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JUSTIFICATION
OF THE ZETA-FUNCTION RENORMALIZATION
IN RIGID STRING MODEL

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1   Introduction

A consistent method to treat the divergences in quantum field theory is known to be the following [1]. Divergent expressions must at first be regularized, for example, by the Pauli-Willars method, then subtractions justified by transition to physical (observable) parameters of the theory should be done. After that the regularization is to be removed.

Together with this approach there are widely used methods that do not apply explicit regularization and renormalization but which nevertheless give finite answer. First of all, it is the zeta function technique. The main idea of this approach is the following [2-4]. One assumes that the divergent sum $\sum_n \omega_n$ of eigenvalues of the operator[1] determining the dynamics in the model under consideration is equal to the value of the zeta function for this operator, $\zeta(s)$, when $s \to -1$. At first the function $\zeta(s)$ is defined by the formula $\zeta(s) = \sum_n \omega_n^{-s}$ for $\operatorname{Re} s > 1$, and then it is analytically continued to $\operatorname{Re} s \leq 1$ possibly save for isolated points. In the case of the Dirichlet boundary conditions for the two-dimensional Laplace and Helmholtz operators this function appears to be the Riemann ζ-function or the Epstein-Hurwitz ζ-function, respectively. These functions are widely used in calculations of the Casimir energy in field [4,5] and string models [6].

Undoubtedly such a formal method to treat divergences needs justification in each particular case [7,8]. The more so, there are examples when analytic continuation leads to ambiguities [9]. To justify this approach, it is necessary to show that it gives the same

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[1]Most commonly this operator is the Laplace operator $(-\triangle)$. The zeta function regularization is usually applied to the Euclidean version of the models where one has to do with elliptic operators [3].

results as the standard renormalization procedure with regularization and subtraction. It is this problem that will be considered in the present paper in the framework of one-loop calculation of the interquark potential (or the Casimir energy) in the rigid string model. This model is chosen because here both the Riemann and Epstein-Hurwitz ζ-functions are employed.

The interquark potential generated by a rigid string was studied in a number of papers by making use of the perturbation theory and variational estimation of the functional integral (see, for example, paper [10] and references therein). These results are well-known. Therefore attention will be basically paid to development of the consistent procedure of renormalization and to justification, on this basis, the results obtained by ζ-function method.

The layout of the paper is as follows. In Section 2, the interquark potential generated by a rigid string is calculated in the one-loop approximation, the standard method of analytic continuation of the Riemann and Epstein-Hurwitz ζ-functions being used. In Section 3, the consistent regularization of the divergences and the string tension renormalization are carried out. Unlike the ζ-function method, the finite expression for the string potential is derived here uniquely. Moreover in our approach the renormalized string energy at zero temperature is obtained in terms of the spectral representation that can be directly generalized to a finite temperature. In the Conclusion (Section 4), the obtained results are discussed in short. Auxiliary material concerning the details of the calculations is given in Appendices A and B.

2   Interquark potential generated by rigid string in one-loop approximation

We consider the most simple example of the application of the Riemann and Epstein-Hurwitz ζ-functions. This is the calculation of the interquark potential generated by the Polyakov-Kleinert string [11,12] in the one-loop approximation [13]. In spite of its simplicity, this example demonstrates the main features of the approach.

For our purpose the quadratic approximation to the Polyakov-Kleinert string action is sufficient

$$S^\beta = M_0^2 \int\limits_0^\beta dt \int\limits_0^R dr \left[1 + \frac{1}{2}\mathbf{u}\left(1 - \frac{\alpha}{M_0^2}\Delta\right)(-\Delta)\mathbf{u}\right]. \qquad (2.1)$$

Here $\mathbf{u}(t,r) = (u^1(t,r), u^2(t,r), \ldots, u^{D-2}(t,r))$ are the transverse string coordinates in D-dimensional space-time, M_0^2 is the string tension, and α is a dimensionless parameter characterizing the string rigidity, $\alpha > 0$, R is the distance between quarks connected by string, i.e. the string length. The Euclidean action is considered, therefore the operator Δ in (2.1) is the two-dimensional Laplace operator $\Delta = \partial^2/\partial t^2 + \partial^2/\partial r^2$. The "time" variable t ranges in the interval $0 \leq t \leq \beta$, where $\beta = 1/T$ is the inverse temperature.

The action (2.1) should be completed by boundary conditions for string coordinates at the points $r = 0$ and $r = R$. Usually a string with fixed ends is considered

$$\mathbf{u}(t,0) = \mathbf{u}(t,R) = 0. \qquad (2.2)$$

This corresponds to the static interquark potential. The string potential $V(R)$ is defined in a standard way

$$\exp[-\beta V(R)] = \int [D\mathbf{u}]\exp\left\{-S^\beta[\mathbf{u}]\right\}, \quad \beta \to \infty. \qquad (2.3)$$

The functional integral in (2.3) is taken over string coordinates $\mathbf{u}(t,r)$ that satisfy periodic conditions in the time variable t

$$\mathbf{u}(t,r) = \mathbf{u}(t+\beta,r). \qquad (2.4)$$

Inserting (2.1) into (2.3) one obtains after the functional integration when $\beta \to \infty$

$$V(R) = M_0^2 R + \frac{D-2}{2\beta}\text{Tr}\ln(-\Delta) + \frac{D-2}{2\beta}\text{Tr}\ln\left(1 - \frac{\alpha}{M_0^2}\Delta\right). \quad (2.5)$$

For calculating the traces in (2.5) the eigenvalues of the operators $(-\triangle)$ and $[1 - (\alpha/M_0^2)\triangle]$ with the boundary conditions (2.2) and the periodicity conditions (2.4) are needed

$$(-\triangle)\phi_{nm} = \lambda_{nm}\phi_{nm},$$

$$\left(1 - \frac{\alpha}{M_0^2}\triangle\right)\psi_{kl} = \xi_{kl}\psi_{kl}, \qquad (2.6)$$

