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Summary

The known integral formulation of the Mach principle [2-6]

$$g_{ik} = \frac{8\pi\kappa}{c^4} \int G^{\alpha\beta}_{ik}(x,y) T_{\alpha\beta}(y) \sqrt{-g(y)}\, d^4 y + \Lambda_{ik} \qquad (2)$$

with the requirement

$$\Lambda_{ik} = 0 \qquad (4)$$

represents a rather complicated mathematical formalism in which many aspects of the physical content of theory are not clear.

Below an attempt is made to consider the integral representation (2), (4) for the most simple case of conformally flat spaces

$$ds^2 = \phi^2 (c^2 dt^2 - dx^2 - dy^2 - dz^2).$$

The fact that in this formalism there is only one scalar function $\phi(x)$ makes it possible to analyse in more detail many physical peculiarities of this representation of the Mach principle: the absence of asymptotically flat spaces, compatibility of the condition (4) with the presence of the local free radiation, problems of inertia and gravity, constraints on state equations, etc.

I. One of the possible ways of mathematical representation of the Mach principle is the method of integral equations in general relativity (GR). This method is based on the known statement of Einstein that the gravitational field should be completely defined by the energy tensor of matter [1] and consists in the requirement that the metric obeys the integral equations [2-6]:

$$g_{ik}(x) = \frac{8\pi\kappa}{c^4} \int G^{\alpha\beta}_{ik}(x,y) T_{\alpha\beta}(y) \sqrt{-g(y)}\, d^4 y , \qquad (1)$$

where $G^{\alpha\beta}_{ik}(x,y)$ is a bitensor (indices α, β refer to the point y and indices i,k refer to the point x) [7], the tensor $T_{\alpha\beta}$ describes the matter and is constructed of the energy-momentum tensor of matter. The tensor $G^{\alpha\beta}_{ik}(x,y)$ is an analog of the Green functions of the known equations.

Condition (1) is imposed on the metric in addition to the Einstein equations. Generally speaking, the Einstein equations themselves are equivalent to the following integral equations

$$g_{ik}(x) = \frac{8\pi\kappa}{c^4} \int G^{\alpha\beta}_{ik}(x,y) T_{\alpha\beta}(y) \sqrt{-g(y)}\, d^4 y + \Lambda_{ik}(x). \qquad (2)$$

In equations (2) the term Λ_{ik} characterizes the metric in the absence of matter ($T_{\alpha\beta} = 0$) and, as follows from the Einstein equation, it differs, generally speaking, from zero. It should be emphasized that due to nonlinearity of the Einstein equations the quantities $G^{\alpha\beta}_{ik}(x,y)$ and $\Lambda_{ik}(x)$ must be rather complicated functionals of the metric g_{ik}

$$G_{ik}^{\alpha\beta}(x,y) = G_{ik}^{\alpha\beta}(x,y \mid g_{mn}, g_{\mu\nu}),$$

$$\Lambda_{ik}(x) = \Lambda_{ik}(x \mid g_{mn}, g_{\mu\nu}), \tag{3}$$

therefore eq. (2) is the nonlinear integral equation. As will be shown, $\Lambda_{ik}(x)$ is represented by an integral over the surface covering the whole space of a cosmological model. At the same time the condition

$$\Lambda_{ik}(x) = 0 \tag{4}$$

is the boundary condition imposed on solutions of the Einstein equations and due to the tensor nature of $\Lambda_{ik}(x)$ it is covariant.

The main difficulty of the analysis of the theory in form (1) is the nonuniqueness of choice of the differential operator of the equation for the Green function $G_{ik}^{\alpha\beta}$:

$$D_{ik}^{mn}(x) G_{mn}^{\alpha\beta}(x,y) = \delta_{(i}^{(\alpha} \delta_{k)}^{\beta)} \frac{\delta^4(x-y)}{\sqrt{-g(x)}}, \tag{5}$$

and therefore the nonuniqueness of choice of the Green function itself. Unfortunately, the proposed variants of the theory, reveal only separate aspects of the problem /2-5/. This nonuniqueness is due to the fact that the only known indisputable requirement imposed on the operator D

$$D_{ik}^{mn} g_{mn} = R_{ik} - \frac{1}{2} g_{ik} R, \tag{6}$$

or

$$D_{ik}^{mn} g_{mn} = R_{ik} \tag{6$'$}$$

is extremely weak. This nonuniqueness extends naturally to the free terms Λ_{ik} and results in that the same cosmological model considered by different authors sometimes appears to be consistent with the Mach principle and sometimes not /2,6/.

Due to the fact that equations (1,5) are, probably, more complicated than the Einstein equations themselves it is rather difficult to analyse their physical consequences. It is not clear to what extent this formalism is adequate to that understood as the Mach principle.

