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SIX-DIMENSIONAL SPACE-TIME TRANSFORMATIONS

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1. INTRODUCTION

The generalization of Einstein theory of relativity treating time on a level with space as three-dimensional vector [1],

$$(\hat{\mathbf{x}})_\mu = (-\mathbf{x}, c\hat{t})^T_\mu \quad , \quad (\hat{\mathbf{x}})^\mu = (\mathbf{x}, c\hat{t})^{T\mu}, \tag{1}$$

has to be independent on a choice of a reference frame. That is assured by the explicit relativistic invariance of all its relations, particularly, rules describing transformations from the given frame to others. Such rules generalizing the known Lorentz formulae have been considered in papers [1] — [7] and were applied to a solution of some particular problems [8].

We survey the properties of these transformations, especially, their link with the demand of the time irreversibility governing the behavior of all real physical systems. The generalized Lorentz transformations will be considered for both the sub- and the superluminal velocities because the latters are important in the microscopic space-time regions, where many modern theories predict the appearance of faster-than-light objects.

2. MULTITIME GALILEI TRANSFORMATIONS

Such transformations link two inertial reference frames if we assume that separately both the space and the time intervals $(\Delta x_k \Delta x^k)^{1/2}$ and $(\Delta t_k \Delta t^k)^{1/2}$ are universal quantities independend neither on the frame velocity nor on the point $\hat{\mathbf{x}}$ where they are measured. If we introduce a

[1] Here and in what follows vectors in x- and t-subspaces are denoted, respectively, by bold face and by a "hat". Six-dimensional vectors will be denoted at once by bold face and a "hat". In manuscripts it is convenient to use the notations $\bar{x}$, $\hat{x}$ and $\hat{\bar{x}}$. All matrix will be denoted by capital letters. We shall also suppose that the Latin and Greek indices take values $k = 1, ..., 3, \mu = 1, ..., 6$).

six-dimensional velocity of a relative frames motion

$$\hat{\mathbf{v}} = \frac{\partial \hat{\mathbf{x}}}{\partial \hat{\tau}} = (\hat{\tau}\hat{\nabla})\hat{\mathbf{x}} = \sum_{i=1}^{3} \tau_i \frac{\partial \hat{\mathbf{x}}}{\partial t} = \frac{d\hat{\mathbf{x}}}{dt}, \tag{2}$$

$$(\hat{\mathbf{v}})_\mu \equiv (d\hat{\mathbf{x}}/dt)_\mu = (-\mathbf{v}, c\hat{\tau})^T_\mu \quad , \quad (\hat{\mathbf{v}})^\mu = (\mathbf{v}, c\hat{\tau})^{T\mu}, \tag{3}$$

then the relation

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}\hat{\tau}\hat{t}\,, \hat{t}' = \hat{t}, \tag{4}$$

where $\hat{\tau}$ is the time trajectory of the frame at rest satisfies, obviously, the condition of the space and time intervals constancy [2].

In a more general case when rotations in x- and t-subspaces are taken into account

$$\mathbf{x}' = \mathbf{R}\mathbf{x} - (\mathbf{R})\mathbf{v}\hat{\tau}\hat{t}\,, \hat{t}' = \hat{R}\hat{t} \tag{5}$$

where $\mathbf{R}$ and $\hat{R}$ are the respective rotation operators.

In one-time approximation these transformations coincide with the Galileiean formula.

3. GENERALIZATION OF LORENTZ TRANSFORMATION

The invariance of the vector length, $c^2\hat{t}^2 - \mathbf{x}^2$, under the linear transformation

$$\hat{\mathbf{x}}' = \Lambda\hat{\mathbf{x}}, \tag{6}$$

i. e. $\hat{\mathbf{x}}'^T\mathbf{G}\hat{\mathbf{x}}' = \eta\hat{\mathbf{x}}^T\mathbf{G}\hat{\mathbf{x}}$ or $\hat{\mathbf{x}}^T\Lambda^T\Lambda\hat{\mathbf{x}} = \eta\hat{\mathbf{x}}^T\mathbf{G}\hat{\mathbf{x}}$ where $\eta = +1$ for a subluminal and $\eta = -1$ for a superluminal transformation velocity, demands the relations