Using the Fourier expansion we find [13]

$$\lambda_{nm} = \Omega_n^2 + \omega_m^2,$$

$$\xi_{nk} = \Omega_n^2 + \tilde{\omega}_k^2, \qquad (2.7)$$

where $\Omega_n = 2\pi n/\beta$, $n = 0, \pm 1, \pm 2, \ldots$ are the Matsubara frequencies, $\omega_m = m\pi/R$, $m = 1, 2, \ldots$ are positive roots of the equation

$$\sin(\omega R) = 0, \qquad (2.8)$$

and $\tilde{\omega}_k = \sqrt{(k\pi/R)^2 + M_0^2/\alpha}$, $k = 1, 2, \ldots$ are those of the equation

$$\sin\left(R\sqrt{\tilde{\omega}^2 - M_0^2/\alpha}\right) = 0. \qquad (2.9)$$

Summation over the Matsubara frequencies Ω_n can be accomplished by making use of the known methods [13]. Upon taking the limit $\beta \to \infty$, the potential generated by the string assumes the form

$$V(R) = M_0^2 R + (D - 2)(E_C^{(1)} + E_C^{(2)}), \qquad (2.10)$$

where $E_C^{(1)}$ и $E_C^{(2)}$ are the Casimir energies corresponding to both the modes of the rigid string oscillations

$$E_C^{(1)} = \frac{\pi}{2R}\sum_{n=1}^{\infty} n, \qquad (2.11)$$

$$E_C^{(2)} = \frac{\pi}{2R}\sum_{n=1}^{\infty}\sqrt{n^2 + a^2}, \quad a^2 = \frac{M_0^2 R^2}{\alpha\pi^2}. \qquad (2.12)$$

Summation of the divergent series (2.11) and (2.12) by analytic continuation of the ζ-function is now commonly used. Nevertheless we remind the main steps of this approach in short.

We begin with the first sum (2.11). According to the scheme outlined in the Introduction, we first have to consider the function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \text{Re } s > 1, \qquad (2.13)$$

and then to continue it analytically to the region Re $s < 1$. In this case $\zeta(s)$ is the Riemann ζ-function. Analytic continuation of the function (2.13) to the rest of the complex plane s, with the exception of the point $s = 1$, is performed by the contour integral [14]

$$\zeta(s) = -\frac{\Gamma(1 - s)}{2\pi i}\int_C \frac{(-z)^{s-1}}{1 - e^{-z}}dz, \qquad (2.14)$$

where the contour C is shown in Fig. 1. This contour should avoid the points $z = \pm 2n\pi i$ $(n = 1, 2, 3, \ldots)$ because of the multiplier $\Gamma(1 - s)$ in (2.14). The Riemann ζ-function has a simple pole at $s = 1$ with the residue equal to 1

$$\zeta(s) = \frac{1}{s - 1} + \gamma + \ldots, \qquad (2.15)$$

where γ is the Euler constant

$$\gamma = \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \ln N\right). \qquad (2.16)$$

The ζ-function defined by integral (2.14) satisfies the reflection formula [14]

$$\zeta(1 - s) = 2\,(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s). \qquad (2.17)$$

According to the scheme outlined above we have to attribute the value $\zeta(-1)$ to the sum of the divergent series (2.11) . For $s = -1$

the integral representation (2.14) gives

$$\zeta(-1) = \frac{1}{2\pi i} \int\limits_C \frac{(-z)dz}{z^3(1 - e^{-z})}. \qquad (2.18)$$

Since the integrand is single-valued in the plane z, the integration contour in Fig. 1 can be closed. As a result, $\zeta(-1)$ is equal to the integrand residue at $z = 0$. To find the residue, we can use the definition of Bernoulli numbers [14]

$$\frac{t}{e^t - 1} = 1 - \frac{1}{2}t + B_1\frac{t^2}{2} - B_2\frac{t^4}{4!} + \ldots,$$

where $B_1 = 1/6$, $B_2 = 1/30, \ldots$. Thus we obtain

$$\zeta(-1) = -B_1\frac{1}{2} = -\frac{1}{12}. \qquad (2.19)$$

Finally we attribute the value

$$E_C^{(1)} = \frac{\pi}{2R}\sum_{n=1}^{\infty} n = \frac{\pi}{2R}\zeta(-1) = -\frac{\pi}{24R}. \qquad (2.20)$$

to the sum of the divergent series (2.11). In the theory of divergent series [15] this summation of the series (2.11) is referred to as the *Ramanujan summation*. Obviously, this method is not universal. For example, it cannot be applied directly to a divergent series $\sum_{n=1}^{\infty} n^{-1}$ because the zeta-function has a pole at the point $s = 1$ (see Eq. (2.15)). Admitting the convention about the rejection of the pole singularity, as it is usually done in the analytic regularization method[2], we get

$$\sum_{n=1}^{\infty} \frac{1}{n} = \gamma.$$

From (2.15), (2.16) it follows that the pole of the Riemann ζ-function at the point $s = 1$ is responsible for the logarithmic divergences.

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[2]It is implicitly assumed that the singularities are taken away by renormalization of parameters in the theory under consideration.

Summarizing, we arrive at the conclusion that the Riemann ζ-function method enables one to obtain the finite value of the Casimir energy (2.11) without explicit regularization, pole singularity rejection, and explicit renormalization. However in the case of divergent series (2.12) the analytic continuation technique requires additional assumptions.

To sum the series (2.12), we have to consider the Epstein-Hurwitz zeta-function $\zeta_{EH}(s, p)$ defined by the formula

$$\zeta_{EH}(s, a^2) = \sum_{n=1}^{\infty} (n^2 + a^2)^{-s}, \qquad (2.21)$$

where $s > 1/2$. Let us remind briefly how to accomplish an analytic continuation of this series to the region $s \leq 1/2$.

