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UNITARITY AND AMPLITUDES FOR HIGH ENERGIES

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1 Introduction

The standard proof of the well-known Froissart upper bound on the total cross-section is based on the combination of three conditions: unitarity, analyticity of the amplitude of elastic scattering in the Martin-Lehmann ellipse and polynomial boundedness of amplitudes in local quantum field theory (see, for example, [1, 2]). So that the Froissart upper bound is considered to be the intrinsic property of local quantum field theory. The generalization of this proof on nonlocal theories where amplitudes have nonpolynomial bound for large momentum was done in [3]. In this paper we want to show that these restrictions for amplitudes of physical processes on the mass shell and the total cross-section can be obtained taking into account the unitarity condition and the natural assumption that amplitudes are convex down functions of transfer momentum at least in some vicinity of the point $t = 0$. The last assumpion is consistent with experiment and it is fulfilled in all known theoretical models of local and nonlocal theories. The idea of the proof is to consider the S-matrix as a unitary operator defined on the Hilbert space of physical states. This method is more general in comparison with the standard methods [1], in which the main object of investigation is the amplitude of elastic scattering considered in the form of decomposition over partial waves.

Another reason of this work is to remove the main objection against the relativistic nonlocal quantum field theory in which the form-factors (see [4, 5]) and propagators (see [6]) are entire analytic functions decreasing in the Euclidean region of momentum space. The problem consists in that perturbation coefficients of any amplitudes represented in the form of perturbation series

$$\mathcal{M}(s_{ij}) = \sum_{n=n_0}^{\infty} g^n \mathcal{M}_n(s_{ij}), \qquad \mathcal{M}_n(s_{ij}) \sim e^{n s_{ij}^a} \qquad \text{for} \quad s_{ij} \to +\infty$$

with $s_{ij} = (k_i + k_j)^2$. It means that the strong coupling regime in nonlocal theories appears for $g \exp(s_{ij}^a) \sim 1$, and we should use some methods to sum perturbation series. In local renormalizable theories the strong coupling regime appears for $g \ln s_{ij} \sim 1$ and from practical point of view this boundary usually is quite far.

However in the nonlocal quantum theory it is proved [4] that the S-matrix is unitary in each perturbation order and analytical properties of amplitudes in momentum complex space are the same in local and nonlocal theories in each perturbation order except the infinity. We hope that these properties will be valid for exact amplitudes. Then the question arises, if the unitarity condition makes all amplitudes to be bounded for high energies. This paper gives a positive answer to this question.

2 The Froissart upper bound

The well-known Froissart upper bound (see, for example, [1]) on the total cross-section is based on the following arguments. The imaginary part of the amplitude of elastic scattering

$$\mathcal{I}(s,t) = \frac{1}{8\pi}\,\mathrm{Im}\,\mathcal{M}(s,t) = \frac{s^{1/2}}{k}\sum_{l=0}^{\infty}(2l+1)a_l(s)P_l(\cos\theta),$$

where $\cos\theta = 1 + \frac{2t}{s-4m^2}$ $(t<0)$, is considered for which

- the unitarity condition is used in the form

$$0 \le a_l(s) \le 1,$$

- the function $\mathcal{M}(s,t)$ should be analytic in the Martin-Lehmann ellipse and, therefore, it is bounded for

$$|\cos\theta| < 1 + \frac{2t_0}{s-4m^2} \sim 1 + \frac{2t_0}{s},$$

  where $t_0 = 4m^2$, for any fixed large s,

- in the local quantum field theory the amplitude increases like a polynomial

$$|\mathcal{M}(s,t)| < s^N \qquad (1)$$

  where the number N does not depend on s

Then, the inequality

$$P_l(x) > \frac{C}{\sqrt{1+l}}\left[1+\sqrt{2x-2}\right]^l \sim \frac{C}{\sqrt{l}}\left[1+2\sqrt{\frac{t_0}{s}}\right]^l$$

for $x = 1 + \frac{2t_0}{s} > 1$ and large s gives

$$C\sqrt{l}\cdot a_l(s)\cdot\left[1+2\sqrt{\frac{t_0}{s}}\right]^l < 2l\cdot a_l(s)\cdot P_l\left(1+\frac{2t_0}{s}\right) < s^N,$$

so that

$$a_l(s) < \frac{C}{\sqrt{l}}\exp\left\{N\ln s - 2\sqrt{t_0}\frac{l}{\sqrt{s}}\right\}.$$