$$\Lambda^T\mathbf{G}\Lambda = \eta\mathbf{G} \tag{7}$$

$$\Lambda^{-1} = \eta\mathbf{G}\Lambda^T\mathbf{G} \tag{8}$$

[2]In eq. (3) the relation $t^2 = \sum_{i=1}^3 t_i^2$ and $\partial t/\partial t_i = t_i/t = \tau_i$ where $\hat{t}$ is the time vector of the moving frame radial trajectory. The product $\hat{\tau}\hat{t}$ is the projection of the time vector of the considered event on the time axis of the motionless frame.

to be fulfilled where $\mathbf{G} = \begin{pmatrix} -I & O \\ O & I \end{pmatrix}$ is the metric matrix, $\mathbf{I} = \begin{pmatrix} 1 & O \\ O & 1 \end{pmatrix}$ is the three-dimensional unity matrix.

Let us introduce now the three-dimensional matrices $\mathbf{S}$ and $\mathbf{T}$ linking, respectively, only space and time co-ordinates and the matrices of composite transformations $\mathbf{P}$ and $\mathbf{Q}$:

$$\Lambda = \begin{pmatrix} \mathbf{S} & \mathbf{P} \\ \mathbf{Q} & \mathbf{T} \end{pmatrix}. \tag{9}$$

From eq.(8) we have then

$$\Lambda^{-1} = \eta \begin{pmatrix} \mathbf{S}^T & -\mathbf{Q}^T \\ -\mathbf{P}^T & \mathbf{T}^T \end{pmatrix}. \tag{10}$$

As seen in the motionless frame,

$$\hat{\mathbf{o}}_1 = (0, c\hat{t}_1)^T = (0, c\hat{\tau}_1)^T t_1 \tag{11}$$

$$\hat{\mathbf{o}}_2 = (\mathbf{x}_2, c\hat{t}_2)^T = (\mathbf{v}_2, c\hat{\tau}_2)^T t_2 \tag{12}$$

are co-ordinates of its and the moving frame origins. Here $\mathbf{x}_2 = \mathbf{v}_2 t_2$ and

$$d\mathbf{x}_2/d\hat{\tau}_2 = (\hat{\tau}_2\hat{\nabla})\mathbf{x}_2 = \partial\mathbf{x}_2/\partial t_2 \tag{13}$$

(see eq. (2)) is the projection of the velocity of the moving frame on its trajectory $\hat{t}_2(t_2)$.

Respectively,

$$\mathbf{o}'_1 = (\mathbf{x}', c\hat{t}'_1)^T = (\mathbf{v}'_1, c\hat{\tau}'_1)^T t'_1 \tag{14}$$

$$\mathbf{o}'_2 = (0.c\hat{t}'_2)^T = (0, c\hat{\tau}'_2)^T t'_2 \tag{15}$$

are co-ordinates of the frame origins, as seen in the moving reference frame [3]. Since the scalar products of the space-time vectors $\hat{\mathbf{x}}_1\hat{\mathbf{x}}_2$ and

[3]All quantities in this frame will be marked by a prime. We shall use also the designations $v = v_2$, $v' = v'_1$ and $\gamma = (1 - v^2/c^2)^{-1/2}$, $\gamma' = (1 - v'^2/c^2)^{-1/2}$ in the subluminal case, $\gamma = (v^2/c^2 - 1)^{-1/2}$, $\gamma' = (v'^2/c^2 - 1)^{-1/2}$ for superluminal velocities.

the six-dimensional frame origins velocities $\hat{\mathbf{u}}_i = \gamma_i \hat{\mathbf{v}}_i$ are invariants, i. e. $\hat{\mathbf{x}}_1\hat{\mathbf{x}}_2 = \eta\hat{\mathbf{x}}'_1\hat{\mathbf{x}}'_2$ and $\hat{\mathbf{u}}_1\hat{\mathbf{u}}_2 = \eta\hat{\mathbf{u}}'_1\hat{\mathbf{u}}'_2$, and the space components $\mathbf{x}_1 = \mathbf{x}'_2 = \mathbf{u}_1 = \mathbf{u}'_2 = 0$, we have the relations

$$\hat{\tau}_1\hat{\tau}_2 t_1 t_2 = \eta\hat{\tau}'_1\hat{\tau}'_2 t'_1 t'_2 \tag{16}$$

$$\gamma\hat{\tau}_1\hat{\tau}_2 = \eta\gamma'\hat{\tau}'_1\hat{\tau}'_2. \tag{17}$$