Using the integral representation for the Euler gamma function [14]

$$(n^2 + a^2)^{-s}\Gamma(s) = \int\limits_0^{\infty} t^{s-1}e^{-(n^2+a^2)t}dt, \qquad (2.22)$$

we can replace each term in series (2.21) by the integral

$$\zeta_{EH}(s, a^2) = \frac{1}{\Gamma(s)} \int\limits_0^{\infty} t^{s-1}\sum_{n=1}^{\infty} e^{-t(n^2+a^2)}dt. \qquad (2.23)$$

The Jacobi θ-function appearing in (2.23) $\theta(t) = \sum_{n=1}^{\infty} e^{-n^2 t}$ has the property [14]

$$\theta(t) = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\pi}{t}} + \sqrt{\frac{\pi}{t}}\theta(\pi^2/t). \qquad (2.24)$$

Substituting (2.24) into (2.23) we obtain

$$\zeta_{EH}(s, a^2) = -\frac{(a^2)^{-s}}{2} + \frac{\sqrt{\pi}\Gamma(s - 1/2)}{2\Gamma(s)}(a^2)^{-s+1/2} +$$

$$+ \frac{\sqrt{\pi}}{\Gamma(s)} \sum_{n=1}^{\infty} \int_{0}^{\infty} t^{s-3/2} \exp\left(-ta^2 - \frac{\pi^2 n^2}{t}\right) dt. \qquad (2.25)$$

The multiplier $\exp(-ta^2 - \pi^2 n^2/t)$ ensures the convergence of the integral in (2.25) for all s. Then this integral is expressed in terms of the modified Bessel function $K_\nu(z)$ having the integral representation [16]

$$\int_{0}^{\infty} x^{\nu-1} \exp\left(-\frac{\gamma}{x} - \delta x\right) dx = 2\left(\frac{\gamma}{\delta}\right)^{\nu/2} K_\nu(2\sqrt{\gamma\delta}), \qquad (2.26)$$

$$K_{-\nu}(z) = K_\nu(z).$$

In the case under consideration, γ and δ are positive quantities: $\gamma = \pi^2 n^2$, $\delta = a^2$. Now we can rewrite formula (2.25) as follows

$$\zeta_{EH}(s,a^2) = -\frac{(a^2)^{-s}}{2} + \frac{\sqrt{\pi}\Gamma(s-1/2)}{2\Gamma(s)}(a^2)^{-s+1/2} +$$

$$+ \frac{2\pi^s}{\Gamma(s)}(a^2)^{-s/2+1/4} \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2}(2\pi n\sqrt{a^2}). \qquad (2.27)$$

The series obtained converges for all s as the modified Bessel function has the asymptotics [16]

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}, \quad |z| \to \infty. \qquad (2.28)$$

Therefore, the singularities of the function $\zeta_{EH}(s,a^2)$ are due to the singularities of $\Gamma(s-1/2)$ in (2.27), i.e. $\zeta_{EH}(s,a^2)$ has first order poles at the points

$$s = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \cdots. \qquad (2.29)$$

Thus formula (2.27) affords an analytic continuation of (2.21) to the region $s \leq 1/2$ except for the points (2.29). Since $\zeta_{EH}(s,a^2)$ has a

pole at $s = -1/2$, function (2.27) can be used for obtaining only the regularized Casimir energy $E_C^{(2)\,reg}$

$$E_C^{(2)\,reg} = -\frac{M_0}{4\sqrt{\alpha}} - \frac{M_0^2 R}{8\pi\alpha}\Gamma(-1) - \frac{M_0}{2\pi\sqrt{\alpha}} \sum_{n=1}^{\infty} n^{-1} K_1\left(\frac{2nM_0 R}{\sqrt{\alpha}}\right). \qquad (2.30)$$

In order for the regularization to be removed the first and second terms in the right-hand side of (2.30) should be omitted (see, for example, [17–20])

$$E_C^{(2)\,ren} = -\frac{M_0}{2\pi\sqrt{\alpha}} \sum_{n=1}^{\infty} n^{-1} K_1\left(\frac{2nM_0 R}{\sqrt{\alpha}}\right). \qquad (2.31)$$

Rejection of the first term in (2.30) proportional to $\Gamma(-1)$ is natural in the analytic continuation method[3]. As for the second term $-M_0/(4\sqrt{\alpha})$, its rejection seems to be rather arbitrary. Usually this is motivated by the fact that this term is independent of R and, as a consequence, does not contribute to the Casimir force. However, this argument does not explain the rejection of the R-independent term in the interquark potential, i.e. in $E_C^{(2)}(R)$. In the general case those terms may be essential for the description of quark-quark interaction inside hadrons. Only the consistent renormalization with preliminary regularization and subsequent subtraction can justify the rejection of both the first and second terms in the right-hand side of (2.30). This will be demonstrated in the next Section.

3 Renormalization of the string tension and removal of the divergences

Let us calculate the interquark potential (2.5), (2.10) applying the standard renormalization technique. The initial model includes two parameters: string tension M_0^2 and a dimensionless constant α

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[3]If fields are considered in a bounded space region, then this procedure is interpreted as the subtraction of infinite space contribution [21].

characterizing the string rigidity. In the one-loop approximation, only the string tension is renormalized.

The renormalized string potential at large distances should coincide with its classical expression[4]

$$V^{ren}(R)\,|_{R\to\infty} = M^2 R, \qquad (3.1)$$

where M^2 is the renormalized string tension, whose explicit expression for be obtained further. Starting with (2.10) and taking into account the necessity to regularize all the divergent expressions, we represent $V^{ren}(R)$ as

$$V^{ren}(R) = M_0^2 R + (D-2)\left[E_C^{(1)\,reg}(R,\Lambda) + E_C^{(2)\,reg}(R,\Lambda)\right]\Big|_{\Lambda\to\infty} =$$

$$= M_0^2 R + (D-2)\left\{\left[E_C^{(1)\,reg}(R,\Lambda) + E_C^{(2)\,reg}(R,\Lambda)\right] - \right.$$

$$\left. - \left[E_C^{(1)\,reg}(R\to\infty,\Lambda) + E_C^{(2)\,reg}(R\to\infty,\Lambda)\right]\right\}\Big|_{\Lambda\to\infty} +$$

$$+ (D-2)\left[E_C^{(1)\,reg}(R\to\infty,\Lambda) + E_C^{(2)\,reg}(R\to\infty,\Lambda)\right]\Big|_{\Lambda\to\infty} =$$

$$= M^2 R + (D-2)\left[E_C^{(1)\,ren}(R) + E_C^{(2)\,ren}(R)\right], \qquad (3.2)$$

where Λ is a regularization parameter; M^2 is the renormalized value of the string tension,

$$M^2 = M_0^2 + \frac{D-2}{R}\left[E_C^{(1)\,reg}(R\to\infty,\Lambda) + E_C^{(2)\,reg}(R\to\infty,\Lambda)\right]\Big|_{\Lambda\to\infty}; \qquad (3.3)$$

and $E_C^{(i)\,ren}$, $i = 1,2$ are the renormalized Casimir energies (2.11) and (2.12)

$$E_C^{(i)\,ren}(R) = \left[E_C^{(i)\,reg}(R,\Lambda) - E_C^{(i)\,reg}(R\to\infty,\Lambda)\right]\Big|_{\Lambda\to\infty}, \quad i=1,\,2. \qquad (3.4)$$

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[4]In the framework of a string model, the potential linearly rising at large distances is the classical string energy considered as a function of its length R when $R \to \infty$ [6]. On the microscopic level (QCD level), the very appearance of the collective string degrees of freedom is interpreted as a complicated nonperturbative effect in quantum dynamics of gluon and quark fields closely related to nontrivial properties of the QCD vacuum.

