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Потенциалы однобозонного обмена в релятивистском конфигурационном представлении

Рассмотрена релятивистская система двух фермионов. В качестве квазипотенциалов взяты потенциалы однобозонного обмена, преобразованные к трехмерному виду в терминах пространства Лобачевского. Осуществлен переход к релятивистскому координатному пространству. Предложено релятивистское обобщение спин-орбитальных и тензорных сил.

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One-Boson-Exchange Potentials in Relativistic Configurational Representation

The two-fermion relativistic system is considered. The one-boson exchange potentials transformed to the three-dimensional form in terms of the Lobachevsky space are taken as quasipotentials. The transition to the relativistic coordinate space is realized. The relativistic generalization is proposed for spin-orbital and tensorial forces.

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I. INTRODUCTION

The subject of the paper is the description of interaction of two relativistic particles with spin 1/2. In ref. /1/ it was shown that for the description of spinless particles it is very suitable to pass to the relativistic configurational representation by using the harmonic expansion on the Lorentz group. In the two-body problem there are widely used matrix elements of the relativistic scattering amplitude in the one-boson exchange approximation. They are taken to be relativistic one-boson exchange potentials (OBEP).

Our aim is to find the explicit form of such relativistic OBEP in the relativistic configurational representation for interaction of the particles with spin. This paper thus may be considered as a sequel to paper /2/. In /2/ for the two-fermion interaction the 4-dimensional matrix elements of the relativistic scattering amplitude, corresponding to the Feynman diagram of the one-boson exchange (OBEP), were transformed to the 3-dimensional form in terms of the Lobachevsky space. Such a transformation is analogous to the Foldi-Wouthuysen transformation, however, it does not deal with the expansion of the interaction terms in powers of v^2/c^2.

On separating the Wigner rotation $D^{1/2}\{V^{-1}(\Lambda_p, k)\}$ originated from the relativistic spin kinematics, the

Feynman matrix element of the one-pion exchange (notations are taken from [3]) can be represented in the form:

$$\frac{\bar{u}^{\sigma_1}(\vec{p}_1)\gamma_5 u^{\sigma_1'}(\vec{k}_1)\bar{u}^{\sigma_2}(\vec{p}_2)\gamma_5 u^{\sigma_2'}(\vec{k}_2)}{\mu^2-(p-k)^2} =$$

$$= \langle \vec{p}_1\sigma_1;\, \vec{p}_2\sigma_2 | T^{(2)}_{PS} | \vec{k}_1\sigma_1';\, \vec{k}_2\sigma_2' \rangle =$$

$$= \sum_{\sigma_{1p}\sigma_{2p}} \overset{*}{\phi}{}^{\sigma_1} \overset{*}{\phi}{}^{\sigma_2}\, T^{(2)}_{PS}(\vec{k}(-)\vec{p})\,\phi_{\sigma_{1p}}\phi_{\sigma_{2p}}\, D^{1/2}_{\sigma_{1p}\sigma_1'}\{V^{-1}(\Lambda_{p_1},k_1)\}\times$$
$$\times D^{1/2}_{\sigma_{2p}\sigma_2'}\{V^{-1}(\Lambda_{p_2},k_2)\}, \qquad (1)$$

where the relativistic amplitude

$$T^{(2)}_{PS}(\vec{k}(-)\vec{p}) = g^2\,\frac{4(\vec{\sigma}_1\vec{\kappa}_1)(\vec{\sigma}_2\vec{\kappa}_2)}{\mu^2+4\vec{\kappa}^2} \qquad (2)$$

does not differ in form from the nonrelativistic potential

$$V_{PS}(\vec{k}-\vec{p}) = \frac{\vec{\sigma}_1(\vec{k}_1-\vec{p}_1)\,\vec{\sigma}_2(\vec{k}_2-\vec{p}_2)}{\mu^2+(\vec{k}-\vec{p})^2} = \frac{4(\vec{\sigma}_1\vec{\kappa}_{1e})(\vec{\sigma}_2\vec{\kappa}_{2e})}{\mu^2+4\vec{\kappa}_e^2}$$

$$\kappa_e = \frac{\vec{k}-\vec{p}}{2}. \qquad (3)$$

used in the meson theory of nuclear forces. The two-component Pauli spinors ϕ_σ are normalized by the condition $\overset{*}{\phi}{}^{\sigma_1}\phi_{\sigma_2} = \delta_{\sigma_1\sigma_2}$.