It is clear only that in the representation (1) the absence of matter $(T_{ik} = 0)$ results in the absence of the space $(g_{ik} = 0)$. (Strictly speaking, this formalism is constructed in such a way that this requirement be satisfied). It is clear as well, that asymptotically flat solutions (of a type of the Schwarzschild solution) do not satisfy requirements of the formalism /2,6/. In this formalism, it is difficult to obtain and to analyse other consequences of the Mach principle within the given representation.

Therefore it seems interesting to apply the above procedure to spaces of a simpler special type, for instance to conformally flat spaces and then to analyse various consequences of the given representation of the Mach principle.

II.1. As has been pointed out, the main reason for it being difficult to realize and to analyse the Mach principle in the form of boundary conditions (4) is the extremely complicated structure of the Einstein equations for ten independent components of the gravitational potential g_{ik}. However, the consideration of spaces of the conformally flat type is considerably simplified from the formal aspect because in their metric

$$g_{ik}(x) = \phi^2(x) \eta_{ik} \tag{7}$$

the quantities η_{ik} (the metric of the flat space) are defined with an accuracy to the choice of a reference frame and they may be considered to be known. Thus, the whole information on the gravitational field is covered by the only "potential" $\phi(x)$. At the same time, despite such a formal simplification, the physical aspect is shown not to be essentially distorted and to become more clear.

2. Since only one "potential" $\phi(x)$ should be defined the Einstein equations

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi\kappa}{c^4} T_{ik} , \qquad (8)$$

which are considered as the equations for $\phi(x)$, are not all independent. To avoid overdetermination and to make the form of the equations more symmetric we take the contraction of the Einstein equations

$$R = -\frac{8\pi\kappa}{c^4} T , \quad T = T_{ik} g^{ik} \qquad (9)$$

as the equations for $\phi(x)$.

For metric (7) the Christoffel symbols are as follows [9]

$$\Gamma^{\ell}_{ik} = \gamma^{\ell}_{ik} + \delta^{\ell}_{i} \frac{\partial}{\partial x^k} \ell n\, \phi + \delta^{\ell}_{k} \frac{\partial}{\partial x^i} \ell n\, \phi - \eta_{ik} \eta^{\ell m} \frac{\partial}{\partial x^m} \ell n\, \phi, \quad (10)$$

where γ^{ℓ}_{ik} are the Christoffel symbols in a space with the metric η_{ik}. Using these symbols we find the scalar curvature

$$R = -6 \frac{\Box \phi}{\phi^3} , \qquad (11)$$

where $\Box\phi$ is the d'Alembertian of ϕ covariant with respect to the metric η_{ik}. Hence the explicit form of eq. (9) is

$$\Box \phi = \frac{4\pi\kappa}{3c^4} T \phi^3 . \qquad (12)$$

Naturally, this equation can be found with the use of the Lagrangian formalism, as well.

3. Equation (12) is nonlinear in ϕ, however, a specific type of nonlinearity makes it possible to obtain an integral equation by analogy with the linear theory [10].

Indeed, equation (12) can be rewritten identically in the form

$$\Box \phi(x) = \frac{4\pi\kappa}{3c^4} \int T(y) \phi^3(y) \delta^4(x-y) \sqrt{-\eta(y)}\, d^4 y , \qquad (13)$$

then, dividing by $\Box$ we get

$$\phi(x) = \frac{4\pi\kappa}{3c^4} \int T(y) \phi^3(y) G(x,y) \sqrt{-\eta(y)}\, d^4 y + \Lambda(x), \qquad (14)$$

where $G(x,y)$ and $\Lambda(x)$ obey the equations

$$\Box G(x,y) = \delta^4(x-y), \qquad \Box \Lambda(x) = 0. \qquad (15)$$

The explicit form of Λ is defined by the conventional method (see, e.g., ref. [11]):

$$\Lambda(x) = -\int_S \{ \phi,_i G - \phi G,_i \} \sqrt{-\eta}\, dS^i , \qquad (16)$$

where S is a boundary of the domain over which the integration in the first term of eq. (14) is made. Finally the equation for ϕ takes the form

$$\phi(x) = \frac{4\pi\kappa}{3c^4} \int T \phi^3 G(x,y) \sqrt{-\eta}\, d^4 y - \int \{ \phi,_i G - \phi G,_i \} \sqrt{-\eta}\, dS^i . \qquad (17)$$

The two integrals in the r.h.s. of (17) have the following meaning: The volume integral allows for a contribution to $\phi(x)$ from all sources localized in the volume. The surface integral allows for a contribution to $\phi(x)$ both from sources which are outside the volume and from the free radiation which comes from infinity and is not produced by any sources.