One can see that for $a_l(s) \ll 1$ for

$$l > L = C\sqrt{s}\ln s$$

and

$$\mathcal{I}(s,0) < \frac{s^{1/2}}{k}\sum_{l=0}^{L}(2l+1) \approx 2L^2 \sim Cs\ln^2 s.$$

The final result, the so called Froissart upper bound, can be obtained according to the optical theorem

$$\sigma_{tot}(s) \sim \frac{\mathcal{I}(s,0)}{s} \le C(\ln s)^2. \qquad (2)$$

Essentially, we can see that this proof is based on the analyticity and locality, and practically does not make use of the unitarity.

We can repeat all these arguments in the nonlocal theory (see [3]) where instead of (1) we have

$$|\mathcal{M}(s,t)| < e^{r(s)}, \qquad (3)$$

where the function $r(s)$ increases like $r(s) \sim s^a$ or more rapidly. We have $a_l(s) \ll 1$ for $l > L = C\sqrt{s}r(s)$ and

$$\mathcal{I}(s,0) < \frac{s^{1/2}}{k}\sum_{l=0}^{L}(2l+1) \approx 2L^2 \sim Csr^2(s).$$

Finally, for the cross-section one gets

$$\sigma_{tot}(s) \sim \frac{\mathcal{I}(s,0)}{s} \le Cr^2(s). \qquad (4)$$

However, this bound is too rough. We shall see that the unitarity condition leads to a weaker upper bound.

3 Unitarity on the mass shell

We shall consider an one-component scalar field for simplicity. The generalization to other fields does not introduce essential difficulties.

The Hilbert space of physical states consists of vectors

$$F = f_0|0\rangle + \sum_{n=1}^{\infty} \int \prod_{j=1}^{n} d\vec{k}_j f_n(\vec{k})|\vec{k}_1, ..., \vec{k}_n\rangle \in \mathcal{H}$$

where

$$|\vec{k}_1, ..., \vec{k}_n\rangle = \frac{1}{\sqrt{n!}} a^+_{\vec{k}_1} \cdot ... \cdot a^+_{\vec{k}_n}|0\rangle,$$

where $|0\rangle$ is the vacuum, for which $\langle 0|0\rangle = 1$. The norm of F is defined

$$||F||^2 = \langle F, F\rangle = |f_0|^2 + \sum_{n=1}^{\infty} \int \prod_{j=1}^{n} d\vec{k}_j |f_n(\vec{k})|^2 < \infty$$

The S-matrix can be represented in the form

$$\begin{aligned} S &= I + iT, \\ T &= \sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty} \int \prod_{j=1}^{n_1} \frac{d\vec{k}_j}{\sqrt{2\omega_j}} \int \prod_{l=1}^{n_2} \frac{d\vec{p}_l}{\sqrt{2\omega_l}} \\ &\cdot \frac{1}{\sqrt{n_1!}} a^+_{\vec{k}_1} \cdot ... \cdot a^+_{\vec{k}_{n_1}} T_{n_1 n_2}(k,p)\, a_{\vec{p}_1} \cdot ... \cdot a_{\vec{p}_{n_2}} \frac{1}{\sqrt{n_2!}} \end{aligned} \tag{5}$$

where $k_j = (\omega_j, \vec{k}_j)$, $\quad \omega_j = \sqrt{m^2 + \vec{k}_j^2}$, $\quad \vec{k}_j \in \mathbf{R}^3$.

