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DESCRIPTION OF FRIEDMANN OBSERVABLES IN QUANTUM UNIVERSE

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1 Introduction

Hope of solving fundamental problems of cosmology of the early Universe by help of quantum gravity [1, 2, 3, 4, 5] has stimulated the development of the Hamiltonian approach to the theory of gravity and cosmological models of the Universe. A lot of papers and some monographs (see, e.g. [6, 7]) have been devoted to the Hamiltonian description of cosmological models of the Universe. The main peculiarity of the Hamiltonian theory of gravity is the presence of nonphysical variables and constraints due to the diffeomorphism invariance of the theory. Just this peculiarity is an obstacle for the solution of the important conceptual problems
- interpretation of the wave function and its non-normalizability ,
- relations between the observational cosmology (the Hubble law and red shift) and the Dirac observables in the Hamiltonian description of the classical and quantum cosmologies.

One of the possible solutions of these problems in the Hamiltonian approach is to reduce the initial constraint system to the unconstrained one by the full separation of pure gauge degrees of freedom from physical ones [8]. In the present paper, we would like to apply the recently developed method of the Hamiltonian reduction of singular systems with the full separation of the gauge sector [9, 10] to a standard cosmological model of the Universe filled in by dust and radiation to investigate the problems listed above and to compare our reduced quantization with the extended approach [3, 4, 5].

In section 2, the Hamiltonian version of standard model identical to classical cosmology is given together with the Wheeler-DeWitt (WDW) equation which follows from the model presented.

In section 3, we apply the Hamiltonian gaugeless reduction developed in [9, 10] to construct the Dirac observables in the classical theory.

Section 4 is devoted to quantization of the reduced system.

2 The Hamiltonian version of the Standard Model

Let us consider the Friedmann standard model beginning from the Hilbert – Einstein action

$$W = \int d^4x \sqrt{-g} \left[-\frac{^{(4)}R(g_{\mu\nu})}{16\pi G} + \mathcal{L}_{matter} \right]. \tag{1}$$

Following Friedmann we suppose the homogeneous distribution of the matter described by the Lagrangian $\mathcal{L}_{matter}$ and, therefore, use the Friedmann – Robertson – Walker (FRW) metric

$$(ds)^2 = a^2(t)[N_c^2 dt^2 - \gamma_{ij} dx^i dx^j]. \tag{2}$$

Here $a(t)$ is a cosmic scale factor, $\gamma_{ij} dx^i dx^j$ is the metric of the three-dimensional space of the constant curvature (we shall call it the "conformal" one)

$$^{(3)}R(\gamma_{ij}) = \frac{-6k}{r_o^2}; \quad k = 0, \pm 1. \tag{3}$$

In this paper, we restrict ourselves to a closed space $k = +1$ to avoid difficulties connected with an infinite volume and boundary conditions. In this case, the parameter r_o characterizes the volume of the three-dimensional conformal space

$$V_c = \int d^3x \sqrt{-\gamma} = 2\pi^2 r_o^3. \tag{4}$$

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We kept the variable N_c, in (2), in contrast with the Friedmann formulation of the standard model, where the component N_c is removed by the definition of the Friedmann time in comoving frame

$$dT = a(t)N_c dt. \tag{5}$$

Variable N_c allows us to preserve the main peculiarity of the Einstein theory (in the considered case) – the invariance with respect to reparametrization of the coordinate time

$$t \rightarrow t' = t'(t) \tag{6}$$

and to reproduce the Einstein – Friedmann equation for the homogeneous distribution of dust and radiation by the variation of the Einstein – Hilbert action (1) for the FRW metric [11, 12]

$$W = \int dt \left\{ \sum_l P_l \dot{A}_l - \left[p_a \dot{a} - \frac{1}{2} \frac{d}{dt}(p_a a) \right] - N_c \mathcal{H}_c(p_a, a, \mathcal{H}_R, M_D) \right\}. \tag{7}$$