Since for the Mach principle all the observed masses of the Universe are important, the integration over region in (17) should be understood as the limit transition to integration over the whole space of the cosmological model [6].

4. In accordance with the above programme the Mach principle holds for the models for which

$$\int \{ \phi,_i G - \phi G,_i \} \sqrt{-\eta}\, dS^i = 0, \qquad (18)$$

or

$$\phi(x) = \frac{4\pi\kappa}{3c^4} \int T(y)\phi^3(y)G(x,y)\sqrt{-\eta(y)}\,d^4y \,. \tag{19}$$

Thus, the expression with which we associate the formal representation of the Mach principle, is a certain boundary condition.

Consider it in more detail. Because the integration in (17) is made over the whole space of one or another cosmological model all the sources T turn out to be taken into account by the first term in r.h.s of (17). Hence the free term Λ is not connected with any material sources of a given cosmological model /6/: it describes the wave of the field ϕ to which material source can never and nowhere be related in the course of the whole evolution of a given cosmological model. In other words, after the limit transition the free term should obey the homogeneous equation

$$\Box\Lambda = 0 .$$

at any point of a given cosmological model (i.e., everywhere and always) in the course of the whole evolution. Thus, the boundary condition (18) eliminates the "globally"-free wave (in the above sense). In the particular case of the absence of matter throughout the whole space of a given model in the course of the whole evolution it simply means the degeneration of the concept of space and time that is completely consistent with the qualitative considerations of the Mach principle (eq. (19) gives $\phi = 0$ at $T = 0$).

On the other hand, it cannot be stated that condition (19) eliminates any free wave. Really, if a source T exists in some bounded space-time region the integral in (19) is not zero, generally speaking, outside the region occupied by the source, as well. At the same time, in the free-of-source region the field ϕ obeys the homogeneous field equation and in this region an observer may consider it as a locally-free wave. Thus, in the given representation as well the Mach principle admits of the waves for which somewhere and at some time there exists a source of a type of T (as an example see subject 6).

This statement is closely connected with the fact that the Mach principle in the form of equation (19) or boundary condition (18), as we shall see below, put some constraints on sources T too. It may happen, however, that no physically admissible source obeys this condition. Then the whole theory will be incompatible, in this sense, with the Mach principle. In what follows examples will be given which give evidence that in this sense the given theory has a physical content and realizes the Mach principle in a specific variant.

5. To calculate just the function $G(x,y)$ we take the coordinate system when the metric η_{ik} has the following form

$$\eta_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{20}$$

Derivatives covariant with respect to η_{ik} reduce to partial ones. Choosing the retarded potentials, from eq. (15) we get the following expressions /10/

$$G(x,y) = \frac{\delta\left(t' - t + \frac{|\vec{r} - \vec{r}\,'|}{c}\right)}{4\pi\,|\vec{r} - \vec{r}\,'|} \tag{21}$$

and for the static problems

$$G(x,y) = \frac{1}{4\pi\,|\vec{r} - \vec{r}\,'|} .$$

Now eq. (19) takes the form

$$\phi(\vec{r}, t) = \frac{\kappa}{3c^4} \int T(\vec{r}\,', \tau)\,\phi^3(\vec{r}\,', \tau)\,\frac{d\vec{r}\,'}{|\vec{r} - \vec{r}\,'|} , \tag{22}$$

$$\tau = t - \frac{|\vec{r} - \vec{r}\,'|}{c}$$

and one may proceed to consideration of specific physical consequences of this equation.

6. What we have said in subject 4 about locally-free fields can be illustrated by the following example. Given the source

$$T(r,t) = T_0 \frac{\delta(r-a)}{r^2}\delta(t), \tag{23}$$

then eq. (22) results in the following equation

$$\phi(r,t) = a^2 \sqrt{\frac{3c^2}{2\pi\kappa T_0}} \frac{1}{r} \theta(ct-|r-a|)\theta(a+r-ct),$$

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{24}$$

This solution has a character of a running wave and does not vanish outside the region occupied by the source (i.e., for $r \neq a$, $t > 0$) at the same time obeying the homogeneous wave equation

$$\Box \phi = 0. \tag{25}$$

The condition (18) forbids us to add arbitrary solutions of homogeneous eq. (25), i.e. globally free waves, to eq. (24) which we might do using the differential equation (12) only.