To regularize divergent series (2.11) and (2.12), we substitute them by finite sums that can be represented in terms of the Cauchy integrals [14]

$$\frac{1}{2\pi i}\oint_C z\frac{f'(z)}{f(z)}dz = \sum_k n_k a_k - \sum_l p_l b_l. \qquad (3.5)$$

Here $f(z)$ is an analytic function having, in a region surrounded by contour C, zeroes of order n_k at points $z = a_k$ and poles of order p_l at points $z = b_l$. As a function $f(z)$ we substitute the right-hand sides of frequency equations (2.8) and (2.9) into (3.5) and choose the contour C so as to include N first positive roots of the corresponding equations. Functions (2.8) and (2.9) have zeroes of the first order on the real axis and have no poles. Therefore only the first sum with $n_k = 1$ remains in the right-hand side of (3.5).

First, we obtain the regularized Casimir energy (2.11)

$$E_C^{(1)\,reg}(R) = \frac{R}{4\pi i}\oint_C \omega\,\frac{\cos(\omega R)}{\sin(\omega R)}\,d\omega, \qquad (3.6)$$

where the contour C is shown in Fig. 2. All the singularities of the integrand in (3.6) being situated on the real axis, it is possible to deform the contour C to C' continuously (see Fig. 2) Now the regularization parameter is the radius Λ of the semicircle entering into the contour C'.

To determine the counterterms according to (3.1)–(3.4), it is necessary to find the asymptotics of $E_C^{(1)\,reg}(R)$ for $R \to \infty$ and fixed Λ. On the semicircle of radius Λ (Fig. 2) the asymptotics of the integrand for $R \to \infty$ is the integrand itself because of its oscillating character. Consequently, the result of integration along this part of the counter C' is completely absorbed by the counterterm and does not give any finite contribution to $E_C^{(1)ren}(R)$. Now let us turn to the

integral along the interval $(-i\Lambda, i\Lambda)$ on the imaginary axis

$$E_C^{(1)\,reg}(R, \Lambda) = -\frac{R}{4\pi} \int_{-\Lambda}^{\Lambda} y\, \frac{\cosh(Ry)}{\sinh(Ry)}\, dy. \qquad (3.7)$$

To find the asymptotics needed, we integrate in (3.7) by parts

$$E_C^{(1)\,reg}(R, \Lambda) = -\frac{1}{4\pi} \int_{-\Lambda}^{\Lambda} y\, d\left(\ln|\sinh(Ry)|\right) =$$

$$= -\frac{\Lambda}{2\pi} \ln[\sinh(\Lambda R)] + \frac{1}{2\pi} \int_0^\Lambda dy\, \ln[\sinh(Ry)].$$

When $R \to \infty$

$$E_C^{(1)\,reg}(R \to \infty, \Lambda) = -\frac{\Lambda}{2\pi} \ln[\sinh(\Lambda R)] + \frac{1}{2\pi} \int_0^\Lambda (Ry - \ln 2)\, dy. \quad (3.8)$$

Inserting (3.8) into (3.4) we obtain the finite value for the renormalized Casimir energy (2.11)

$$E_C^{(1)\,ren}(R) = \frac{1}{2\pi} \int_0^\infty \ln\left(1 - e^{-2R\omega}\right)\, d\omega = -\frac{R}{\pi} \int_0^\infty \frac{\omega\, d\omega}{e^{2R\omega} - 1}. \quad (3.9)$$

The last formula is derived by integrating by parts. It is interesting to note that (3.9) is expressed in terms of the value of the Riemann ζ-function at the point $s = 2$. Really,

$$\int_0^\infty \frac{\omega\, d\omega}{e^{a\omega} - 1} = \int_0^\infty \frac{\omega e^{-a\omega}}{1 - e^{-a\omega}}\, d\omega = \sum_{n=1}^\infty \int_0^\infty \omega e^{-an\omega}\, d\omega =$$

$$= \frac{\Gamma(2)}{a^2} \sum_{n=1}^\infty \frac{1}{n^2} = \frac{1}{a^2} \zeta(2). \qquad (3.10)$$

In view of this, Eq. (3.8) can be rewritten as

$$E_C^{(1)\,ren}(R) = -\frac{1}{4\pi R} \zeta(2). \qquad (3.11)$$

Thus, under consistent renormalization, the sum of divergent series (2.11) is also defined through the Riemann ζ-function, but now another range of its definition is used, namely, the region $\mathrm{Re}\,s > 1$. Here $\zeta(s)$ is defined by convergent series (2.13).

With the help of the Riemann reflection formula (2.17) the value of ζ-function at $s = 2$ entering into (3.11) can be expressed through $\zeta(-1)$

$$\zeta(2) = -2\pi^2 \zeta(-1). \qquad (3.12)$$

Final renormalized formula for the Casimir energy (2.11) assumes the same form as that obtained by analytic continuation of the Riemann ζ-function (Eq. (2.20))

$$E_C^{(1)\,ren}(R) = \frac{\pi}{2R} \zeta(-1) = \frac{\pi}{2R}\left(-\frac{1}{12}\right) = -\frac{\pi}{24R}. \qquad (3.13)$$

Thus, there is a complete agreement between two outlined approaches to the calculation of the finite value of $E_C^{(1)}$.

