrt{\frac{2}{\pi}}\,\chi_q r\, e^{\frac{\pi\alpha}{2sh2\chi_q}}\left|\Gamma\left(1-\frac{i\alpha}{sh2\chi_q}\right)\right| e^{-ir\chi_q}\,\Phi\left(1+\frac{i\alpha}{sh2\chi_q},2,2ir\chi_q\right), \qquad (4.9)$$

which is given by formula

$$\varphi_{q0}(r)\underset{r\to\infty}{\sim}\sqrt{\frac{2}{\pi}}\sin\left(\chi_q r+\frac{\alpha}{sh2\chi_q}\ln 2\chi_q r+\delta_0(\chi_q)\right), \qquad (4.10)$$

we obtain the expression for the matrix element of the S-matrix

$$S_0(\chi_q)=e^{2i\delta_0(\chi_q)}=\frac{\Gamma(1-i\alpha/sh2\chi_q)}{\Gamma(1+i\alpha/sh2\chi_q)} \qquad (4.11)$$

We consider now the equation for a harmonic oscillator. In dimensional quantities it has the form

$$\left(-\frac{\hbar^2}{m}\frac{d^2}{dr^2}+\frac{m\omega^2}{4}r^2-mc^2\chi_q^2\right)\varphi_{q0}(r)=0. \qquad (4.12)$$

We introduce the operators

$$\hat{q}=\sqrt{\frac{m\omega}{2\hbar}}\,r, \qquad (4.13a)$$

$$\hat{\pi}=-i\frac{\partial}{\partial q}=c\sqrt{\frac{2m}{\hbar\omega}}\,\hat{\chi}=-i\sqrt{\frac{2\hbar}{m\omega}}\frac{d}{dr}, \qquad (4.13b)$$

$$[\hat{\pi},\hat{q}]=-i, \qquad (4.13c)$$

as well as the creation and annihilation operators

$$a^+=\frac{1}{\sqrt{2}}(q-i\pi),\quad a^-=\frac{1}{\sqrt{2}}(q+i\pi), \qquad (4.14a)$$

$$[a^-,a^+]=1. \qquad (4.14b)$$

The initial equation can be written now as

$$\hat{\varkappa}^2\varphi_{q0}=\frac{1}{2}(\hat{\pi}^2+\hat{q}^2)\varphi_{q0}=\frac{mc^2}{\hbar\omega}\chi_q^2\varphi_{q0}=\varkappa_q^2\varphi_{q0} \qquad (4.15)$$

Because of the commutation relation (4.14b) and formula

$$\hat{\varkappa}^2=a^-a^+-1/2 \qquad (4.16)$$

all excited S-states can be obtained from the ground state $\varphi_0^{(0)}$:

$$\hat{\varkappa}^2\varphi_0^{(0)}=\varkappa_q^{(0)2}\varphi_0^{(0)}, \qquad (4.17a)$$

$$a^-\varphi_0^{(0)}=0, \qquad (4.17b)$$

by n-multiple action of the creation operator a^+:

$$\varphi_0^{(n)}(q)=\frac{(a^+)^n\varphi_0^{(0)}(q)}{\sqrt{n!}}=\frac{q\,e^{-q^2/2}}{2\sqrt{2n!/\pi\,\Gamma(n+3/2)}}\,\Phi(-n,3/2,q^2). \qquad (4.18)$$

The energy spectrum

$$E_n=2mc^2 ch\left(\sqrt{\frac{\hbar\omega}{mc^2}(2n+3/2)}\right) \qquad (4.19)$$

in the nonrelativistic limit goes to the S-states spectrum of the usual isotropic oscillator

$$E_n-2mc^2\to\hbar\omega(2n+3/2). \qquad (4.20)$$

We also present the relation, connecting the root-mean-square deviations of the relativistic relative distance $\overline{\Delta r}$ and the rapidity $\overline{\Delta\chi}$ in the ground state (cf. /12/)

$$(\overline{\Delta r})^2(\overline{\Delta\chi})^2=\frac{3}{4}\frac{\hbar^2}{m^2c^2}. \qquad (4.21)$$