What we have said in subsect 4 about the restrictions imposed by the Mach principle on a source is illustrated by the following example. Given the following distribution of matter

$$T(r) = T_0(a^2 - r^2)\theta(R-r); \quad T, a, R - \text{const}. \tag{26}$$

The differential eq. (12) then gives

$$\phi(r) = \begin{cases} c^2\sqrt{\dfrac{3}{2\pi\kappa T_0}} \dfrac{3}{a^2+3r^2}, & r < R, \\ \dfrac{\text{const}_1}{r} + \text{const}_2, & r > R \end{cases} \tag{27}$$

for arbitrary R and const_1 and const_2 defined on the basis of some additional considerations (behaviour on the boundary of a body, that at infinity, etc.). From the integral equation we get

$$\phi(r) = \begin{cases} c^2\sqrt{\dfrac{3}{2\pi\kappa T_0}} \dfrac{3}{a^2+3r^2}, & r < R, \\ \dfrac{3}{2} c^2 \sqrt{\dfrac{1}{2\pi\kappa T_0}} \dfrac{1}{ar}, & r > R \end{cases} \tag{28}$$

under the condition $R = a/\sqrt{3}$.

It may happen, of course, that in some cases the Mach principle does not impose any restrictions on a source. For instance, due to a special type of nonlinearity of the field equation of our theory, delta-type sources linearize eq. (12). One example of this kind was given at the beginning of this subsect. For another example:

$$T(r) = T_0 \frac{\delta(r-a)}{4\pi r^2} \tag{29}$$

eq. (12) gives

$$\phi(r) = c_1 \frac{c_2 r + c_3 - |r-a|}{r}; \quad c_1, c_2, c_3 - \text{const} \tag{30}$$

and from eq. (22) we have

$$\phi(r) = c^2\sqrt{\frac{3a}{\kappa T_0}} \frac{a + r - |r-a|}{2r} \tag{31}$$

for arbitrary a and T_0 (however $a = 0$, i.e., point like particles, are forbidden; at $a = 0$ we have $\phi = 0$ which is seen from eq. (31) and, of course, can be checked by direct calculation).

A peculiar dependence $\phi \sim T_0^{-1/2}$ should be noted. Since, however, this dependence is not continuous when $T_0 \to 0$, the limit of ϕ as $T_0 \to 0$ should be defined directly from the cubic equation for ϕ, which gives $\phi = 0$ at $T_0 = 0$ and is completely consistent with the Mach principle.

It should be noted also that $\phi(x)$ in its sense is defined up to a constant factor whose variations simply mean the scale transformation of coordinates which does not result in any physical consequences. This arbitrariness corresponds to an arbitrary choice of an additive constant of a potential of the Newton theory (into the equation of motion in the Newton approximation there enters $\ln \phi$ and the scale factor reduces to an additive constant).

7. Consider now a problem of inertia of an object. Since the Mach principle relates the accelerated motion not to absolute space but to all other bodies of the Universe, the inertia of a body in this motion is defined by the presence of remaining bodies so that for an isolated body in the Universe the concept of mass makes no sense now, according to /1,12,13,14/.

An explicit concept of mass of a body is tightly connected with the presence of flat asymptotic form of a surrounding space. No such an asymptotic behaviour exists for the above cited examples.

This result is of extreme generality. Consider an arbitrary restricted distribution of matter. At sufficiently large distances from this distribution the problem becomes spherically symmetric and eq. (12) gives

$$\phi(r) = \frac{a}{r} + b \;; \quad a, b - \text{const}, \tag{32}$$

i.e., at infinity we have, generally speaking, a flat space

$$ds^2 = b^2(c^2 dt^2 - dx^2 - dy^2 - dz^2). \tag{33}$$

Requiring that condition (18) be satisfied

$$\lim_{r' \to \infty} \left\{ \frac{1}{4\pi|\vec{r} - \vec{r}\,'|} \frac{\partial}{\partial r'}\left(\frac{a}{r'} + b\right) - \left(\frac{a}{r'} + b\right) \frac{\partial}{\partial r'} \frac{1}{4\pi|\vec{r} - \vec{r}\,'|} \right\} 4\pi r^2 = 0, \tag{34}$$

we obtain

$$b = 0 \tag{35}$$

Thus, that we have called a realization of the Mach principle excludes not only solutions at $T = 0$ but also the asymptotically solutions in total agreement with qualitative consideration of the Mach principle*. But, let us consider dynamics in finite regions of a given world, i.e., in those regions where the quadratic form ds^2 has sense and, $g_{ik} \neq 0$.

A specific property of dynamics is that it is impossible to neglect the effect of a test body on the metric, since in the opposite case concepts of inertia and gravitational mass do not arise.