The quantity $\vec{\kappa}$ defined in [2] is called the half momentum transfer (an analog of the half velocity of a particle introduced in [4]) and is related to the transfer momentum in the Lobachevsky space

$$\vec{\Delta} = \vec{k}(-)\vec{p} \equiv \Lambda_p^{-1}\vec{k} = \vec{k} - \frac{\vec{p}}{M}\left(k_0 - \frac{\vec{k}\cdot\vec{p}}{p_0+M}\right) = M\,\mathrm{sh}\,\chi_\Delta\,\frac{\vec{\Delta}}{|\vec{\Delta}|}$$

$$\Delta_0 = \sqrt{M^2+\vec{\Delta}^2} = (\Lambda_p^{-1}k)_0 = (k_0p_0-\vec{k}\vec{p})/M = M\,\mathrm{ch}\,\chi_\Delta \qquad (4)$$

as follows

$$\vec{\kappa} = \vec{\Delta}\sqrt{\frac{M}{2(\Delta_0+M)}} = M\,\mathrm{sh}\,\frac{\chi_\Delta}{2}\,\frac{\vec{\Delta}}{|\vec{\Delta}|}. \qquad (5)$$

The formulation in terms of the Lobachevsky space is natural when the momenta in the matrix element (1) are on the mass shell, i.e., their components are linked by the relation

$$p_0^2 - \vec{p}^2 = M^2. \qquad (6)$$

This equation gives the three-dimensional surface of the hyperboloid which upper sheet is used as a model of the Lobachevsky space. In the nonrelativistic limit, when the curvature of the Lobachevsky space tends to zero and this space turns into the Euclidean space, one has $\vec{\kappa}\to\vec{\kappa}_e$.

The Feynman matrix element (1) in form (2) is a direct geometrical generalization [5] of the quantummechanical potential (3) obtained via changing the Euclidean half momentum transfer $\vec{\kappa}_e$ by its analog in the Lobachevsky space. The kinematical Wigner rotation in (1) is interpreted as a rotation at the angle between the old direction of the spin vector and the new one. The latter occurs after the parallel translation of the spin vector in the space of negative curvature along the triangle composed of vectors $\vec{p}$, $\vec{k}$ and $\vec{\Delta} = \vec{k}(-)\vec{p}$. In the nonrelativistic limit the Wigner rotation is absent: $D^{1/2}\{V^{-1}(\Lambda_p,k)\}\to 1$. Analogously, the amplitude of the vector meson exchange (with the mass μ) in the c.m.s. $(\vec{p}_1=-\vec{p}_2=\vec{p};\ \vec{k}_1=-\vec{k}_2=\vec{k})$ takes the form [2]:

$$<\vec{p}\sigma_1;-\vec{p}\sigma_2 \mid T_V^{(2)} \mid \vec{k}\sigma_1';-\vec{k}\sigma_2'> =$$

$$= g_V^2 \frac{\bar{u}^{\sigma_1}(\vec{p})\gamma_\mu u^{\sigma_1'}(\vec{k})\bar{u}^{\sigma_2'}(-\vec{p})\gamma^\mu u^{\sigma_2'}(-\vec{k})}{\mu^2-(p-k)^2}$$

$$= \sum_{\sigma_{1p}\sigma_{2p}} \overset{*}{\phi}{}^{\sigma_1} \overset{*}{\phi}{}^{\sigma_2} T_V^{(2)}(\vec{k}(-)\vec{p};\vec{p})\phi_{\sigma_{1p}}\phi_{\sigma_{2p}} \times$$

$$\times D^{1/2}_{\sigma_{1p}\sigma_1'}\{V^{-1}(\Lambda_p,k)\}D^{1/2}_{\sigma_{2p}\sigma_2'}\{V^{-1}(\Lambda_p,k)\}, \qquad (7)$$

where the amplitude obtained after separation of the Wigner rotation

$$T_V^{(2)}(\vec{k}(-)\vec{p};\vec{p}) = -g_V^2\frac{4M^2}{\mu^2+4\vec{\kappa}^2} -$$