The Lagrangian of the matter and the corresponding energy $\mathcal{H}_c$ are chosen to reproduce the equation of state of radiation with dynamical variables $(P_l(t), A_l(t))$ and dust at rest (in the comoving frame (5)) with the total mass M_D:

$$\mathcal{H}_c(p_a, a, \mathcal{H}_R, M_D) = -\left(\frac{p_a^2}{2\beta} + \frac{ka^2}{2r_o^2}\beta \right) + a(t)M_D + \mathcal{H}_R \tag{8}$$

$$\mathcal{H}_R = \frac{1}{2} \sum_l (P_l^2 + \omega_l^2 A_l^2). \tag{9}$$

The constant β is

$$\beta = V_c \frac{6}{8\pi G} = \frac{3\pi r_o^3}{2M_{Pl}^2}, \qquad (M_{Pl} = 1.22 \, 10^{19} GeV). \tag{10}$$

We kept here also the time surface term of action (1).

It is easy to verify that the variation principles applied to the action (7) reproduce the Friedmann evolution of the Universe (filled in by matter with equation of state for dust and radiation) in the comoving frame (5).

The equations of motion for variables of radiation A_l, P_l

$$\dot{A}_l = N_c \{ \mathcal{H}_c, A_l \} = N_c \frac{\partial \mathcal{H}_R}{\partial P_l} \; ; \dot{P}_l = N_c \{ \mathcal{H}_c, P_l \} = -N_c \frac{\partial \mathcal{H}_R}{\partial A_l} \tag{11}$$

lead to the integral of motion

$$\dot{\mathcal{H}}_R = \frac{d}{dt} \mathcal{H}_R = 0 \;\Rightarrow\; \mathcal{H}_R \mid_{eq.m.} = E_R. \tag{12}$$

E_R being a value of $\mathcal{H}_R$ on the equations of motion. The equation for variable N_c

$$\frac{\delta W}{\delta N_c} = 0 \;\Rightarrow\; \mathcal{H}_c(p_a, a, E_R, M_D) = 0 \tag{13}$$

is treated as a constraint and allows us to express the momentum p_a in terms of the cosmic scale factor a and parameters E_R, M_D

$$p_{a(\pm)} = \pm \bar{p}(a, E_R, M_D) = \pm (2\beta)^{1/2} [aM_D + E_R - \frac{a^2}{2r_o^2}]^{1/2}. \tag{14}$$

The evolution of the scale factor a in the comoving frame (5) follows from the equation for p_a

$$\frac{\delta W}{\delta p_a} = 0 \;\Rightarrow\; \frac{ada}{aNdt} \equiv \frac{ada}{dT} = \frac{p_a}{\beta} \;\Rightarrow\; dT(E_R, a) = \frac{\beta ada}{\bar{p}(a, E_R, M_D)} \tag{15}$$

and completely reproduces the evolution of observables in the standard Friedmann model of the Universe. These observables are
the red shift as a function of the present time T_o

$$z_0(T_o, d_F) = \frac{a(T_o)}{a(T_o - d_F/c)} - 1 = H_o d_F/c + ... \tag{16}$$

and a distance between the Earth and a cosmic object

$$d_F(T) = a(T)d_c, \tag{17}$$

where d_c is a distance in the conformal space with stationary metric γ_{ij} and

$$H_o = \frac{1}{a(T_o)} \frac{da(T_o)}{dT_o} \tag{18}$$

is the Hubble parameter.

On the other hand, the quantization of the scale factor

$$i[p_a, a] = \hbar \; ; \; p_a \rightarrow \hat{p}_a = \frac{\hbar}{i} \frac{d}{da} \tag{19}$$

converts the energy balance equation (13) into the Wheeler · DeWitt equation [3, 4]

$$[-\frac{\hbar^2}{2a\beta} \frac{d^2}{da^2} + \beta \frac{a}{2r_o^2} - \frac{E_R}{a} - M_D]\Psi_{WDW} = 0 \tag{20}$$

(for the Einstein energy (8) in the comoving frame version $\mathcal{H}_E = \mathcal{H}_c/a$).

The main problem of our paper is to find the connection between the classical Friedmann observables (5), (16), (18) and the wave function of the Universe (20) and to establish a bridge between the observational and quantum cosmologies.