Indeed, the field is as follows

$$\phi(r) = \frac{a}{r}. \tag{36}$$

Let us now introduce in the space with metric defined by this field a test particle which does not perturb the metric, then the force acting on the particle will be

$$\vec{f} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \left\{ -\operatorname{grad} \ln \sqrt{g_{00}} \right\} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{1}{r} : \frac{\vec{r}}{r}, \tag{37}$$

where m is the mass of a test particle, v is its velocity; for $v \ll c$ we have the following equations of motion

$$\dot{\vec{r}} = -\frac{c^2}{r} \cdot \frac{\vec{r}}{r}. \tag{38}$$

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*A specific space obtained in this way ($g_{ik} \to 0$ if $r \to \infty$) seems to be unreasonable to be classified following Petrov: the Weyl tensor (of conformal curvature) is zero identically.

Thus the equations of motion do not include any parameters of a central body: whatever the central gravitational body might be taken when observing the motion of a test particle it is impossible to distinguish one body from another, i.e., there is no such a quantity of a central body which would specify the motion of a test particle (e.q., its acceleration). Since, on the other hand such a parameter of a gravitational body, by definition, is the gravitational mass of the body then in virtue of the equivalence principle this result means that a single body (noninteracting with other bodies isolated of them) has no inertia in total agreement with the sense of the Mach principle.

And what is more, the mass of a central body can be shown to be zero. If one uses the results of ref. /8/ then the inertial mass of a body at rest can be defined through zeroth component of its 4-momentum in the following way:

$$M = -\frac{c^2}{16\pi\kappa} \int \frac{\partial}{\partial x^m} [(-g)(g^{00} g^{\alpha m} - g^{0\alpha} g^{0m})] df_\alpha , \qquad (39)$$

that gives, for the spherically-symmetric field:

$$M = \lim_{r \to \infty} \frac{c^2}{\kappa} \phi^3 \frac{\partial \phi}{\partial r} r^2 \qquad (40)$$

and for the given field *

$$M = \lim_{r \to \infty} \left(-\frac{c^2}{\kappa} \frac{a^4}{r^3} \right) = 0 . \qquad (41)$$

Things are changed essentially when we take into account the influence of a test particle on a metric. To determine the field of two particles (central and test) within a given theory it is impossible to represent them

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*It is the same as in the case of a closed world; though this model cannot be considered as three-dimensionally closed. Another fundamental property of a closed world, viz., electric neutrality, has no adequacy in this case (due to the scalar character of the theory).

as mathematical points, i.e., to give a δ-type distribution of matter:

$$T(\vec{r}) = T_1 \delta(\vec{r}) + T_2 \delta(\vec{r} - \vec{R}), \qquad (42)$$

since a field of a particle within our theory (as is seen from subsect *6.*) tends to zero if the particle sizes tend to zero. Therefore we will consider that both particles are of a radius ρ with

$$\rho \ll R , \qquad (43)$$

where R is the distance between particles,

$$T(\vec{r}) = T_1 \frac{\delta(r-\rho)}{4\pi r^2} + T_2 \frac{\delta(|\vec{r}-\vec{R}|-\rho)}{4\pi(\vec{r}-\vec{R})^2} . \qquad (43')$$

An appropriate field has the form

$$\phi(\vec{r}) = \frac{\kappa T_1}{3c^4} \int \frac{\delta(r'-\rho)}{4\pi r'^2} \phi^3(\vec{r}') \frac{d\vec{r}'}{|\vec{r}-\vec{r}'|} +$$

$$+ \frac{\kappa T_2}{3c^4} \int \frac{\delta(|\vec{r}-\vec{R}|-\rho)}{4\pi(\vec{r}'-\vec{R})^2} \phi^3(\vec{r}') \frac{d\vec{r}'}{|\vec{r}-\vec{r}'|} =$$

$$= \frac{\kappa T_1}{3c^4} \phi^3(\vec{r}^*) \int \frac{\delta(r'-\rho)}{4\pi r'^2} \frac{d\vec{r}'}{|\vec{r}-\vec{r}'|} + \qquad (44)$$

$$+ \frac{\kappa T_2}{3c^4} \phi^3(\vec{r}^{**} + \vec{R}) \int \frac{\delta(|\vec{r}-\vec{R}|-\rho)}{4\pi(\vec{r}'-\vec{R})^2} \frac{d\vec{r}'}{|\vec{r}-\vec{r}'|} ,$$

where

$$|\vec{r}^*| = |\vec{r}^{**}| = \rho \ll R , \qquad (45)$$

and finally, between those particles

$$\phi(\vec{r}) = \frac{\kappa T_1}{3c^4}\phi^3(\vec{r}^*)\frac{1}{r} + \frac{\kappa T_2}{3c^4}\phi^3(\vec{R}+\vec{r}^{**})\frac{1}{|\vec{R}-\vec{r}|}. \quad (46)$$