$$-g_V^2\frac{4(\vec{\sigma}_1\vec{\kappa}_1)(\vec{\sigma}_2\vec{\kappa})-4(\vec{\sigma}_1\vec{\sigma}_2)\vec{\kappa}^2}{\mu^2+4\vec{\kappa}^2} -$$

$$-g_V^2\frac{8p_0\kappa_0}{M^2}\frac{i(\vec{\sigma}_1+\vec{\sigma}_2)[\vec{p}\times\vec{\kappa}]}{\mu^2+4\vec{\kappa}^2} -$$

$$-g_V^2\frac{8}{M^2}\frac{p_0^2\kappa_0^2+2p_0\kappa_0(\vec{p}\cdot\vec{\kappa})-M^4}{\mu^2+4\vec{\kappa}^2} -$$

$$-g_V^2\frac{8}{M^2}\frac{(\vec{\sigma}_1\vec{p})(\vec{\sigma}_1\vec{\kappa})(\vec{\sigma}_2\vec{p})(\vec{\sigma}_2\vec{\kappa})}{\mu^2+4\vec{\kappa}^2} \qquad (8)$$

has the form of a direct geometrical generalization of the Breit potentials [2,8].

In the second part of the paper we report in brief the main ideas concerning the relativistic configurational representation required to deduce new results. In the third and fourth sections we derive the relativistic analogs of spin-orbital, spin-spin and tensor forces and discuss their properties and distinction from the nonrelativistic case. In Appendix I we give the calculations for deducing the expression for tensor forces, and in Appendix II we prove the hermiticity of the obtained relativistic potentials.

II. THE RELATIVISTIC CONFIGURATIONAL REPRESENTATION. COULOMB AND YUKAWA POTENTIALS

The relativistic configurational representation (RCR) was determined in [1]. In the nonrelativistic case the configurational representation is introduced by the Fourier-transformation which has the meaning of the expansion over the unitary representation of the Galilean group, i.e., function $\exp(i\vec{q}\vec{r})$. The RCR is introduced via expansions over the unitary irreducible representation (UIR) of the Lorentz group. The use of such an expansion is justified since the Lorentz group is the group of motion of the Lobachevsky space realized on the upper sheet of hyperboloid (6). So, if we want to associate the local in the Lobachevsky space quasipotential (2) (8) with some local expressions in the coordinate space, we should make a transition to it by means of expansions on the group of motion of the Lobachevsky space.

The method of expansions on the Lorentz group is well known [9] and was already employed by a number of authors [10] in the elementary particle physics. However, these authors did not treat the group parameter (which plays a role of the relative coordinate in our approach) as a relativistic generalization of the relative coordinate and the expansion over UIR of the Lorentz group itself as a relativistic generalization of the Fourier to the coordinate space. In ref. [1] the harmonic analysis on the Lorentz group was used in the form given in [10]. The main object is the introduced by Shapiro [10] functions

$$\xi(\vec{p};\vec{n},r) = \left(\frac{p_0 - \vec{p}\cdot\vec{n}}{M}\right)^{-1-irM} \tag{9}$$

$$p_0 = \sqrt{M^2 + \vec{p}^{\,2}}; \quad \vec{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi;\ \cos\theta);\ \vec{n}^{\,2} = 1$$

composing the complete and orthogonal system in the Lobachevsky space

$$\frac{1}{(2\pi)^3}\int \xi^*(\vec{p};\vec{n},r)\,\xi(\vec{p};\vec{n}',r')\,d\Omega_p = \delta^3(\vec{r}-\vec{r}\,')$$

$$\frac{1}{(2\pi)^3}\int \xi^*(\vec{p};\vec{n},r)\,\xi(\vec{k};\vec{n},r)\,d\vec{r} = \delta^3(\vec{p}-\vec{k})\sqrt{1+\vec{p}^{\,2}/M^2}$$

$$\vec{r} = r\cdot\vec{n};\quad \vec{r}\,' = r'\cdot\vec{n}\,';\quad d\Omega_p = \frac{d\vec{p}}{\sqrt{1+\vec{p}^{\,2}/M^2}}$$

$$d\vec{r} = r^2 dr\, d\omega_{\vec{n}} = r^2 dr \sin\theta\, d\theta\, d\phi \tag{10}$$