Before we turn to consideration of the series (2.12), let us make a short remark concerning formula (3.9). Discarding the minus sign, the integrand in (3.9) has a form of the Planck energy distribution in the spectrum of one-dimensional black-body with temperature $1/2R$.

The renormalized value of the Casimir energy $E_C^{(2)}$ (see Eqs. (2.12), (3.4)) can be obtained in the same way as it was done above. Substitution of the frequency equation (2.9) into (3.5) gives

$$E_C^{(2)\,reg}(R, \Lambda) = \frac{R}{4\pi i} \oint_C \frac{\cos(\sqrt{\omega^2 - \omega_0^2})}{\sin(\sqrt{\omega^2 - \omega_0^2})} \frac{\omega^2}{\sqrt{\omega^2 - \omega_0^2}}\, d\omega, \qquad (3.14)$$

where $\omega_0^2 = M_0^2/\alpha$. When choosing the contour C one has to take into account the branch points of the integrand $\omega = \pm\omega_0$. To select the single-valued branch of the function, we make a cut connecting

the branch points along the real axis. After that the contour can be chosen as shown in Fig. 3. Again integration along the semicircle of radius Λ contributes only to the counterterm. The integrals along the edges of the cut are mutually cancelled, and the contribution I_1 of integration around the branch point $\omega = \omega_0$ is equal to $-\omega_0/4 = -M_0/(4\sqrt{\alpha})$ (see Appendix A). It should be noted that I_1 is exactly equal to the first term in formula (2.30) for $E_C^{(2)\,reg}$. The sum of integral I_2 along the interval $(-i\Lambda, i\Lambda)$ of the imaginary axis and I_1 is

$$E_C^{(2)\,reg}(R, \Lambda) = -\frac{\Lambda}{2\pi} \ln\left[\sinh\left(R\sqrt{\Lambda^2 + \omega_0^2}\right)\right] +$$

$$+ \frac{1}{2\pi} \int_0^\Lambda dy \ln\left[\sinh\left(R\sqrt{y^2 + \omega_0^2}\right)\right] - \frac{\omega_0^2}{4}. \qquad (3.15)$$

Integration by parts is already done here. Formula (3.4) requires an asymptotics $E_C^{(2)\,reg}(R \to \infty, \Lambda)$. From (3.15) it follows that

$$E_C^{(2)\,reg}(R \to \infty, \Lambda) = -\frac{\Lambda}{2\pi} \ln\left[\sinh\left(R\sqrt{\Lambda^2 + \omega_0^2}\right)\right] +$$

$$+ \frac{1}{2\pi} \int_0^\Lambda dy \left(R\sqrt{y^2 + \omega_0^2} - \ln 2\right) - \frac{\omega_0^2}{4}. \qquad (3.16)$$

The constant term $-\omega_0^2/4$ is preserved here to satisfy condition (3.1) which defines the behavior of the string potential at infinity. Otherwise this term would appear in the right-hand side of (3.1), but that is physically unacceptable. At large distances string potential should be determined by its classical value only. Inserting (3.15) and (3.16) into (3.4) we find for $i = 2$

$$E_C^{(2)\,ren}(R) = \frac{1}{2\pi} \int_0^\infty d\omega \ln\left(1 - e^{-2R\sqrt{\omega^2 + \omega_0^2}}\right) =$$

$$= -\frac{R}{\pi} \int_0^\infty \frac{\omega^2 d\omega}{\sqrt{\omega^2 + \omega_0^2}} \frac{1}{e^{2R\sqrt{\omega^2 + \omega_0^2}} - 1}. \qquad (3.17)$$

It is interesting to the compare the formula derived with an analogous expression for $E_C^{(1)\,ren}(R)$ (see Eq. (3.9)). Formula (3.17) can be obtained by changing the variable frequency in (3.9) to $\sqrt{\omega^2 + \omega_0^2}$. This completely corresponds to the fact that Eq. (3.9) deals with oscillations of the massless (two-dimensional) scalar field on the segment $[0, R]$ while Eq. (3.17) treats oscillations of the same field, but with the mass equal to $\omega_0 = M_0/\sqrt{\alpha}$ (see field equations (2.6)).

At first sight, the expression obtained for $E_C^{(2)\,ren}(R)$ by making use of the consistent renormalization of the string tension does not coincide with that derived by analytic continuation of the Epstein-Hurwitz zeta function (see formula (2.31)). This is not true, however. Equations (3.17) and (2.31) are completely equivalent. To show this, let us expand the logarithm in (3.17)

$$E_C^{(2)\,ren}(R) = -\frac{1}{2\pi} \sum_{n=1}^\infty n^{-1} \int_0^\infty d\omega\, e^{-2nR\sqrt{\omega^2 + \omega_0^2}}. \qquad (3.18)$$

By changing the variable $\omega = \omega_0 \sinh t$, the integral is reduced to the tabular one [16]

$$K_\nu(z) = \int_0^\infty e^{-z \cosh t} \cosh(\nu t)\, dt$$

with $z = 2n\omega_0 R$. Finally we deduce the series (2.31) from (3.17)

$$E_C^{(2)\,ren}(R) = \frac{1}{2\pi} \int_0^\infty d\omega \ln\left(1 - e^{-2R\sqrt{\omega^2 + \omega_0^2}}\right) =$$

$$= -\frac{\omega_0}{2\pi} \sum_{n=1}^\infty n^{-1} K_1(2n\omega_0 R), \qquad (3.19)$$

where $\omega_0 = M_0/\sqrt{\alpha}$. Thus we found an integral representation for the series (2.31). This series is convenient for investigating the behavior of the Casimir energy $E_C^{(2)\,ren}(R)$ at large distances. Taking

into consideration (2.28) we get

$$E_C^{(2)\,ren}(R)\Big|_{R\to\infty} \simeq -\frac{1}{4}\left(\frac{\omega_0}{\pi R}\right)^{1/2} e^{-2\omega_0 R}. \qquad (3.20)$$

The integral representation (3.17) enables one to study the asymptotics of $E_C^{(2)\,ren}(R)$ at small R. From (3.17) it follows that $E_C^{(2)\,ren}(R)$ has a singularity when $R = 0$. For small R the main contribution to this integral is given by large ω, therefore one can neglect here the dependence on ω_0. This immediately gives the asymptotics of $E_C^{(2)ren}(R)$ for $R \to 0$

$$E_C^{(2)\,ren}(R)\Big|_{R\to 0} \simeq \frac{1}{2\pi}\int\limits_0^\infty d\omega \ln\left(1 - e^{2\omega R}\right) = -\frac{\pi}{24R}. \qquad (3.21)$$