Neglecting now the sizes of particles as compared with the distance between them and putting $|\vec{r}-\vec{R}| = \rho$ we have

$$\phi(\vec{R}) = \frac{\kappa T_1}{3c^4}\phi^3(0)\frac{1}{R} + \frac{\kappa T_2}{3c^4}\phi^3(R)\frac{1}{\rho}. \quad (47)$$

Next suppose that R is so large that the first term in the r.h.s. of (47) is a small correction to the second one; then neglecting the second and higher order corrections we have the following solution of the cubic eq. (47)

$$\phi(R) = +\sqrt{\frac{3\rho}{\kappa T_2}c^2} - \frac{\kappa T_1}{6c^4}\phi^3(0)\frac{1}{R}. \quad (48)$$

Taking the sign + in the first term of the r.h.s. we change the coordinates

$$\tilde{x}^i = \sqrt{\frac{3\rho}{\kappa T_2}c^2}\, x^i \quad (49)$$

and allowing for the definition of $\phi(\vec{r})$ we have

$$\tilde{\phi}(\tilde{R}) = 1 - \frac{\kappa T_1}{6c^4}\phi^3(0)\frac{1}{\tilde{R}}. \quad (50)$$

From the geodesic equation in the nonrelativistic limit and in virtue of a smallness of the term $\sim 1/\tilde{R}$ we obtain the equation of motion of the second particle

$$\ddot{\vec{\tilde{R}}} = -\frac{\kappa M}{\tilde{R}^2}\frac{\vec{\tilde{R}}}{\tilde{R}}, \quad (51)$$

where $M = \frac{T_1}{6c^2}\phi^3(0)$ has the meaning of mass of

the first particle. In this way we have arrived at the Newton law with the essential account of both-particle field.

The same result can be illustrated in the following way. Given two concentric massive spheres

$$T(r) = T_1\frac{\delta(r-a)}{4\pi r^3} + T_2\frac{\delta(r-b)}{4\pi r^2}, \quad a, b = \text{const}, \quad a < b. \quad (52)$$

The solution to integral equation is

$$\phi(r) = \frac{\kappa T_1}{3c^4}\phi^3(a)\frac{r+a-|r-a|}{2ar} + \frac{\kappa T_2}{3c^4}\phi^3(b)\frac{r+b-|r-b|}{2br} \quad (53)$$

and in the gap between the spheres (i.e., for $a \le r \le b$)

$$\phi(r) = \frac{\kappa T_1}{3c^4}\phi^3(a)\frac{1}{r} + \frac{\kappa T_2}{3c^4}\phi^3(b)\frac{1}{b}. \quad (54)$$

It is clear that for constant $\phi(a)$ and $\phi(b)$ from eq. (54), one gets the system

$$\phi(a) = \frac{\kappa T_1}{3c^4 a}\phi^3(a) + \frac{\kappa T_2}{3c^4 b}\phi^3(b),$$

$$\phi(b) = \frac{\kappa T_1}{3c^4 b}\phi^3(b) + \frac{\kappa T_2}{3c^4 b}\phi^3(b). \quad (55)$$

As can be seen, this system has the solution

$$\phi(a) < 0, \qquad \phi(b) > 0. \quad (56)$$

Indeed, with the notations $\phi(a) = -\alpha < 0$ and $\phi(b) = \beta > 0$ from (56) one can easily obtain (regarding eqs. (55) as identities) the following consistent inequalities (taking into account that $a < b$)

$$T_2 \beta^3 < \frac{b}{a} T_1 \alpha^3, \qquad T_2 \beta^3 > T_1 \alpha^3 . \tag{57}$$

Renormalizing in a proper way the coordinates, from (54) we get the Schwarzschild-type field

$$\tilde{\phi}(\tilde{r}) = 1 - \frac{\kappa M}{c^2 \tilde{r}}, \quad M = - \frac{T_1}{3c^2} \phi^3(a) > 0, \tag{58}$$

which gives rise to the Newton equation of motion

$$\ddot{\vec{r}} = - \frac{\kappa M}{\tilde{r}^2} \frac{\vec{\tilde{r}}}{\tilde{r}} . \tag{59}$$

At the same time, outside the larger sphere (i.e., for $r \geq b$) the field is of the form which does not result in the Newton law

$$\phi(r) = \left[\frac{\kappa T_1}{3c^4} \phi^3(a) + \frac{\kappa T_2}{3c^4} \phi^3(b) \right] \frac{1}{r} . \tag{60}$$

Both these examples give evidence that the Newton law (and hence, mass) is meaningful within the gravity theory based on integral equations as long as there is interaction between several bodies and consequently the mass of isolated body cannot be introduced.