The functions $\xi(\vec{p};\vec{n},r)$ realizing infinite-dimensional UIR of the Lorentz group play the role of plane waves when passing to RCR. The nonrelativistic limit

$$\xi(\vec{p};\vec{n},r) \to e^{i(\vec{p}\cdot\vec{n})r} \equiv e^{i\vec{p}\cdot\vec{r}}.$$

The partial expansion for $\xi(\vec{p};\vec{n},r)$ has the form [1,10]:

$$\xi(\vec{p};\vec{n},r) = \sum_{\ell=0}^{\infty}(2\ell+1)\,i^{\ell}\,p_\ell(\mathrm{ch}\chi_p,r)\,P_\ell\left(\frac{\vec{p}\cdot\vec{n}}{|\vec{p}|}\right) \tag{11}$$

where the radial functions

$$p_\ell(\mathrm{ch}\chi_p,r) = (-i)^{\ell}\sqrt{\frac{\pi}{2\,\mathrm{sh}\chi_p}}\,\frac{\Gamma(irM+\ell+1)}{\Gamma(irM+1)}\,P^{-\frac{1}{2}-\ell}_{-\frac{1}{2}-irM}(\mathrm{ch}\chi_p) =$$

$$= i^{\ell}\,\frac{\Gamma(-irM+1)}{\Gamma(-irM+\ell+1)}\,(\mathrm{sh}\chi_p)^{\ell}\left(-\frac{d}{\mathrm{sh}\chi_p\, d\chi_p}\right)^{\ell} p_0(\mathrm{ch}\chi_p,r)$$

$$p_0(\mathrm{ch}\chi_p,r) = \frac{\sin rM\chi_p}{rM\ \mathrm{sh}\chi_p} \tag{12}$$

in the nonrelativistic limit $M \gg 1$ $\chi_p \ll 1$ transform to the spherical Bessel functions $p_\ell(\mathrm{ch}\chi_p,r) \to j_\ell(pr)$.

In ref. [1] the authors have found the operator of free Hamiltonian for the plane waves (9)

$$\hat{H}_0\,\xi(\vec{p};\vec{n},r) = p_0\,\xi(\vec{p};\vec{n},r). \tag{13}$$

The relativistic Hamiltonian is the second-order finite-difference operator

$$\hat{H}_0 = M\,\mathrm{ch}\left(\frac{i}{M}\frac{\partial}{\partial r}\right) + \frac{i}{r}\,\mathrm{sh}\left(\frac{i}{M}\frac{\partial}{\partial r}\right) - \frac{\Lambda_{\theta,\phi}}{2Mr^2}\,e^{\frac{i}{M}\frac{\partial}{\partial r}} \tag{14}$$

with the step proportional to the particle Compton wave length $1/M$. In the nonrelativistic limit

$$\exp\left(\frac{i}{M}\frac{\partial}{\partial r}\right) - 1 \to \frac{i}{M}\frac{\partial}{\partial r}$$

and $\hat{H}_0$ transforms into the free Hamiltonian of the Schrödinger equation. Analogously there has been found the momentum operator [11]

$$\hat{P}\,\xi(\vec{p};\vec{n},r) = \vec{p}\,\xi(\vec{p};\vec{n},r) \tag{15}$$

(for its explicit form see Appendix).

To complete this section, note an important property of the new relativistic coordinate. Its modulus is the relativistic invariant since it parametrizes the eigenvalues of the Casimir operator of the Lorentz group $\hat{C} \equiv \frac{1}{4} M_{\mu\nu} M^{\mu\nu} = \vec{N}^2 - \vec{M}^2 = 1/M^2 + r^2$. It was shown in /12/ that to the standard invariant definition of the mean square radius of a system has been given the group-theoretical sense of the Casimir operator of the Lorentz group because there holds the following relation

$$\langle r_0^2 \rangle = \frac{6}{F(0)} \frac{\partial F(t)}{\partial t} \Big|_0 = \frac{1}{F(0)} [\hat{C} F(t)]_{t=0} ,$$

where F(t) is the invariant form factor of the system. Thus, the distance from the centre of the system is defined by the eigenvalues of $\hat{C}$

$$\langle r_0^2 \rangle = \frac{6}{F(0)} \frac{\partial F(t)}{\partial t} \Big|_{t=0} = \frac{\int \left(\frac{1}{M^2} + r^2 \right) F(r) d\vec{r}}{\int F(r) d\vec{r}} , \qquad (16)$$

where $F(r)$ is the transform of F(t) in the new r -space. This relation between the new coordinate and the system mean-square radius will be important for the interpretation of the results obtained in Sect. IV.