3 The Dirac observables in classical and quantum theories

The Einstein – Hilbert action (1) (and, of course, its Hamiltonian version (7)) describes the first class constrained system according to the Dirac classification [8].

To construct the Dirac observables [8] of the first class constrained systems, we fulfil gauge-less Hamiltonian reduction [9, 10] by using the canonical transformation to new variables, so that the constraints become new momenta.

Thus, instead of the extended phase space N_c, a, p_a, A_l, P_l and the initial action invariant under reparametrizations of the coordinate time ($t \longmapsto t' = t'(t)$), we hope to get the reduced phase space which contains only the fields of matter described by the reduced Hamiltonian. These quantities (the fields of matter and the reduced Hamiltonian) are invariant under the time reparametrizations and, therefore, are the Dirac observables [8] by definition.

By means of the canonical transformation

$$(p_a, a) \rightarrow (\Pi, \eta) ; \qquad \{p_a, a\}|_{(\Pi, \eta)} = 1 \qquad (21)$$

we convert the gravitational part of the Einstein – Friedmann constraint (13) to the new momentum

$$\left(\frac{p_a^2}{2\beta} + \frac{ka^2}{2r_o^2}\beta \right) - aM_D = \Pi . \qquad (22)$$

Equations (21) and (22) have two solutions (for $k = +1$)

$$a(\Pi, \eta) = \pm \bar{a}(\Pi, \eta) ; \quad \bar{a}(\Pi, \eta) = \left[\sqrt{2\Pi \frac{r_o^2}{\beta}} S(\eta) + M_D \frac{r_o^2}{\beta}(1 - C(\eta)) \right] , \qquad (23)$$

$$p_a(\Pi, \eta) = \pm \bar{p}_a(\Pi, \eta) ; \quad \bar{p}_a(\Pi, \eta) = \left[\sqrt{2\Pi\beta} C(\eta) + M_D r_o S(\eta)) \right] , \qquad (24)$$

where

$$S(\eta) = \sin \frac{\eta}{r_o} ; \ C(\eta) = \cos \frac{\eta}{r_o} . \qquad (25)$$

In terms of the new canonical variables (Π, η) the action (7) for two solutions of eqs.(21), (22) has the forms

$$W^{(\pm)} = \int dt \left\{ \sum_l P_l \frac{d}{dt} A_l - N_c(\mathcal{H}_R - \Pi) \pm (\Pi \frac{d}{dt}\eta + \frac{M_D}{2}\frac{d}{dt}T(\Pi, \eta)) \right\} , \qquad (26)$$

where the function $T(\Pi, \eta)$

$$T(\Pi, \eta) = \int_o^\eta dx\bar{a}(\Pi, x) . \qquad (27)$$

coincides with the Friedmann time (15) in the parametric form (23),(27), where E_R is changed by Π. So, we get the following equations of motion for the "cosmic sector" (η, Π, N_c)

$$\frac{\delta W^{(\pm)}}{\delta \eta} = 0 \quad \Rightarrow \quad \dot{\Pi} = 0 \qquad (28)$$

$$\frac{\delta W^{(\pm)}}{\delta N_c} = 0 \quad \Rightarrow \quad \Pi = \mathcal{H}_R \qquad (29)$$

$$\frac{\delta W^{(\pm)}}{\delta \Pi} = 0 \quad \Rightarrow \quad N_c dt = d\eta \qquad (30)$$

and for "matter sector" (A_l, P_l).

$$\frac{\delta W^{(\pm)}}{\delta B_l} = 0 \quad \Rightarrow \quad \frac{d}{N_c dt} B_l = \{\mathcal{H}_R, B_l\} ; \quad (B_l = A_l, P_l) . \qquad (31)$$

Equations (29), (30) and (31) mean that the new cosmological variable η converts into the invariant time parameter of evolution of matter fields, and its canonical momentum Π converts into the reduced Hamiltonian. Equation (30) determines the conformal time η.