The S-matrix is renormalized and satisfies the stability conditions:

$$S|0\rangle = |0\rangle, \qquad S|\vec{k}\rangle = |\vec{k}\rangle,$$

i.e. the vacuum loops are removed and one-particle states are renormalized.

$$T_{n_1 n_2}(k,p) = \sum_{v=0}^{\min(n_1,n_2)} \sum_{\{s,l\}} M_{s_1 m_1}(k,p) \cdot ... \cdot M_{s_v n_v}(k,p),$$

$$M_{sm}(k,p) = \delta^{(4)}\left(\sum_{j=1}^{s} k_j - \sum_{l=1}^{m} p_l\right) \mathcal{M}_{sm}(k,p).$$

The functions $\mathcal{M}_{sm}(k,p)$ contain connected Feynman graphs and depend on the relativistic invariant momentum variables on the mass shell

$$s_{ij} = (k_i - k_j)^2 > 0, \qquad s_{lm} = (p_l - p_m)^2 > 0, \qquad t_{il} = (k_i - p_l)^2 < 0.$$

We shall assume that the functions $\mathcal{M}_{sm}(s_{ij}, t_{il})$ are differentiable and convex down functions at least in vicinities of the points $t_{il} = 0$.

If the S-matrix is unitary

$$S^+S = SS^+ = I,$$

we have the evident inequality

$$||T|| = ||S - I|| \leq ||S|| + ||I|| \leq 2. \tag{6}$$

Thus, if the S-matrix is unitary the inequality

$$|\langle F, TF\rangle| \leq 2\langle F, F\rangle \tag{7}$$

is valid for any $F \in \mathcal{H}$. This inequality is the main tool of our investigation.

4 Elastic scattering.

Let us consider the amplitude of elastic scattering. We introduce

$$F = F_2 = \int d\vec{k}_1 \int d\vec{k}_2\, f_2(\vec{k}_1, \vec{k}_2)|\vec{k}_1, \vec{k}_2\rangle. \tag{8}$$

One obtains

$$\langle F, F\rangle = \langle F_2, F_2\rangle = \int\int d\vec{k}_1 d\vec{k}_2\, |f_2(\vec{k}_1, \vec{k}_1)|^2$$

and

$$\begin{aligned} &\langle F, TF\rangle = \langle F_2, TF_2\rangle \\ &= \int\int \frac{d\vec{k}_1 d\vec{k}_2}{\sqrt{2\omega_1 2\omega_2}} \int\int \frac{d\vec{p}_1 d\vec{p}_2}{\sqrt{2E_1 2E_2}} \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \\ &\cdot f_2(\vec{k}_1, \vec{k}_2)\mathcal{M}_{22}(k_1, k_2; p_1, p_2) f_2(\vec{p}_1, \vec{p}_2) \end{aligned} \tag{9}$$

where $\mathcal{M}_{22}(k_1, k_2; p_1, p_2) = \mathcal{M}(s,t)$, $s = (k_1 + k_2)^2$ and $t = (k_1 - p_1)^2$

To extract the behaviour of the amplitude $\mathcal{M}(s,t)$ as $s \to \infty$ and $t \to 0$ we introduce the following function

$$f_2(\vec{k}_1, \vec{k}_2) = \frac{1}{\varepsilon^{3/2}} \cdot \phi\left(\frac{(\vec{k}_1 - \vec{q})^2}{\varepsilon}\right) \cdot \phi\left(\frac{(\vec{k}_2 + \vec{q})^2}{\varepsilon}\right) \tag{10}$$

where $\phi(u)$ is a function with a finite support

$$\text{supp } \phi(u) = \left\{u : 0 < u < \frac{1}{2}\right\}$$

and it is normalized by the condition

$$\int \frac{d\vec{k}}{\varepsilon^{3/2}} \phi^2\left(\frac{\vec{k}^2}{\varepsilon}\right) = 2\pi \int_0^{1/2} du \sqrt{v} \phi^2(u) = 1$$

so that $\langle F, F\rangle = 1$.