In ref. /1/ the transform of the meson propagator has been found in the new coordinate space. Due to the spherical symmetry of the propagator the transformation via functions (9) takes the form

$$V(\vec{r}) = \frac{1}{(2\pi)^3} \int \xi(\vec{\Delta}; \vec{n}, r) \frac{d\Omega_\Delta}{\mu^2 - (p-k)^2} =$$

$$= 4\pi \int \frac{\sin rM\chi_\Delta}{rM \operatorname{sh} \chi_\Delta} \frac{\operatorname{sh}^2 \chi_\Delta \, d\chi_\Delta}{\mu^2 - 2M^2 + 2M\sqrt{M^2 + \vec{\Delta}^2}} \qquad (17)$$

of the relativistic generalization of the Yukawa potential

Consider the Yukawa potential for $\mu^2 < 4M^2$. On performing the finite-difference differentiation we obtain, like in the nonrelativistic case, the scalar and tensor parts:

$$V(\vec{r}) = V_S(r)(\vec{\sigma}_1 \vec{\sigma}_2) + V_T(r) S_{1,2} , \qquad (27)$$

where

$$V_S(r) = \frac{1}{3} \left[\mu^2 V_{Yik.}(r) - 8\pi \frac{\delta(1/M^2 + r^2)}{r} \delta(\vec{n}) \right] \qquad (28)$$

and

$$V_T(r) = \frac{1}{3} \frac{r^2}{(r + \frac{i}{M})(r + 2\frac{i}{M})} \left[\mu^2 + 3\frac{\mu}{r}\left(1 - \frac{\mu^2}{2M^2}\right) \frac{\operatorname{th}(rMa)}{\sqrt{1 - \mu^2/4M^2}} + \right.$$

$$\left. + \frac{1}{r^2} \frac{3 - 2\frac{\mu^2}{M^2}(1 - \mu^2/4M^2) - \frac{3}{2\operatorname{ch}(rMa)}}{1 - \mu^2/4M^2} \right] V_{Yuk.}(r). \qquad (29)$$

The potential (27) with such V_S and V_T in the nonrelativistic limit reduces to expression (23). So in the relativistic expression (28) there appears the δ -function of $1/M^2 + r^2$, instead of $\delta(\vec{r})$ in (23). As was mentioned at the end of Sect. II (see formula (16)), it is just the combination $1/M^2 + r^2$ that measures the mean-square radius of a system in the relativistic case. Therefore to the point $1/M^2 + r^2 = 0$ there corresponds the system centre in full analogy with the nonrelativistic case $r^2 = 0$.

Like the above considered spin-orbital interaction, the potential V_T (29) does not contain at origin the singularities higher than the Yukawa potential itself (17) (in contrast to the nonrelativistic expression (23)).

CONCLUSION

It is shown that in the relativistic theory the consistent use of the language of the Lobachevsky geometry and the harmonic analysis on the Lorentz group allows one to construct the three-dimensional formalism similar by form to that of quantum mechanics. Indeed, the Feynman matrix elements of the scattering amplitude, corresponding to the one-boson exchange, in the momentum space can be represented as a direct geometrical relativistic generalization of quantum-mechanical potentials. To this end one needs only to extract the kinematic Wigner rotation which performs the transfer of all spin indices onto one momentum /13/. Then the remaining dynamic part takes an absolute form in the sense of transition from the Euclidean to the Lobachevsky geometry.

The use of the relativistic configurational representation allows one to conserve this three-dimensional nature of description and the nonrelativistic spin structure of expressions in the coordinate space too. The derived images of the relativistic OBEP may be treated as a relativistic generalization of the spin-orbital and tensor forces.

An important difference of the obtained relativistic potentials (21) and (28),(29) from their nonrelativistic analogs consists in that in the consistent relativistic formalism the account inclusion of spin does not result in increasing of the order of singularity at the origin, i.e., the appearance of terms of the type r^{-3} and r^{-5}.