Thus, consistent regularization of the divergent series (2.12) and subsequent renormalization of the string tension justify the rejection of the singular (pole) term and R-independent constant in Eq. (2.30) when analytic continuation of Epstein-Hurvitz ζ-function is used. It is worthwhile to emphasize an important advantage of the proposed regularization by contour integration and subsequent subtraction. In this way we obtain the spectral representation for string energy at zero temperature (see Eqs. (3.9) and (3.17) in contrast to analytic continuation of ζ-functions (Eqs. (2.20) and (2.31)). Proceeding from this spectral representation one can immediately derive the string free energy at finite temperature. To this end one must pass from integration to summation over the Matsubara frequencies $\Omega_n = 2\pi n T$, $n = 0, \pm 1, \pm 2, \ldots$. Practically it is done by the substitution

$$d\omega \to 2\pi T\, d\omega \sum_{n=0}^\infty{}' \delta(\omega - \Omega_n), \qquad (3.22)$$

where T is the temperature (see Appendix B). The prime of the sum sign means that the term with $n = 0$ should be multiplied by $1/2$.

For either quantity $E_C^{(i)}(R)$ $i = 1, 2$ we have obtained two integral representations (see Eqs. (3.9) and (3.17)). Substitution (3.22) in

these formulas with logarithmic functions gives us the *free energy* at finite temperature (Appendix B). For example,

$$F^{(2)}(R,T) = 2T\sum_{n=0}^\infty{}' \ln\left(1 - e^{-2R\sqrt{\Omega_n^2 + \omega_0^2}}\right). \qquad (3.23)$$

Taking the limit $\omega_0 \to 0$ in (3.23) one can obtain the free energy $F^{(1)}(R,T)$ that diverges due to the term with $n = 0$ (see Appendix B). Making the substitution (3.22) in the second version of the spectral representations (3.9) and (3.17) we arrive at the *internal energy* at temperature T

$$U^{(1)}(R,T) = -4\pi RT^2 \sum_{n=0}^\infty{}' \frac{n}{\exp(4\pi nRT) - 1}, \qquad (3.24)$$

$$U^{(2)}(R,T) = -8\pi^2 RT^3 \sum_{n=0}^\infty{}' \frac{n^2}{\sqrt{\Omega_n^2 + \omega_0^2}} \frac{1}{\exp(2R\sqrt{\Omega_n^2 + \omega_0^2}) - 1}. \qquad (3.25)$$

Both the energies, $U^{(i)}(R,T)$, $i = 1, 2$ are well defined. The last two equations prove to be convenient for investigating the behaviour of the internal energies at large and small T. Let us demonstrate this using Eq. (3.24). At large T the main contribution to (3.24) comes from the first term with $n = 0$

$$U^{(1)}(R, T \to \infty) = -\frac{T}{2}. \qquad (3.26)$$

At small T the Euler-Maclaurin formula

$$\sum_{n=0}^\infty{}' f(n) = \int\limits_0^\infty f(x)\,dx - \frac{1}{12}f'(0) \qquad (3.27)$$

can be used. In the case under consideration

$$f(x) = \frac{x}{\exp(4\pi TRx) - 1} \quad\text{and}\quad f'(0) = -\frac{1}{2}. \qquad (3.28)$$

As a result, we obtain for small T

$$U^{(1)}(R,T) \approx -\frac{\pi}{24R} - \frac{\pi T^2 R}{6}. \qquad (3.29)$$

4 Conclusion

The experience of treating the divergences shows that a correct result can be obtained by applying practically any regularization and renormalization procedures provided that the prescriptions are properly modified. Therefore, when evaluating such methods, those should be preferred which are closer to the quantum field theory. Only in the framework of this approach one succeeds in formulation of a consistent renormalization procedure. Besides, quantum field formalism provides a clear and simple transition from zero temperature calculations to those at finite temperature [23]. In view of this, contour integration has an obvious advantage. At first it was proposed as a simple method for calculating the van der Waals forces between dielectrics [24] (see also [13, 17, 25, 26]). However its relation to the formalism of Green functions is not elucidated properly. And this problem is undoubtedly worth investigating.

Appendix A. Investigation of the contour integral determining $E_C^{(2)\,reg}(R, \Lambda)$

Let us consider integral (3.14) for individual parts of contour C shown in Fig. 3. Integration along the semicircle of radius Λ contributes only to the counterterm, therefore we do not analyze it here. When going from the upper edge of the cut to the lower one, the integrand does not change. As a result, integrals along these two parts of the contour C are cancelled mutually due to opposite directions of integration. Only integration around the branch point and along the interval of the imaginary axis $(-i\Lambda, i\Lambda)$ lead to finite contributions. While integrating around the branch point $\omega = \omega_0$ we introduce usual variables $\omega - \omega_0 = \rho e^{i\phi}$ with $\rho \to 0$, and in terms of them we have $\omega^2 - \omega_0^2 = (\omega + \omega_0)(\omega - \omega_0) \simeq 2\omega_0 \rho e^{i\phi}$, $\cos(R\sqrt{\omega^2 - \omega_0^2}) \simeq 1$, $\sin(R\sqrt{\omega^2 - \omega_0^2}) \simeq R\sqrt{\omega^2 - \omega_0^2}$. Taking this into account we deduce

$$I_1 = -\frac{R}{4\pi i} \int_0^{2\pi} \frac{\omega_0^2 \rho e^{i\phi} i\, d\phi}{2\omega_0 \rho e^{i\phi} R} = -\frac{\omega_0}{4} = -\frac{M_0}{4\sqrt{\alpha}}. \qquad (A.1)$$