Two parameters $\vec{q}$ and ε are chosen

$$Q^2 = \vec{q}^{\,2} + m^2 \gg m^2, \qquad \text{and} \qquad \varepsilon \ll Q^2.$$

For $\varepsilon \to 0$

$$|f_2(\vec{k}_1, \vec{k}_2)|^2 \to \delta(\vec{k}_1 - \vec{q})\delta(\vec{k}_2 + \vec{q}).$$

Let us intoduce new variables in the integrals (9) where the function f_2 is given by 10

$$\vec{k}_1 \to \vec{k}_1 + \vec{q}, \quad \vec{k}_2 \to \vec{k}_2 - \vec{q}, \quad \vec{p}_1 \to \vec{p}_1 + \vec{q}, \quad \vec{p}_2 \to \vec{p}_2 - \vec{q}$$

Then the integration region is bounded

$$(\vec{k}_j)^2 \le \frac{\varepsilon}{2}, \qquad (\vec{p}_j)^2 \le \frac{\varepsilon}{2}, \qquad (j = 1, 2).$$

For large $Q \gg m^2$ one can get

$$\omega_{\pm} = \sqrt{m^2 + (\vec{k} \pm \vec{q})^2} = Q \pm (\vec{k}\vec{n}) + O\left(\frac{\varepsilon}{Q^2}\right) = Q\left[1 + O\left(\frac{\sqrt{\varepsilon}}{Q}\right)\right],$$

$$\vec{n} = \frac{\vec{q}}{Q}, \qquad (\vec{n})^2 = \frac{\vec{q}^{\,2}}{Q^2} \to 1.$$

Thus we obtain for $Q \to \infty$

$$\begin{aligned}\langle F, TF\rangle \to & \frac{1}{(\pi\varepsilon)^3 s} \cdot \int\int d\vec{k}_1 d\vec{k}_2 \int\int d\vec{p}_1 d\vec{p}_2 \\ & \cdot\phi\left(\frac{\vec{k}_1^2}{\varepsilon}\right) \phi\left(\frac{\vec{k}_2^2}{\varepsilon}\right) \phi\left(\frac{\vec{p}_1^2}{\varepsilon}\right) \phi\left(\frac{\vec{p}_2^2}{\varepsilon}\right) \\ & \cdot\delta^{(3)}\left(\vec{k}_1 + \vec{k}_2 - \vec{p}_1 - \vec{p}_2\right) \delta\left(([\vec{k}_1 - \vec{k}_2 - \vec{p}_1 + \vec{p}_2]\vec{n})\right) \\ & \cdot\mathcal{M}\left(s, -\left[(\vec{k}_1 - \vec{p}_1)^2 - (\vec{n}(\vec{k}_1 - \vec{p}_1))^2\right]\right) \\ = & \frac{\pi\varepsilon}{2s} \cdot \int_0^1 du\, \Phi^2(u) \mathcal{M}(s, -u\varepsilon) = \frac{\pi}{2s} \cdot \int_0^\varepsilon dv\, \Phi^2\left(\frac{v}{\varepsilon}\right) \mathcal{M}(s, -v).\end{aligned}$$

The function $\Phi(u)$ satisfies

$$\begin{aligned}&\Phi(u) = \Phi(\vec{k}^{\,2})|_{\vec{k}^2 = u} = \int d\vec{p}\, \phi\left(\vec{p}^{\,2}\right) \phi\left((\vec{k} + \vec{p})^2\right), \\ &\text{supp } \Phi(u) = \{u : 0 \le u \le 1\}, \qquad \Phi(0) = 1, \quad |\Phi(u)| \le 1.\end{aligned}$$

Thus, for large $s \to \infty$ the inequality (7) leads to

$$\left|\int_0^\varepsilon du\, \Phi^2\left(\frac{u}{\varepsilon}\right) \mathcal{M}(s, -u)\right| \le \frac{4}{\pi} \cdot s, \qquad (0 < \varepsilon \ll s) \tag{11}$$

for positive functions $\Phi(u)$ and any value of the parameter ε.

This inequality for $t < t_1 < 0$ gives

$$|\mathcal{M}(s,t)| < C(t_1) \cdot s, \qquad (|t| > |t_1| > 0). \tag{12}$$

in comparison with

$$|\mathcal{M}(s,t)| < C(t_0) \cdot s(\ln s)^{3/2}$$

mentioned in [1].