Comparing two latter relations, we see that the expression

$$\gamma' t_1/t'_1 = \gamma t'_2/t_2 \tag{18}$$

doesn't depend on any time turn and, therefore, its left and right side must have the some values as in the customary Lorentz transformations:

$$t'_1/t_1 = \gamma' \quad , \quad t_2/t'_2 = \gamma. \tag{19}$$

Eq. (17) may be rewritten as

$$\gamma\cos\theta = \eta\gamma'\cos\theta' \tag{20}$$

where θ and θ' are angles between the time trajectories of the reference frames. In comparison with the subluminal four-dimensional case the γ's in (20) are equal only in the particular case when $\theta = \theta'$, i. e. the time direction doesn't change. Only in this particular case $|v'_1| = |v_2|$. For example, if $\theta = 0$, the velocity

$$v'^2 = c^2 + (v^2 - c^2)\cos^2\theta' = \begin{cases} v^2 & \text{for} \quad \theta' = 0 \\ c^2 & \text{for} \quad \theta' = \pi/2 \end{cases} \tag{21}$$

One should stress that in contrast to the space transformations when reference frames can move collinearly, the transformations in t-subspace are associated always with an axis turn and the condition $v'^2 \geq 0$ demands

$$\cos^2\theta' \leq \cos^2\theta\,(v'^2 - c^2)/(v^2 - c^2) \tag{22}$$

Now, in order to calculate the transformation matrix (9), let us consider the relations $\hat{\mathbf{o}}'_i = \boldsymbol{\Lambda}\hat{\mathbf{o}}_i$ and $\hat{\mathbf{o}}_i = \boldsymbol{\Lambda}^{-1}\hat{\mathbf{o}}'_i$. Hence we get

$$c\mathbf{P}\hat{\tau}_1 - \mathbf{v}'\gamma' = \eta\gamma\mathbf{v} + c\mathbf{Q}^T\hat{\tau}'_2 = 0 \tag{23}$$

$$\gamma'\hat{\tau}'_1 - \mathbf{T}\hat{\tau}_1 = \eta\gamma\hat{\tau}_2 - \mathbf{T}^T\hat{\tau}'_2 = 0 \tag{24}$$

$$c\mathbf{P}\hat{\tau}_2 + \mathbf{S}\mathbf{v} = c\mathbf{Q}^T\hat{\tau}'_1 - \mathbf{S}^T\mathbf{v}' = 0 \tag{25}$$

$$\gamma(\mathbf{Q}\mathbf{v} + c\mathbf{T}\hat{\tau}_2) - c\hat{\tau}'_2 = \gamma'(\mathbf{P}^T\mathbf{v}' - c\mathbf{T}^T\hat{\tau}'_1) - c\eta\hat{\tau}_1 = 0 \tag{26}$$

Besides, there are other six relations for the matrices **S**, **T**, **P**, **Q** arising from eqs. (7), (8) and the obvious equality $\boldsymbol{\Lambda}\boldsymbol{\Lambda}^{-1} = \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}$:

$$\mathbf{S}\mathbf{S}^T - \mathbf{P}\mathbf{P}^T = \mathbf{S}^T\mathbf{S} - \mathbf{Q}^T\mathbf{Q} = \eta\mathbf{I} \tag{27}$$

$$\mathbf{T}\mathbf{T}^T - \mathbf{Q}\mathbf{Q}^T = \mathbf{T}^T\mathbf{T} - \mathbf{P}^T\mathbf{P} = \eta\mathbf{I} \tag{28}$$

$$\mathbf{S}\mathbf{Q}^T - \mathbf{P}\mathbf{T}^T = \mathbf{S}^T\mathbf{P} - \mathbf{Q}^T\mathbf{T} = 0. \tag{29}$$