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Appendix A

Using the connection between $S_e(\tau,\chi)$ and $e^{(1,2)}_e(\tau,\chi)$ /6/, we write eq.(3.6) in the form

$$g_{qe}(\tau,\tau') = -\frac{\chi_q}{sh\chi_q}\,\frac{V_e(\tau')}{2\pi}\left[g^{(1)}_{qe}(\tau,\tau') - g^{(2)}_{qe}(\tau,\tau')\right], \tag{A.1}$$

where

$$g^{(1)}_{qe}(\tau,\tau') = \int \frac{e^{(1)}_e(\tau,\chi_k)\,e^{(2)}_e(\tau',\chi_k)}{\chi_q^2 - \chi_k^2 + i\varepsilon}\,d\chi_k\,, \tag{A.2a}$$

$$g^{(2)}_{qe}(\tau,\tau') = \int \frac{e^{(1)}_e(\tau,\chi_k)\,e^{(2)}_e(\tau',\chi_k)}{\chi_q^2 - \chi_k^2 + i\varepsilon}\,d\chi_k\,. \tag{A.2b}$$

The integrand in (A.2a) decreases in the upper half-plane χ_k, the integrand in (A.2b) decreases in the upper half-plane when $\tau > \tau'$ and in the lower half-plane when $\tau < \tau'$. These statements can be easily proved using the following representation for $e^{(1,2)}_\ell(\tau,\chi_k)$:

$$e^{(1,2)}_\ell(\tau,\chi) = \frac{(-i)^\ell}{2^\ell(sh\chi)^\ell}\, e^{\pm i\tau\chi}\, e^{\pm \ell\chi}\, {}_2F_1(\ell,-\ell+i\tau,1-i\tau,e^{\mp 2\chi}) \tag{A.3}$$

From the representation (A.3) it also follows that integrands in (A.2a,b) have infinite chains of 2ℓ-order "kinematic" poles at the points $\chi_k = \pm i\pi n\ (n=1,2,\ldots)$. To exclude these poles, not contibuting to the unitarity condition (1.7), we define the integrals in (A.2a,b) as the contour integrals (see. fig.1). The integration in (A.2b) is carried out either over contour C, or over contour C', depending on the sign of $\tau - \tau'$. Performing the integration by Caushy theorem we obtain the expression (3.6).

Strictly speaking, our definitions are of model character, because we omit a certain quantity in the Fourier transform of the free Green function. The model satisfies all physical requirements what is its main justification. From equation (3.10) with

the Green function (3.6) for a real potential we obtain real scattering phase shifts. The wave function $\psi_{qe}(\tau)$ safisfies the correct boundary conditions of the scattering theory in the relativistic τ -space (3.23).

Appendix B

We give another derivation of the equation (3.2) (cf. /18/).

The wave function $\psi(p_1,p_2)$ of a system of two free particles with equal masses in the momentum representation obeys the equations

$$\begin{aligned}(p_1^2 - m^2)\psi(p_1,p_2) &= 0,\\ (p_2^2 - m^2)\psi(p_1,p_2) &= 0.\end{aligned} \tag{B.1}$$

Passing to new variables Q and P by the formulas

$$p_1 = p + \frac{Q}{2}\,,\quad p_2 = -p + \frac{Q}{2}\,, \tag{B.2}$$

we write the system (A.1) in the form

$$\begin{aligned}\left[\left(\frac{Q}{2}+p\right)^2 - m^2\right]\psi(Q,p) &= 0,\\ \left[\left(\frac{Q}{2}-p\right)^2 - m^2\right]\psi(Q,p) &= 0.\end{aligned} \tag{B.3}$$