5 The total cross-section

Now let us consider the function $\mathcal{I}(s,t) = \text{Im } \mathcal{M}(s,t)$. This function is a differentiable and convex down function in some vicinity of $t = 0$ so that for small enough ε

$$\mathcal{I}(s,-u\varepsilon) \geq \mathcal{I}(s,0) - u\varepsilon \frac{d}{dt}\mathcal{I}(s,t)|_{t=0}. \tag{13}$$

Then we get the inequality

$$\int_0^1 du\, \Phi^2(u)\mathcal{I}(s,-u\varepsilon) \geq c_1\mathcal{I}(s,0) - \varepsilon c_2 \frac{d}{dt}\mathcal{I}(s,t)|_{t=0}. \tag{14}$$

where c_1 and c_2 are positive constants.

Inequalities (11) and (13) give

$$\mathcal{I}(s,0) < \frac{C}{4} \cdot \frac{s}{\varepsilon[1-\varepsilon c b(s)]},$$
$$b(s) = \frac{d}{dt} \ln \mathcal{I}(s,t)|_{t=0} \tag{15}$$

so that

$$\mathcal{I}(s,0) < \frac{C}{4} s b(s) \min_{\varepsilon c b(s)} \frac{1}{\varepsilon c b(s)[1-\varepsilon c b(s)]} = C s b(s). \tag{16}$$

If the function $b(s) \to \text{const}$, we should choose $\varepsilon = \text{const}$; if the function $b(s) \to \infty$, we should choose $\varepsilon = \frac{1}{cb(s)}$.

According to the optical theorem, for $s \to \infty$

$$\sigma_{tot}(s) \sim \frac{\mathcal{I}(s,0)}{s} \leq C b(s). \tag{17}$$

If the amplitude $\mathcal{M}(s,t)$ has the Regge behavior

$$\mathcal{M}(s,t) \sim \Gamma(t)\left(\frac{s}{s_0}\right)^{\alpha(t)},$$

where $\alpha(t) = -at$ for small t, then $b(s) = a \ln s$ and

$$\sigma_{tot}(s) \leq C \ln s. \tag{18}$$

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7 The optical theorem

$$T_{2\to n} = \langle \vec{k}_1, \vec{k}_2 | T | \vec{p}_1, \ldots, \vec{p}_n \rangle = \frac{(2\pi)^4 \delta\left(k_1 + k_2 - \sum_{j=1}^n p_j\right) \mathcal{M}_{2n}(k,p)}{(2\pi)^{3/2(n+2)} \sqrt{2\omega_1 2\omega_2} \prod_{j=1}^n \sqrt{2E_j}}$$

$$\sigma_{2\to n}(s) = \frac{1}{J(s)} \int \delta\left(k_1 + k_2 - \sum_{j=1}^n p_j\right) \frac{|\mathcal{M}_{2n}(k,p)|^2}{(2\pi)^{3n-4}} \prod_{j=1}^n \frac{d\vec{p}_j}{2E_j}$$

$$J(s) = \rho_1 \rho_2 v_{12} 2\omega_1 2\omega_2 (2\pi)^6 = 2\sqrt{s(s-4m^2)}, \qquad s = (k_1 + k_2)^2$$

$$\sigma_{tot}(s) = \sum_{n=2}^{\left[\frac{\sqrt{s}}{m}\right]} \sigma_{2\to n}(s)$$

$$i(T - T^*) = TT^*$$

$$2\text{Im } \mathcal{M}_{22}(s,0) = \sum_{n=2}^{\left[\frac{\sqrt{s}}{m}\right]} \int \delta\left(k_1 + k_2 - \sum_{j=1}^n p_j\right) \frac{|\mathcal{M}_{2n}(k,p)|^2}{(2\pi)^{3n-4}} \prod_{j=1}^n \frac{d\vec{p}_j}{2E_j} = 2\sqrt{s(s-4m^2)}\sigma_{tot}(s)$$

$$\sigma_{tot}(s) = \frac{\text{Im } \mathcal{M}_{22}(s,0)}{\sqrt{s(s-4m^2)}} \to \frac{\text{Im } \mathcal{M}_{22}(s,0)}{s}$$

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