The further definition of the transformation matrix $\boldsymbol{\Lambda}$ depends on a choice of the relative motion of the considered reference frames. Let us consider the case when their space axes are collinear, the velocity space components $\mathbf{v} = (v, 0, 0)^T$, $\mathbf{v}' = (v', 0, 0)^T$, however, the time axes intersect in the motionless and in the moving frames respectively at angles θ and θ': $\hat{\tau}_1 = \hat{\tau}'_2 = (1, 0, 0)^T$, $\hat{\tau}_2 = (\cos\theta, \sin\theta, 0)^T$, $\hat{\tau}'_1 = (cos\theta', \sin\theta', 0)^T$. More general cases of the relative motion can be obtained by means of customary three-dimensional rotations in x- and t-subsoaces. We assume also that at an intersection point the frame origins coincide.

As the motion occurs only along the x-axis, the co-ordinates y and z remain unavailable, the matrix elements

$$\mathbf{S}_{i,k} = (s - 1)\delta_{1i}\delta_{2k} + \delta_{i,k}. \tag{30}$$

Inserting this expresses and the chosen values of the six-velocities into the relations (23) — (26) and (27) — (29), we get the equations for the calculation of all $\boldsymbol{\Lambda}$-matrix elements. E.Cole and S. Buchanan were the first who have derived and solved such an equation system [3, 7].

One can convince oneself by means of substitution that the desired transformation matrix has the form [4].

$$
\Lambda = \begin{pmatrix} s & 0 & 0 & \gamma' v'/c & p & 0 \\ 0 & \eta^{1/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^{1/2} & 0 & 0 & 0 \\ -\eta\gamma v/c & 0 & 0 & \eta\gamma\cos\theta & \eta\gamma\sin\theta & 0 \\ q & 0 & 0 & \gamma'\sin\theta' & t & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta^{1/2} \end{pmatrix} \qquad (31)
$$

The element s is the solution of the quadratic equation

$$
s^2(1-a^2) - 2sa\gamma\gamma' vv'/c^2 - \gamma^2\gamma'^2 + \gamma^2 + \eta(a^2-\gamma^2) = 0, \qquad (32)
$$

where $a = \gamma\cos\theta$, and the other parameters are defined by the relations

$$
t^2 = s^2 + a^2 - \gamma^2 - \gamma'^2 + 1, \qquad (33)
$$

$$
p^2 = s^2 - \gamma'^2 \quad , \quad q^2 = s^2 - \gamma^2, \qquad (34)
$$

The customary four-dimensional Lorentz formulae are recovered, as it can be easy checked, if $\theta' = \theta = 0$ and $v' = -v$. Particularly, if the reference frame speed is superluminal, the transversal co-ordinates y' and z' automatically obtain imaginary values iy, iz as it was supposed by E. Recami and his co-worker (see surveys [9, 10]).

Contrary to the one-time theory where the Lorentz transformation is completely determined by the demand of the invariance of space-time vector lengths in the multitime case, we have to fix additionally the values of v' and θ'. In other words, if in the one-time theory the knowledge of the reference frame speed $\mathbf{v}$ determines uniquely the co-ordinates $(\mathbf{x}', ct')$ of every event in this frame, in the multitime world one has to inquire the moving observer how he sees our reference frame. Such a situation

[4]The matrix elements P_{11} and Q_{11} are obtained from eqs (23). The eqs (33) and (34) follow from (27) and (28). The quadratic equation for s has been derived by means of a combination of eqs (25) and (34).

takes place even in the case of a motion of the both frames along the same time axis ($\theta = 0$) when, it would seem, no information is needed about any other time trajectories. From this point of view it is quite reasonable to impose the conditions [5]

$$
v' = -v \quad , \quad \theta' = \theta \qquad (35)
$$

The relations (32) — (34) give in this case:

$$
s = \gamma\omega(\gamma + \cos\theta) \quad , \quad q = -p = vc^{-1}\gamma^2\omega\sin\theta \quad , \quad t = 1 - \gamma^2\omega\sin^2\theta \qquad (36)
$$

where $\omega = 1/(1+\gamma\cos\theta)$.