The integral I_1 is exactly equal to the first term in (2.30) which is independent of R. When integrating along the imaginary axis, trigonometrical functions in (3.14) become hyperbolic ones

$$\begin{aligned} I_2 &= \frac{R}{4\pi i} \int_\Lambda^{-\Lambda} \frac{(-y^2)\, i\, dy}{i\sqrt{y^2 + \omega_0^2}} \frac{\cosh(R\sqrt{y^2 + \omega_0^2})}{\sinh(R\sqrt{y^2 + \omega_0^2})} dy = \\ &= -\frac{1}{2\pi} \int_0^\Lambda y\, d\left[\ln\sinh\left(R\sqrt{y^2 + \omega_0^2}\right)\right]. \end{aligned} \qquad (A.2)$$

Summing (A.1) and (A.2) and integrating by parts one arrives at formula (3.15)

Appendix B. Transition to finite temperature in infinite system of noninteracting oscillators

Let us consider an infinite system of noninteracting oscillators with eigenfrequencies ω_n, $n = 1, 2, \ldots$ determined by the equation

$$f(\omega, R) = 0. \qquad (B.1)$$

Roots of this equation are assumed to be situated on the real axis in the complex plane ω. This set of oscillators arises, for example, in quantization of a scalar field defined on the line segment $[0, R]$. Boundary conditions imposed on this field result in frequency equation (B.1). Without loss of generality, for relatistically invariant system one can admit that the function f satisfies the condition

$$f(-\omega, R) = f(\omega, R). \qquad (B.2)$$

The free energy of this system is given by

$$F(R, T) = \sum_{n=1}^{\infty} \left[\frac{\omega_n}{2} + T \ln \left(1 - e^{-\omega_n/T} \right) \right]. \qquad (B.3)$$

At zero temperature this formula obviously turns into the energy of zero point oscillations $E_C = \sum_{n=1}^{\infty} \omega_n/2$.

In the general case sum (B.3) diverges, therefore to obtain the finite value for the free energy, we have to use the renormalization procedure discussed in Section 3. First, the infinite sum should be represented as the contour integral

$$\sum_{n=1}^{\infty} g(\omega_n) = \frac{1}{2\pi i} \oint_C g(z) \frac{f'(z, R)}{f(z, R)} dz, \qquad (B.4)$$

where

$$g(\omega) = \frac{\omega}{2} + T \ln \left(1 - e^{-\omega/T} \right) = \frac{\omega}{T} - T \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\omega/T}.$$

The contour C, as in Section 3, surrounds the first N roots of Eq. (B.1).

As shown in Section 3, only integration along the imaginary axis gives a finite contribution to the free energy

$$F^{reg}(R, \Lambda) = -\frac{1}{2\pi i} \int_{-\Lambda}^{\Lambda} \left(\frac{iy}{2} - T \sum_{n=1}^{\infty} \frac{1}{n} e^{-iny/T} \right) d \ln[f(iy, R)]. \quad (B.5)$$

On integrating by parts we obtain

$$F^{reg}(R, \Lambda) = \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dy \left(\frac{1}{2} + T \sum_{n=1}^{\infty} e^{-iny/T} \right) \ln[f(iy, R)] = \qquad (B.6)$$

$$= \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dy \left\{ \frac{1}{2} + T \sum_{n=1}^{\infty} \left[\cos \left(\frac{ny}{T} \right) - i \sin \left(\frac{ny}{T} \right) \right] \right\} \ln[f(iy, R)]$$

The off-integral terms are omitted in (B.6) because they contribute only to the counterterm. Taking into account (B.2) we can drop terms with sine functions

$$F^{reg}(R, \Lambda) = \frac{1}{\pi} \int_{0}^{\Lambda} dy \left[\frac{1}{2} + T \sum_{n=1}^{\infty} \cos \left(\frac{ny}{T} \right) \right] \ln[f(iy, R)] \qquad (B.7)$$

The renormalized free energy is obtained by the subtraction

$$\begin{aligned} F^{ren}(R, T) &= \left. \left[F^{reg}(R, \Lambda) - F^{reg}(R \to \infty, \Lambda) \right] \right|_{\Lambda \to \infty} = \\ &= \frac{1}{\pi} \int_{0}^{\infty} dy \left[\frac{1}{2} + T \sum_{n=1}^{\infty} \cos \left(\frac{ny}{T} \right) \right] \ln \left[\frac{f(iy, R)}{f(iy, \infty)} \right] \end{aligned} (B.8)$$

With allowance for the Fourier-series representation of the δ-function

$$\pi T \sum_{n=-\infty}^{\infty} \delta(y - 2\pi n T) = \frac{1}{2} + \sum_{n=1}^{\infty} \cos \left(\frac{ny}{T} \right)$$

integration in (B.8) can be done to produce

$$\begin{aligned} F^{ren}(R, T) &= T \sum_{n=-\infty}^{\infty} \ln \left[\frac{f(2\pi i n T, R)}{f(2\pi i n T, \infty)} \right] = \\ &= 2T \sideset{}{'}\sum_{n=0}^{\infty} \ln \left[\frac{f(2\pi i n T, R)}{f(2\pi i n T, \infty)} \right]. \end{aligned} \qquad (B.9)$$

Now we apply formula (B.9) to the models considered in Section 3. In the case of a scalar field with mass ω_0 on the segment $[0, R]$ we have

$$f(\omega, R) = \sin\left(R\sqrt{\omega^2 - \omega_0^2}\right),$$

and Eq. (B.9) gives

$$F^{(2)}(R, T) = 2T \sum_{n=0}^{\infty}{}' \ln\left(1 - e^{-2R\sqrt{\Omega_n^2 + \omega_0^2}}\right), \qquad \text{(B.10)}$$

where $\Omega_n = 2\pi nT$. The same result was obtained in Section 3 by transition from the integral representation for the Casimir energy at zero temperature to summation over the Matsubara frequencies (see Eq. (3.24)). Proceeding from (B.10) one can derive the *internal energy* of the system under consideration applying thermodynamic rules

$$U^{(2)}(R, T) = -T^2 \left[\frac{\partial}{\partial T} \frac{F^{(2)}(R,T)}{T}\right] = \qquad \text{(B.11)}$$

$$= -8\pi^2 RT^3 \sum_{n=0}^{\infty}{}' \frac{n^2}{\sqrt{\Omega_n^2 + \omega_0^2}} \frac{1}{\exp(2R\sqrt{\Omega_n^2 + \omega_0^2}) - 1}.$$

This equation was derived in Section 3 by a simple substitution (see Eq. (3.25)).