Formally this result is rather obvious. In fact, within the standard theory the Newton law is obtained in an approximation based on the flat asymptotic form cf a space surrounding the restricted distribution of matter.

The integral formalism rejects the solution with the flat asymptotic behaviour. It is clear, however, that the flat asymptotic behaviour is not necessary for the appearance of the Newton force. It is only sufficient that in the region of localization of a test particle the field has the Schwarzschild form

$$\phi(r) = \mathrm{const}_1 + \frac{\mathrm{const}_2}{r} , \tag{61}$$

that is just achieved by allowing for the field of the second interacting body * (as in the conventional theory where the flat asymptotic behaviour of this problem is given initially, the proper field of the test particle results in unimportant corrections only).

Thus, in the given theory inertia of a body actually is completely defined by interactions with other bodies, in agreement with the Mach principle.

8. The problem is also rather interesting concerning the possibility of validity of the Mach principle in the real Universe.

If one assumes that the real Universe is described sufficiently accurately by the Fridmann model then it is necessary to define conditions under which the Fridmann model is consistent with boundary conditions of the type (18). The results of the well-known investigations [2,3,6] are discrepant, as has been noted in the Introduction.

If one sets the equation of matter state in the Fridmann model to be of the form

$$p = \gamma \epsilon , \tag{62}$$

where $\gamma = \mathrm{const}$, p and ϵ are the pressure and energy density of matter then from [2,3] we have

$$p = -\epsilon , \quad (\gamma = -1) \tag{63}$$

and from [6] it follows that

$$\frac{1}{3}\epsilon \leq p \leq \epsilon , \quad \left(\frac{1}{3} \leq \gamma \leq 1 \right) \tag{64}$$

Assuming that the Fridmann model is conformally-

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* In some approximation the proper field of the particle can be considered as independent of a mutual positions of the particles.

flat it appears to be possible to solve the above raised problem within our theory, as well.

Consider the simplest case of the flat model /6/

$$ds^2 = \phi^2(\tau)\{d\tau^2 - d\chi^2 - \chi^2(d\theta^2 + \sin^2\theta\, d\beta^2)\} \qquad (65)$$

$$\phi(\tau) = A_0^{-\frac{m}{2}}\left(\frac{\tau}{m}\right)^m, \qquad m = \frac{2}{3\gamma + 1},$$

$$A_0 = \frac{8\pi\kappa\epsilon_0}{3c^4}, \qquad \epsilon_0 = \epsilon(\tau)\phi^{3\gamma+3}(\tau). \qquad (66)$$

The field has a singularity at $\tau = 0$, thus the integration in (22) will be made for $\tau \geq \alpha > 0$ and then $\alpha \to 0$.

After elementary integration eq. (22) gives

$$A_0^{\frac{m}{2}}\left(\frac{\tau}{m}\right)^m = \lim_{\alpha\to 0}\left\{A_0^{\frac{m}{2}}\left(\frac{\tau}{m}\right)^m - A_0^{\frac{m}{2}}\left(\frac{\alpha}{m}\right)^{m-1}\tau + \frac{1-3\gamma}{2}\left(\frac{1}{m}\right)^{m-1} A_0^{\frac{m}{2}}\alpha^m\right\}, \qquad (67)$$

with $\gamma \neq \frac{1}{3}$, whence

$$\lim_{\alpha\to 0}\left\{\left(\frac{\alpha}{m}\right)^{m-1}\tau - \frac{1-3\gamma}{2}\left(\frac{1}{m}\right)^{m-1}\alpha^m\right\} = 0, \qquad (68)$$

which is possible only for $m > 1$, therefore one obtaines

$$-\frac{1}{3} < \gamma < \frac{1}{3}. \qquad (69)$$

For the open model /6/

$$ds^2 = A^2(\tau)\{d\tau^2 - d\chi^2 - \operatorname{sh}\chi^2(d\theta^2 + \sin^2\theta\, d\beta^2), \qquad (70)$$

$$A(\tau) = A_0^{\frac{m}{2}}\operatorname{sh}^m\left(\frac{\tau}{m}\right) \qquad (71)$$

there is the known substitution /8/

$$r = \text{const}\; e^{\tau}\operatorname{sh}\chi, \qquad ct = \text{const}\; e^{\tau}\operatorname{ch}\chi, \qquad (72)$$

which reduces the model to the conformally flat form

$$ds^2 = \phi^2(r,t)\{c^2dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\beta^2)\}, \qquad (73)$$

where taking the const in (72) to be equal to $(A_0^{1/2}/2)^m$ we have

$$\phi(r,t) = [1 - \frac{A_0}{4}(c^2t^2 - r^2)^{-\frac{1}{m}}]^m. \qquad (74)$$

In the given model the field has a singularity at $c^2t^2 - r^2 = (\frac{A_0}{4})^m$, therefore we integrate eq. (22) over the domain