This is the result of the finite-difference nature of the relativistic operators $\hat{\vec{P}}$ and H_0 in RCR, since the action of the finite-difference operator does not raise the order of singularity. Thus only a new singularity appears on the imaginary axis, at a distance of the order of the particle Compton wave length from the real axis. It is clear that in the nonrelativistic limit these singularities and the order of singularity increases.

It is interesting to note that the potentials with such singularities at complex points have already been introduced by some authors in a pure phenomenological way to describe a number of specific features of the high-energy scattering /14/ (cf. also /15/). Our formalism provides a field-theoretical basis for application of such potentials.

Our further purpose will be to utilize the obtained relativistic OBEP formalism in quasipotential equations for constructing the three-dimensional formalism suitable to the relativistic description of composite particles and the high-energy scattering.

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APPENDIX I

The momentum operators $\hat{\vec{P}}$ in the coordinate space satisfying (15) have the form

$$\hat{P}_x = -\sin\theta\cos\phi\,(Me^{\frac{i}{M}\frac{\partial}{\partial r}} - \hat{H}_0) - i\left(\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)e^{\frac{i}{M}\frac{\partial}{\partial r}}$$

$$\hat{P}_y = -\sin\theta\sin\phi\,(Me^{\frac{i}{M}\frac{\partial}{\partial r}} - \hat{H}_0) - \left(\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)e^{\frac{i}{M}\frac{\partial}{\partial r}} \qquad (\mathrm{I}.1.)$$

$$\hat{P}_z = -\cos\theta(Me^{\frac{i}{M}\frac{\partial}{\partial r}} - \hat{H}_0) + i\,\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}e^{\frac{i}{M}\frac{\partial}{\partial r}}\,.$$

In the nonrelativistic limit the finite difference operator p transforms into the known quantum-mechanical differential operator $\hat{\vec{P}}_{nonrel.}$ written in the spherical coordinates. In accordance with (1.1) the momentum operator can be written as follows

$$\hat{\vec{P}}_j = -\vec{n}_j\left(M e^{\frac{i}{M}\frac{\partial}{\partial r}} - \hat{H}_0\right) + \hat{\vec{Y}}_j(\theta,\phi) e^{\frac{i}{M}\frac{\partial}{\partial r}} \qquad (I.2)$$

The operator $\hat{Y}_j(\theta,\phi)$ containing the differentiation over angles θ and ϕ has the same form as the non-relativistic theory.

From (I.2) it follows that

$$[\vec{r} \times \vec{p}] = \vec{L} \exp\left(\frac{i}{M}\frac{\partial}{\partial r}\right), \qquad (1.3)$$

where $\vec{L}$ coincides with the usual angular momentum operator.

Now we introduce the operator $\vec{V}'_{f.dif.} = \vec{V}_{f.dif.} \exp\left(-\frac{i}{M}\frac{\partial}{\partial r}\right)$ and derive the commutation relation for radial functions $\vec{V}'_{f.dif.}$ and $\vec{r}$ which is necessary for calculations of the tensor forces. The most simple way is to proceed from the relation between $\vec{V}'_{f.dif.}$ and the nonrelativistic gradient operator $\vec{V}_{nonr.}$:

$$\vec{V}'_{f.dif.} = \vec{V}_{nonr.} + \vec{n}\left(\hat{D} - \frac{\partial}{\partial r}\right), \qquad (I.4)$$

where $\hat{D}$ is the finite-difference operator

$$\hat{D} = \frac{1}{2\frac{i}{M}}\left[\left(1 - \frac{i}{rM}\right)\left(1 - \exp\left(-2\frac{i}{M}\frac{\partial}{\partial r}\right)\right) + \frac{\Delta_{\theta,\phi}}{M^2 r^2}\right]. \qquad (I.5)$$

Then we have

$$[\vec{V}'_{f.dif.i}, \vec{r}_i] = \delta_{ij} + \vec{n}_i[\hat{D}, \vec{r}_j] - \vec{n}_i\vec{n}_j. \qquad (I.6)$$