The one-time Lorentz transformation does not contain linear terms of the order v/c. The first relativistic correction is of the order$(v/c)^2$. Another situation takes place in the multitime case. If $v/c \ll 1$ and the parameters $s = 1, q = p = 0$, $t = \cos\theta$, the multitime transformation conserves linear terms:

$$
x' = x - vt_1 - vt_2 \tan\theta/2 \qquad (37)
$$

$$
t'_1 = -vc^{-2}x + t_1\cos\theta + t_2\sin\theta \qquad (38)
$$

$$
t'_2 = vc^{-1}x\tan\theta/2 - t_1\sin\theta + t_2\cos\theta \qquad (39)
$$

$$
y' = y \,,\; z' = z \,,\; t'_3 = t_3 \qquad (40)
$$

Discarding the linear terms we get the expressions coinciding with the particular case of the formulae (5) in which $v_2 = v_3 = 0$, the space rotation operator $\mathbf{R} = 1$ and the time axis of the motionless reference frame $\hat{\tau}_1$ is directed along .

Experiments, as it is known, don't discover any deviations from the standard Lorentz transformation, particularly, no linear terms are observed. This fact says that, if even the outer world is multitemporal, all

[5]Actually, it is enough either one from these two equalities because the retained one is determined by the condition $Det = 1$ or bu an insertion of the submatrices $\mathbf{S}$, $\mathbf{P}$, $\mathbf{Q}$ from (31) into eqs (27).

surrounding us bodies move along the same time trajectory.

4. KINEMATIC EFFECTS

Let us consider the generalized Lorentz transformation for two points $\mathbf{x}_1$ and $\mathbf{x}_2$ at the same time $\hat{t}$. Henceforth we have

$$\Delta x \equiv x_2 - x_1 = s\Delta x, \tag{41}$$

i. e. the length contraction is more strong than in one-time theory:

$$\Delta x/\Delta x' = \begin{cases} \gamma & \text{for} \quad \theta' \simeq 0 \\ \gamma^2\omega & \text{for} \quad \theta' \simeq \pi/2 \end{cases} \tag{42}$$

A link of time intervals defined at fixed space points x and x' in two inertial frames is more complicated and depends on the angle θ. However, if the time axes of the motionless frame are chosen so that an evolution of a phenomenon occurs along t_1-axis of the frame (i. e. time intervals specifying the development $\Delta\hat{t} = (\Delta t_1, 0, 0)^T$), then in a moving reference frame the evolution appears as multitime ones, but the length of the time interval is described by the some relation as in the one-time theory:

$$\Delta t' = (\Delta t_1^2 + \Delta t_2^2)^{1/2} = \gamma \Delta t. \tag{43}$$

The velocity summation law has now the form

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} = \frac{\bar{\Lambda}^{-1} d\hat{\mathbf{x}}'}{\left|c^{-1}\hat{\Lambda}^{-1} d\hat{\mathbf{x}}'\right|} = \frac{c\bar{\Lambda}^{-1} d\hat{\mathbf{x}}'/dt'}{\left|\hat{\Lambda}^{-1} d\hat{\mathbf{x}}'/dt'\right|} = \frac{c\bar{\Lambda}^{-1}\hat{\mathbf{u}}'}{\left|\hat{\Lambda}^{-1}\hat{\mathbf{u}}'\right|} \tag{44}$$

where $dt = (dt_k dt^k)^{1/2}$, $dt' = (dt'_k dt'^k)^{1/2}$. The (6×3)-matrixes $\bar{\Lambda}$ and $\hat{\Lambda}$ are defined by the relation $\Lambda^{-1} = \binom{\bar{\Lambda}}{\hat{\Lambda}}$. Particularly, in the two limiting cases of a collineary space motion

$$\mathbf{u} = \frac{u' + v}{1 + u'v/c^2} \begin{cases} 1, & \text{if} \quad \theta' \simeq 0 \\ \left[1 + \left(\frac{1 + u'v\omega/c^2}{1 + u'v/c^2}\right)^2\right]^{-1/2}, & \text{if} \quad \theta \simeq \pi/2 \end{cases} \tag{45}$$

5. TIME DIRECTION

One of the main principles of contemporary physics is the demand that any theory describing a system with a finite number of interacting objects has to possess a property of the time reversibility, i. e. all its relations and conclusions must be invariant under the change of time direction: $t \to -t$. The equivalence of two time directions allows to introduce into the theory both the bodies with negative and bodies with positive total energies. In temporary physics the latter are expelled by a simple cut out.