The evolution of matter fields can be described by the reduced action (26) (on the solution of equations (29), (30) of the cosmic sector)

$$W^{red(\pm)}(\eta) = \int_o^{\eta(t)} d\eta \left\{ \sum_l P_l \frac{dA_l}{d\eta} \pm \mathcal{H}_R \right\} \pm \frac{M_D}{2}T(\mathcal{H}_R, \eta) . \qquad (32)$$

The last term follows from the surface term, and it does not influence the equations of motion (31).

The Dirac observables, here, are the "matter sector" (31) and the conformal time (30). The partial variation of the action (32) (or (26)) with respect to this time (30) represents the Tolman version [13] of the total energy of the Universe filled in by radiation and dust in the conformal frame of reference

$$E_D^{(\pm)} = -\frac{\partial W^{(\pm)}}{\partial \eta(t)} = \pm (E_R + \frac{M_D}{2}a(E_R, \eta)) . \qquad (33)$$

In this frame interval is determined as

$$(ds)_D^2 = d\eta^2 - \gamma_{ij}dx^i dx^j , \qquad (34)$$

with a stationary space distance, in contrast with the comoving frame (2) with the Friedmann time (5) and observable interval

$$(ds)_F^2 = dT^2 - a^2(T)\gamma_{ij}dx^i dx^j . \qquad (35)$$

The variation of the reduced action (32) with respect to the Friedmann time (15) leads to the Tolman version [13] of the Friedmann energy

$$E_F^{(\pm)} = -\frac{\partial W^{(\pm)}}{\partial T(\eta)} = \pm \left(\frac{E_R}{a(E_R, \eta)} + \frac{M_D}{2} \right) . \qquad (36)$$

The first term of this energy describes the conventional Friedmann evolution of the red shift of "photons" (16) in the process of expansion of the Universe in the comoving frame (35).

Thus, the considered version of the Hamiltonian reduction also describes the conventional observables of the classical FRW cosmology provided the choice of the comoving frame (35) (this means the choice of a corresponding observer). The Friedmann observer, in the comoving frame, sees stationary mass, while the wavelengh of a photon (therefore, the energy of radiation) are changing according to the red shift law (16).

The Dirac observer, in the conformal frame (34), sees constant wavelength of a photon, while all masses in the Universe are changing so that the spectrum of an atom on a cosmic object at moment of emission $\eta_o - d_D/c$ differs from the Earth spectrum at moment of observation η_o.

As it has first been established by Hoyle and Narlikar [14] the red shift also exists in terms of the conformal time

$$z_o = \frac{ma(\eta_o - d_D/c)}{ma(\eta_o)} - 1 = \frac{a(\eta_o - d_D/c)}{a(\eta_o)} - 1 \qquad (37)$$

We see that in any frame of reference the observables are "energy" and "time", but not the cosmological scale factor a and its momentum p_a. This fact explains the difficulty in comparing the Friedmann observables "energy - time" with the result of the WDW quantization in terms of a, p_a.

4 Quantization

Let us quantize the theory (26). First at all note that the commutation relation $i[\hat{\Pi}, \eta] = \hbar$ and the WDW equation in terms of the new variables

$$(\hat{\Pi} - \mathcal{H}_R)\Psi_{WDW} = 0 \qquad (38)$$

do not reflect all information of the classical theory, in particular, about two different solutions (26) with signs ($\pm$) and the dust evolution of the "observable red shift" (37). The latter is hidden in the surface term which contributes to the total energy (33).

These peculiarities of the classical theory can be described, if we use the action (26) to determine the momentum Π.

$$\Pi^{(\pm)} = \frac{\partial L^{(\pm)}}{\partial \dot{\eta}} = \mp(\Pi + \frac{M_D}{2}\bar{a}(\Pi, \eta)). \qquad (39)$$

We should use the class of the wave functions where the constraint (38) is fulfilled.