Further using the property

$$\Delta_{\theta,\phi}\vec{n} = -2\vec{n} \qquad (I.7)$$

we arrive at the commutation relation for radial functions

$$[\vec{V}'_{f.dif.i}, \vec{r}_j] = \delta_{ij} - 2\frac{i}{M}\vec{n}_i\vec{n}_j\hat{D}_{rad}, \qquad (I.8)$$

where $\hat{D}_{rad}$ is the radial part of $\hat{D}$

$$\hat{D}_{rad} = \frac{1}{2\frac{i}{M}}\left[\left(1 - \frac{i}{rM}\right)\left(1 - \exp\left(-2\frac{i}{M}\frac{\partial}{\partial r}\right)\right)\right] \qquad (I.9)$$

It is easy to see that the second term in (I.8) vanishes in the nonrelativistic limit and (I.8) reduces to the usual commutation relation for nonrelativistic quantities:

To calculate the tensor forces we should know the action of $\vec{V}_i\vec{V}_j$ on an arbitrary radial function $f(r)$. Using the relation (I.8) one can find

$$\vec{V}_{f.dif.i}\vec{V}_{f.dif.j} f(r) = \left\{\vec{n}_i\vec{n}_j \frac{r - 2\frac{i}{M}}{r + \frac{i}{M}}\left[\hat{T} - \frac{1}{r}\hat{O}\right] + \delta_{ij}\frac{1}{r}\hat{P}\right\} f(r), \qquad (I.10)$$

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* The relation (I.7) follows from the fact that components of the vector $\vec{n}$ can be represented as a combination of the spherical functions $Y_{1M}(\theta,\phi)$, and the spherical Laplacian acts on them as follows:

$$\Delta_{\theta,\phi} Y_{\ell M}(\theta,\phi) = -\ell(\ell+1) Y_{\ell M}(\theta,\phi).$$

where

$$\hat{P} = e^{\frac{i}{M}\frac{\partial}{\partial r}} \quad \hat{D}_{rad} \quad e^{-\frac{i}{M}\frac{\partial}{\partial r}} = \frac{1}{2\frac{i}{M}} \frac{r}{r+\frac{i}{M}} \left(e^{2\frac{i}{M}\frac{\partial}{\partial r}} - 1 \right)$$

$$\hat{O} = \frac{1}{2(2\frac{i}{M})} \left(e^{2\frac{i}{M}\frac{\partial}{\partial r}} - e^{-2\frac{i}{M}\frac{\partial}{\partial r}} \right) \tag{I.11}$$

$$\hat{T} = \frac{1}{(2\frac{i}{M})} \left(e^{2\frac{i}{M}\frac{\partial}{\partial r}} + e^{-2\frac{i}{M}\frac{\partial}{\partial r}} - 2 \right).$$

In the nonrelativistic limit the operators $\hat{P}$ and $\hat{O}$ transform into the one and the same operator of a usual differentiation $\partial/\partial r$. The operator T transforms into the differentiation operator $\partial^2/\partial r^2$. Expression (I.10) is written in the form close to the nonrelativistic one where the operators of single and double differentiation are replaced by their finite-difference analogs.

APPENDIX II

Here we show the hermiticity of the operators of the free Hamiltonian $\hat{H}_0$ and the momentum P.

Consider first H_0. It appears to consist of two hermitian parts

$$M \operatorname{ch} \frac{i}{M}\frac{\partial}{\partial r} + \frac{i}{r} \operatorname{sh} \frac{i}{M}\frac{\partial}{\partial r} \tag{II.1}$$

and

$$-\frac{\Delta_{\theta,\phi}}{2Mr^2} e^{\frac{i}{M}\frac{\partial}{\partial r}} . \tag{II.2}$$

To prove this assertion we note that the operator $\exp(\frac{i}{M}\frac{\partial}{\partial r})$ is hermitian with the integration measure dr* and the spherical Laplacian $\Delta_{\theta,\phi}$ is hermitian with the measure $d\omega = \sin\theta\, d\theta\, d\phi$.