In the case of massless scalar field ($\omega_0 \to 0$) the term with $n = 0$ in (B.10) diverges

$$F^{(1)}(R, T) = 2T \sum_{n=0}^{\infty}{}' \ln\left(1 - e^{-4\pi nRT}\right) =$$

$$= -T \lim_{n \to 0} \sum_{k=1}^{\infty} \frac{\exp(-4\pi nkRT)}{k} + 2T \sum_{n=1}^{\infty} \ln\left(1 - e^{-4\pi nRT}\right) =$$

$$= -T \sum_{k=1}^{\infty} \frac{1}{k} + 2T \sum_{n=1}^{\infty} \ln\left(1 - e^{-4\pi nRT}\right). \qquad \text{(B.12)}$$
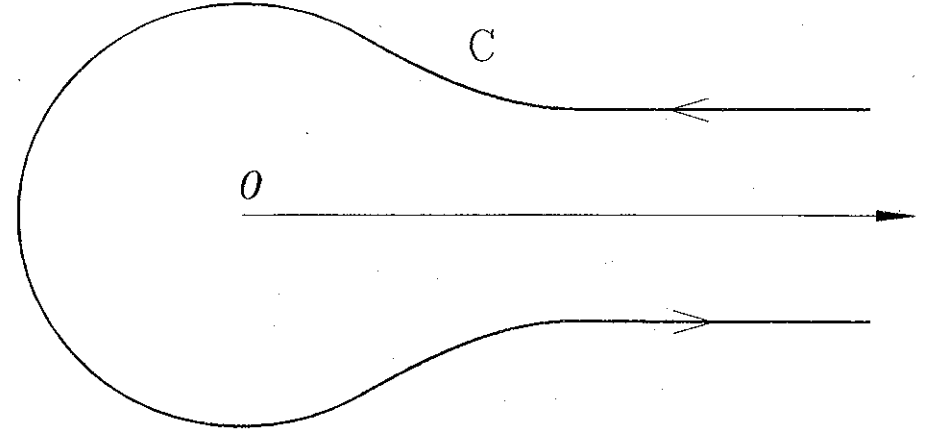

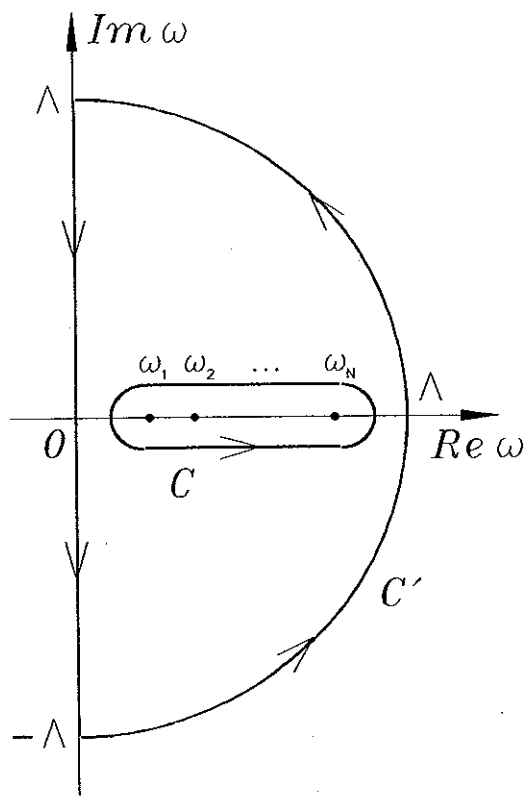Fig. 1. Contour C used in analytic continuation of the Riemann dzeta-function.

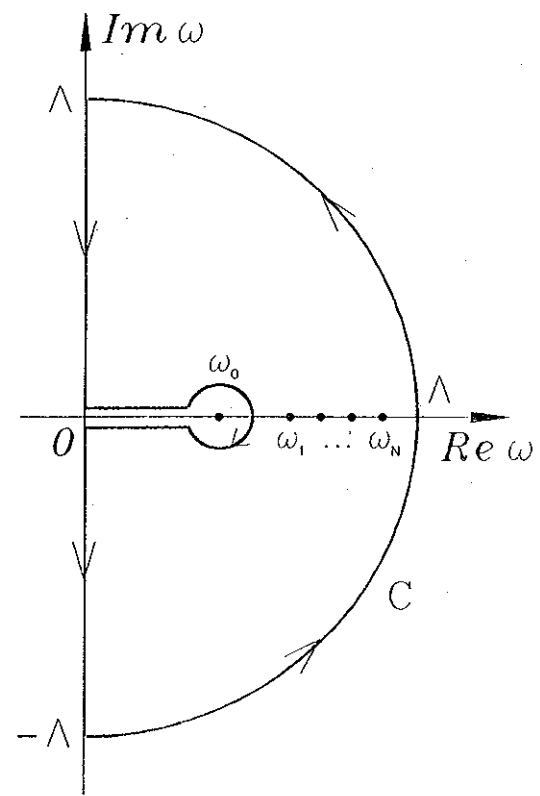Fig. 2. Transformation of the contour in integral (3.6).

Fig. 3. Contour used for summing the roots of Eq. (2.9).

This divergence is a manifestation of the well-known infrared instability of a massless scalar field in two-dimensional space-time. In Section 2 some reasons were given to attribute the Euler constant value, γ to the sum of the divergent series $\sum_{k=1}^{\infty} k^{-1}$. Finally we obtain a finite expression for the free energy of the massless scalar field on the segment $[0, R]$

$$F^{(1)}(R, T) = -\gamma T + 2T \sum_{n=1}^{\infty} \ln \left(1 - e^{-4\pi n RT}\right) . \qquad (B.13)$$

It should be noted that this treatment of infrared divergences in the problem in question is absolutely formal and it needs the physical justification.

However the internal energy of this field is well defined. Putting $\omega_0 = 0$ in (B.11) we get

$$U^{(1)}(R, T) = -4\pi RT^2 \sum_{n=0}^{\infty}{}' \frac{n}{\exp(4\pi n RT) - 1}. \qquad (B.14)$$

In Section 3 the same formula has been derived by a formal substitution (see Eq. (3.24)).

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