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ON THE ALGEBRAIC STRUCTURE OF DIFFERENTIAL CALCULUS ON QUANTUM GROUPS

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I. Introduction

Non-commutative differential calculus on quantum groups initiated and thoroughly worked out by Woronowicz [1] is up to now a subject of active discussions and development. Though meeting some problems [2, 3, 4, 5, 6] with non-classical dimensionalities of spaces of higher-order differential forms (which, in its turn, stimulated very interesting alternative approaches [7, 5, 8]), original Woronowicz's construction remains highly attractive due to both its rich algebraic structure and useful applications. Probably, the best known realization of this scheme is bicovariant differential calculus on the $GL_q(N)$ quantum groups [9, 3, 10].

Closely related but somewhat parallel to Woronowicz's construction is another project [3, 7, 11, 12] that, in particular, has produced a bicovariant algebra of four types of elements: functions on a quantum group, differential forms, Lie derivatives along vector fields, and inner derivations – by exact analogy with classical differential geometry. However, this scheme, as it is, does not seem to be fully motivated by the Hopf-algebraic nature of non-commutative differential calculus.

In the present paper, we suggest an extension of Woronowicz's axiomatics which naturally involves Lie derivatives and inner derivations in a way that respects the Hopf algebra structure of the whole scheme. Actually, in the framework of Woronowicz's noncommutative differential calculus [1, 10] one deals with the differential complex

$$A \xrightarrow{d} \Gamma \xrightarrow{d} \Gamma^2 \longrightarrow \ldots\,, \tag{1}$$

where A is a Hopf algebra (of functions on a quantum group), Γ is its bicovariant bimodule, $\Gamma^2 \equiv \Gamma \wedge \Gamma$ is its second wedge power, and so on. Exterior differential map $d : \Gamma^n \to \Gamma^{n+1}$ is assumed to obey the Leibniz rule

$$d(ab) = (da)b + a\,db \tag{2}$$

and the nilpotency condition $d \circ d = 0$. Brzezinski [13] has shown that

$$\Gamma^\wedge \doteq A \oplus \Gamma \oplus \Gamma^2 \oplus \ldots \tag{3}$$

also becomes a (graded) Hopf algebra with respect to (wedge) multiplication and natural definitions of coproduct and antipode. In what follows,

we want to demonstrate how this Hopf structure can be used to build an associative noncommutative bicovariant algebra containing functions, differential forms, Lie derivatives and inner derivations. Similar algebras have been introduced and studied by several authors [3, 11, 14, 12] (and the idea to a use cross-product for constructing bicovariant differential calculus is due to [7]). Probably, the closest to ours is the approach by P. Schupp [14]. However, some of our results and, especially, starting points appear to be different. So, we propose the construction described below as entirely Hopf-algebra motivated (and, we believe, natural) new approach to the problem.

II. Cross-product of dual Hopf algebras

Notions of mutually dual Hopf algebras and their cross-product will actively be used throughout this paper. Let us recall the corresponding terminology and basic definitions [15, 16, 17, 18, 7]. Let A be a Hopf algebra with associative multiplication, coassociative coproduct

$$\Delta : A \longrightarrow A \otimes A\,, \quad \Delta(a) \doteq a_{(1)} \otimes a_{(2)}\,, \quad \Delta(ab) = \Delta(a)\Delta(b) \tag{4}$$

(we will use the notation

$$a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \doteq (\Delta \otimes id) \circ \Delta(a) = (id \otimes \Delta) \circ \Delta(a)\,, \tag{5}$$

and so on, for multiple coproducts), a counit

$$\varepsilon : A \longrightarrow C\,, \quad \varepsilon(ab) = \varepsilon(a)\varepsilon(b)\,, \quad \varepsilon(a_{(1)})\, a_{(2)} = a_{(1)}\varepsilon(a_{(2)}) = a\,, \tag{6}$$

and an invertible antipode

$$S : A \longrightarrow A\,, \quad S(ab) = S(b)S(a)\,, \quad \Delta(S(a)) = S(a_{(2)}) \otimes S(a_{(1)})\,,$$

$$\varepsilon(S(a)) = \varepsilon(a)\,, \quad S(a_{(1)})a_{(2)} = a_{(1)}S(a_{(2)}) = \varepsilon(a)\,. \tag{7}$$