The hermiticity of (I.1) follows from the following consideration

$$\int dr\, r^2 \psi^*(r) \left[M \operatorname{ch} \frac{i}{M}\frac{\partial}{\partial r} + \frac{i}{r} \operatorname{sh} \frac{i}{M}\frac{\partial}{\partial r} \right] \phi(r) =$$

$$= \frac{M}{2} \sum_n \frac{1}{n!} \int dr\, \psi^*(r) \left[\left(r^2 + \frac{ir}{M}\right) + (-1)^n \left(r^2 - \frac{ir}{M}\right) \right] \times$$

$$\times \left(\frac{i}{M}\frac{\partial}{\partial r} \right)^n \phi(r) = \frac{M}{2} \int dr \left\{ \left[\left(r^2 - \frac{ir}{M}\right) e^{-\frac{i}{M}\frac{\partial}{\partial r}} + \right.\right.$$

$$\left.\left. + \left(r^2 + \frac{ir}{M}\right) e^{\frac{i}{M}\frac{\partial}{\partial r}} \right] \psi^*(r) \right\} \phi(r) =$$

$$= \int dr\, r^2 \left\{ \left[M \operatorname{ch} \frac{i}{M}\frac{\partial}{\partial r} + \frac{i}{r} \operatorname{sh} \frac{i}{M}\frac{\partial}{\partial r} \right] \psi(r) \right\}^* \phi(r). \tag{II.3}$$

Hermiticity of (II.2) follows immediately from hermiticity of $\exp(\frac{i}{M}\frac{\partial}{\partial r})$ and $\Delta_{\theta,\phi}$ with the corresponding measures

$$\int d\vec{r}\, \psi^*(r) \frac{\Delta_{\theta,\phi}}{r^2} e^{\frac{i}{M}\frac{\partial}{\partial r}} \phi(r) =$$

$$= \int d\omega_n dr\, \psi^*(r) \Delta_{\theta,\phi} \exp\left(\frac{i}{M}\frac{\partial}{\partial r}\right) \phi(r) =$$

$$= \int d\vec{r} \left[\frac{\Delta_{\theta,\phi}}{r} \exp\left(\frac{i}{M}\frac{\partial}{\partial r}\right) \psi(r) \right]^* \phi(r). \tag{II.4}$$

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* This follows from hermiticity of the operator $(\frac{i}{M}\frac{\partial}{\partial r})^n$.

Thus, the finite-difference operator of the free Hamiltonian $\hat{H}_0$ given by (I.4) is hermitian with the measure $d\vec{r}$.

The proof of the hermiticity of the operator $\hat{\vec{P}}$ is carried out for the third component $\hat{P}_3$ as the most simple for calculations case. Then using hermiticity of $\hat{H}_0$ and the commutation relation

$$[\hat{H}_0, n_3] = \frac{1}{Mr^2}(\cos\theta + \sin\theta\frac{\partial}{\partial\theta})\exp(\frac{i}{M}\frac{\partial}{\partial r})$$

and integrating by parts over angle θ, we have

$$\int dr\,\psi^*(r) = [-\cos\theta(M\exp(\frac{i}{M}\frac{\partial}{\partial r}) - \hat{H}_0) +$$

$$+ i\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\exp(\frac{i}{M}\frac{\partial}{\partial r})]\phi(r) =$$

$$= \int dr\{[-\cos\theta(M\exp(\frac{i}{M}\frac{\partial}{\partial r}) - \hat{H}_0) +$$

$$+ (\frac{2}{Mr^2}\cos\theta + \frac{1}{Mr^2}\sin\theta\frac{\partial}{\partial\theta} - \frac{2i}{r}\cos\theta)\exp(\frac{i}{M}\frac{\partial}{\partial r}) +$$

$$+ i\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\exp(\frac{i}{M}\frac{\partial}{\partial r}) +$$

$$+ (\frac{2i}{r}\cos\theta - \frac{2\cos\theta}{Mr^2} - \frac{\sin\theta}{Mr^2}\frac{\partial}{\partial\theta})\exp(\frac{i}{M}\frac{\partial}{\partial r})]\times$$

$$\times\psi(r)\}^*\phi(r) =$$

$$= \int dr\{[-\cos\theta(M\exp(\frac{i}{M}\frac{\partial}{\partial r}) - \hat{H}_0) +$$

$$+ \frac{i\sin\theta}{r}\frac{\partial}{\partial\theta}\exp(\frac{i}{M}\frac{\partial}{\partial r})]\psi(r)\}^*\phi(r).$$

Thus we derive the condition of hermiticity

$$\int dr\,\psi^*(r)\,\hat{\vec{P}}_3\,\phi(r) = \int dr\,[\hat{\vec{P}}_3\,\psi(r)]^*\phi(r).$$

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