At the same time the reversibility of time is absent in real physical systems confining an unlimited number of objects. In such cases the preferred time direction (the direction of the system evolution) exists always.

The multitime theory must obey the similar conditions. Its mathematical structure has to be symmetrical with respect to time vectors $\hat{t}$ and $-\hat{t}$, i. e. has to obey the time reversibility along all three axes t_i. The positiveness of masses and energies is guaranteed by the complementary demand (principle of irreversibility) that all real physical motions and processes develop always in the direction of increasing the times t_i. The latter presuppose the existence of a fixed, intrinsic ("relict") time reference frame defined spontaneously by an evolution process which took place in all or, maybe, only in a surrounding us part of the inflating Universe. In this frame all time trajectories are placed in the firth quadrant (oktant, if we consider three time axes). In all other frames one can find intersecting trajectories arranged above and under the t_1-axes (the trajectories τ' and τ'' in Fig., if the t_1 -axes would be parallel to τ) [6].

The existence of the intrinsic time reference frame imposes restrictions

[6] Considering phenomena on his own time trajectory the observer can confine oneself only by proper scalar time t, because all $t_i = f_i(t)$, however, the evolution of outside events (e. g. on the trajectory $\hat{\tau}'$ or $\hat{\tau}''$) has to be described in terms of the times $|\hat{t}_i|$

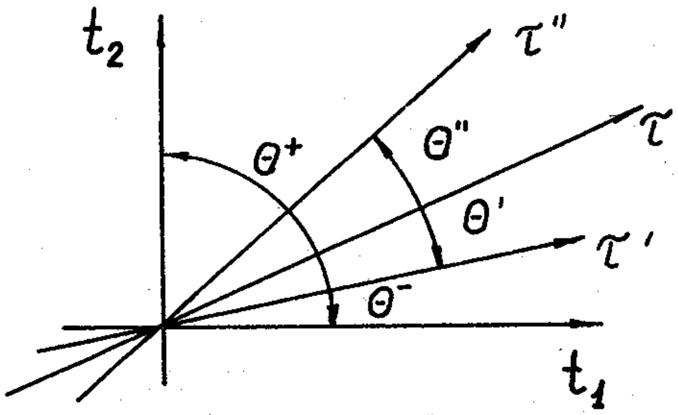

Figure 1: Time trajectories accessible from the trajectory $\hat{\tau}$ have to place inside the firth quadrant.

on the Galileiean and Lorentz transformations. The rotation angles cannot exceed some critical values because at larger angles the evolution of phenomena would be backward in time. For example, an observer moving along a trajectory $\hat{\tau}$ (see the Fig) can go to trajectories $\hat{\tau}'$ and $\hat{\tau}''$ at angles $\theta' \leq \theta^+$ and $\theta'' \leq \theta^-$.

The demand of the conservation of a sign of time by the Lorentz transformations in the customary one-time theory is a trace of these restrictions. While in the multitime theory the energy vectors $\hat{E}_i$ are parallel to the trajectory time vector $\hat{t}$ and, therefore, the time irreversibility results automatically in the positive definiteness of the energies, in the one-time approximation such a link of E_i and t is absent and the demand $E_i \geq 0$ becomes apparent as an independent condition.

If an observer deals only with his own time trajectory, he cannot discover the hidden time co-ordinates. He needs a body or bodies with diverse time trajectories for that. Then, comparing the time order of phenomena in various reference frames he is able to reveal the preferred relict frame. One has to take into account, however, that this frame

is distinguished only with respect to the time direction, the space coordinates obey the principle of relativity.

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