Finally, using the operator

$$\hat{\Pi}^{(\pm)} = \pm\frac{\hbar}{i}\frac{d}{d\eta}$$

we get for the wave function the following spectral decomposition

$$\Psi_D(\eta|A_l) = \sum_{E_R}\left[\alpha^+_{E_R}\Phi^{(+)}_{E_R}(\eta) < E_R|A_l > +\alpha^-_{E_R}\Phi^{(-)}_{E_R}(\eta) < E_R|A_l >^*\right] \qquad (40)$$

where $< E_R|A_l >$ are the production of the Hermite polynomials,

$$\Phi^{(\pm)}_{E_R}(\eta) = exp\{\pm i(E_R\eta + \frac{M_D}{2}\bar{T}(E_R,\eta)\}, \qquad (41)$$

and $\alpha^{(\pm)}$ are the operators of creation of the Universes with the total energy (33)

$$E_R = \sum_l \omega_l(n_l + \frac{1}{2}), \qquad (42)$$

and quantum numbers of occupation n_l of a "photon" with the energy ω_l. We can express the observable red shift (37) in terms of the wave function with quantum numbers (42)

$$z_o = \frac{\mathcal{M}_{E_R}(\eta_o)}{\mathcal{M}_{E_R}(\eta_o - \frac{d_D}{c})} - 1, \qquad (43)$$

where

$$\mathcal{M}_{E_R} = \left[\Phi^{(+)*}_{E_R}(\eta)\frac{\partial}{i\partial\eta}\Phi^{(+)}_{E_R}(\eta) - E_R\right]. \qquad (44)$$

The variation of the wave function (41) with respect to the Friedmann time (15) leads to the energy of the red shift of a "photon" in the comoving frame of the Friedmann observer.

$$\frac{d}{idT(\eta)}\Phi^{(\pm)}_{E_R}(\eta) = \pm\left(\frac{E_R}{a(T)} + \frac{M_D}{2}\right)\Phi^{(\pm)}_{E_R}(\eta). \qquad (45)$$

In contrast with the WDW wave function (20), eq. (45) establishes the direct relation of the Dirac wave function to the observables of the classical theory.

The wave function (40) is normalizable for variables of the Dirac physical sector (A_l, P_l) and has clear physical interpretation. The wave function of the Universe is the conventional wave function of massless excitations at the conformal time multiplied by the wave function of a particle at the Friedmann time with a half mass of the Universe.

The corresponding functional integral representation of the Green function does not contain functional integration over the variable η (as it was excluded from the Dirac sector of physical variables) [15]. This conversion of the variable into the time parameter excludes the infinite gauge factor from the functional integral discussed by Hartle and Hawking [16].

5 Interpretation and conclusion

The aim of the present paper is to investigate relations between the Friedmann cosmological observables and the Dirac physical ones in the Hamiltonian approach to quantization of the Universe using a simple but important example of the homogeneous Universe filled in by dust and radiation.

An essential difference of the research presented here from the analogous papers on the Hamiltonian dynamics of cosmological models is complete separation of the sector of physical invariant variables from the pure gauge sector by application of the gaugeless reduction [9, 10]. The main point is that in the process of reduction one of variables converts into the observable invariant time. We have shown that this conversion of the variable to the time parameter leads to the normalizability of the wave function of the Universe and plays the role of gauge-fixing for removing an infinite factor from the corresponding functional integral. The considered reduction allows us to give the definite mathematical and physical treatment of the wave function and clarifies its relation to the observational cosmology.

The choice of the conformal frame of reference was a crucial to construct the Dirac "observables" in the generalized Hamiltonian approach. These "observable" are connected with the Friedmann observables by conformal transformations with the cosmic scale factor. However, these transformations have singularity at the beginning of the Dirac time. From this point of view, the Dirac "observables" in the frame connected with the radiation seem to be more fundamental than the Friedmann "observables" in the comoving frame connected with massive dust.

At the beginning of the Universe (at time $\eta \to 0$) the energy of particles at rest (forming the dust) in a closed space becomes larger than their masses, and all dust converts into massless excitations with wavelengths of an order of a conformal size of the Universe r_0. For these excitations the region of validity of quantum theory coincides with the size of the Universe. Thus, at the beginning of the Universe, the dust disappears and the Friedmann observables connected with massive dust lose physical meaning. While the Dirac "observer" sees the closed space filled by the homogeneous massless excitations bounded all regions in the Universe. For the Dirac "observer" the difficulties of singularity and horison do not exist.

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