Taking the sum and difference of equations (B.3), we come to the equivalent pair of equations

$$\left(\frac{Q^2}{4} + p^2 - m^2\right)\psi(Q,p) = 0, \tag{B.4a}$$

$$(Qp)\,\psi(Q,p) = 0. \tag{B.4b}$$

In the c.m. system $\vec{Q} = 0$, $\vec{p}_1 = -\vec{p}_2 = \vec{p}$ because of (B.4b) we have

$$\psi(Q,p) = \delta(p_0)\,\psi_{Q_0}(\vec{p}). \tag{B.5}$$

For the class of states with positive energy taking account of (B.5), (B.4a) takes the form

$$(E_q - E_p)\Psi_{Q_0}(\vec{p}) = 0, \qquad \text{(B.6)}$$

where $Q_0 = 2E_q = 2\sqrt{m^2 + \vec{q}^{\,2}}$, $E_p = \sqrt{\vec{p}^{\,2} + m^2}$.

Passing to the rapidities according to formulas (1.16), we have

$$(ch\,\chi_q - ch\,\chi_p)\Psi_q(\vec{p}) = 0, \qquad \text{(B.7)}$$

or, equivalently

$$2\,sh\left(\frac{\chi_q + \chi_p}{2}\right) sh\left(\frac{\chi_q - \chi_p}{2}\right)\Psi_q(\vec{p}) = 0. \qquad \text{(B.8)}$$

Using the relation

$$sh\,z = \frac{z}{\Gamma(1 - iz/\pi)\,\Gamma(1 + iz/\pi)},$$

we will write (B.8) in the form

$$A(\chi_q, \chi_p)(\chi_q^2 - \chi_p^2)\Psi_q(\vec{p}) = 0, \qquad \text{(B.9)}$$

where

$$A(\chi_q, \chi_p) = 2/|\Gamma(1 + i(\chi_q - \chi_p)/\pi)\Gamma(1 + i(\chi_q + \chi_p)/\pi)| \qquad \text{(B.10)}$$

As $A(\chi_q, \chi_p)$ does not vanish for real rapidities, $\Psi_q(\vec{p})$ satisfies the equation

$$(\chi_q^2 - \chi_p^2)\Psi_q(\vec{p}) = 0. \qquad \text{(B.11)}$$

Introducing the interaction into (B.11) according to the equation

$$(\chi_q^2 - \chi_p^2)\Psi_q(\vec{p}) = \frac{1}{(2\pi)^3}\frac{\chi_q}{sh\,\chi_q}\int V(\vec{p}, \vec{k}; E_q)\Psi_q(\vec{k})\,d\Omega_k, \qquad \text{(B.12)}$$

we arrive at equation (3.2)

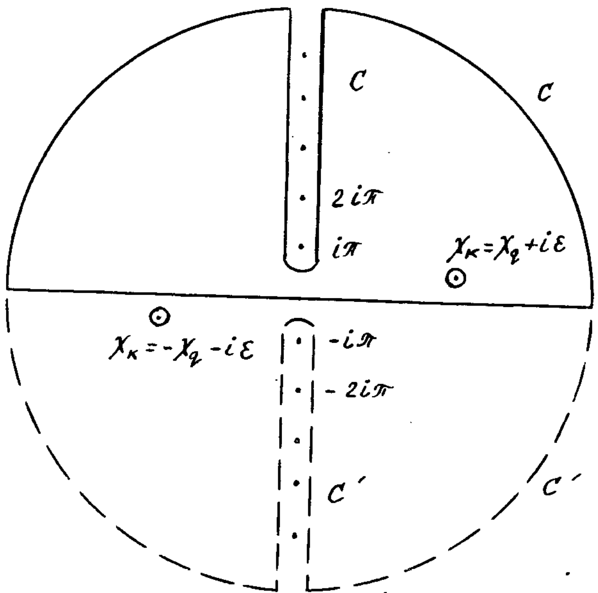

Fig. 1. The contours C and C' in the complex χ_k plane for integrals (A.2a,b).

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