$$c^2t^2 - r^2 \geq \left(\frac{A_0}{4}\right)^m(1+\alpha), \qquad (75)$$

and then $\alpha \to 0$. In so doing, eq. (22) gives ($\gamma \neq \frac{1}{3}$)

$$[1 - \frac{A_0}{4}(c^2t^2 - r^2)^{-\frac{1}{m}}]^m = \lim_{\alpha\to 0}\{[1 - \frac{A_0}{4}(c^2t^2 - r^2)^{\frac{1}{m}}]^m -$$

$$-\left(\frac{\alpha}{m}\right)^{m-1}[1 - \left(\frac{A_0}{4}\right)^m\left(1 - \frac{1-m}{m}\alpha\right)\frac{1}{c^2t^2 - r^2}]\}, \qquad (76)$$

whence

$$\lim_{\alpha\to 0}\left(\frac{\alpha}{m}\right)^{m-1}[1 - \left(\frac{A_0}{4}\right)^m\left(1 - \frac{1-m}{m}\alpha\right)\frac{1}{c^2t^2 - r^2}] = 0, \qquad (77)$$

which is possible for $m > 1$ only, i.e., as in the flat model one has

$$-\frac{1}{3} < \gamma < \frac{1}{3}. \qquad (78)$$

Allowing for the general relation[16]

$$0 \le \gamma \le 1 \qquad (79)$$

we find the Fridmann model is Machian one when

$$0 \le p < \frac{1}{3}\epsilon \qquad (80)$$

at least for the open and flat cases.

Taking into account that according to (67) and (76) the consistency of the model with the Mach principle is basically defined by the beginning of evolution when the difference between all the three types is small one may possibly consider that the obtained result is valid also for the closed model.

That eq. (22) forbids $\gamma = \frac{1}{3}$ is probably due to the scalar character of our theory ($\gamma = \frac{1}{3} \to T = 0$), however, as, in this way, the Universes consisting of photons and neutrinos are ruled out of the consideration it is doubtful whether this fact can be regarded to be important for the real Universe. The following note also should be made: There is a rather widely accepted viewpoint that the Mach principle necessarily requires the three-dimensional closedness of cosmological models. As a rule, this requirement is derived from that the Mach principle forbids asymptotically flat spaces. However, in the proposed representation of the variant of the Mach principle the absence of flat asymptotic behaviour and the three-dimensional closedness, generally speaking, is not the same. In this sense, the flat and open Fridmann models are illustrative enough, in our opinion.

III. An integral form of the gravity equations (in fact, originating from the known paper by Einstein [15]) being applied to the GR in the general case [2–6] seems to be physically nonclear because the formalism is extremely complicated.

The consideration of conformally flat spaces, however, simplifies essentially understanding of the formalism of this variant of the Mach principle.

The considered variant of the Mach principle is characterized by a specific geometrical interpretation and also by a highly specific physical content. The fact that imposing certain boundary conditions on the Einstein equations gives the consequences we are used to obtaining from qualitative consideration of the Mach principle allows one to consider that the integral equation formalism (though requirung further development) is really adequate to the Mach principle and that the generally-covariant boundary conditions imposed on the Einstein equations, in this formalism, really make the general relativity to be a Machian theory, as has been proposed by Einstein.

An Addition to Subsection 3

As has been noted in the Introduction, condition (6) imposed on the operator of equation for the Green function is too weak.

Thus, for the considered conformally flat spaces, as an operator obeying only a condition analogous to (6) one can take an operator of the form

$$D = \alpha \Box + (1 - \alpha)\frac{\Box\phi}{\phi} \qquad (81)$$

($D\phi = \Box\phi$ gives the left-hand side of eq. (12)) with arbitrary (nonzero) α.

To the formalism we have considered there corresponds $\alpha = 1$, to the cases[2,4] $\alpha = -1/3$, to those of refs.[3,5] $\alpha = 1/3$.

Now the Green function satisfies the equation

$$\alpha \Box G + (1 - \alpha)\frac{G}{\phi}\Box\phi = \delta^4(x - y). \qquad (82)$$

Equation (17) formally remains unchanged, however, since $G(x,y)$ depends in one way or another, on α, the meaning of expressions entering into it (and hence of eqs. (18), (19)) changes.

All the formalisms coincide only for $\Box\phi = 0$ (e.g., for empty space and a Schwarzschild-type solution. See the Introduction) whereas for $\Box\phi \neq 0$ the results, generally speaking, are different (e.q., for the Fridmann model. See subsect 8).

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