Algebra A^* is a Hopf dual of A with $<\cdot,\cdot>: A^* \otimes A \longrightarrow C$ being a duality map, if

$$<xy\,,a>=<x \otimes y, \Delta(a)>\,, \quad <x\,,ab>=<\Delta(x), a \otimes b>\,,$$

$$<x, 1>= \varepsilon(x)\,, \quad <1\,, a>= \varepsilon(a)\,, \quad <S(x), a>=<x, S(a)>\,. \tag{8}$$

Here and below $a, b \in A\,, x, y \in A^*$.

One can define left and right covariant actions $A^* \triangleright A$ and $A \triangleleft A^*$ by

$$x \triangleright a = a_{(1)} < x, a_{(2)} > , \quad a \triangleleft x = a_{(2)} < x, a_{(1)} > . \tag{9}$$

As usual, left and right actions imply

$$(xy) \triangleright a = x \triangleright (y \triangleright a) , \quad a \triangleleft (xy) = (a \triangleleft x) \triangleleft y , \tag{10}$$

whereas the covariance (or generalized differential property) means

$$x \triangleright (ab) = (x_{(1)} \triangleright a)(x_{(2)} \triangleright b) , \quad (ab) \triangleleft x = (a \triangleleft x_{(1)})(b \triangleleft x_{(2)}) , \tag{11}$$

i.e., the A^*-actions respect multiplicative structure of A, or, in other words, A is a left (right) A^*-module algebra.

One can use (e.g., left) action (9) of A^* on A to define on their tensor product $A \otimes A^*$ the cross-product algebra $A \rtimes A^*$ [15, 18, 7]. This is an associative algebra with the cross-multiplication rule given by

$$xa = (x_{(1)} \triangleright a) x_{(2)} \equiv < x_{(1)}, a_{(2)} > a_{(1)} x_{(2)} \tag{12}$$

(multiplication inside A and A^* does not change). A cross-product is not a Hopf algebra but exhibits remarkable A^*-module and A-comodule properties [7, 19].

First, $A \rtimes A^*$ is covariant under the right A^*-action of the following form:

$$\text{right } A^*\text{-action:} \quad a \triangleleft x = a_{(2)} < x, a_{(1)} > , \quad y \triangleleft x = y < x, 1 > \equiv \varepsilon(x) y , \tag{13}$$

to be extended on arbitrary products in $A \rtimes A^*$ by the covariance condition

$$(pq) \triangleleft x = (p \triangleleft x_{(1)})(q \triangleleft x_{(2)}) , \quad p, q \in (A \rtimes A^*) . \tag{14}$$

Surely, this needs to be consistent with (12). Let us check it:

$$\begin{aligned}
(ya) \triangleleft x &= (y \triangleleft x_{(1)})(a \triangleleft x_{(2)}) = \varepsilon(x_{(1)}) y < x_{(2)}, a_{(1)} > a_{(2)} \\
&= < x, a_{(1)} > y a_{(2)} = < x, a_{(1)} >< y_{(1)}, a_{(3)} > a_{(2)} y_{(2)} \\
&= \varepsilon(x_{(2)}) < x_{(1)}, a_{(1)} >< y_{(1)}, a_{(3)} > a_{(2)} y_{(2)} \\
&= < y_{(1)}, a_{(2)} > (a_{(1)} \triangleleft x_{(1)})(y_{(2)} \triangleleft x_{(2)}) = (< y_{(1)}, a_{(2)} > a_{(1)} y_{(2)}) \triangleleft x .
\end{aligned} \tag{15}$$

It is known [18] that a covariant right action $F \triangleleft H$ of a Hopf algebra H on an algebra F implies a covariant left coaction $F \to H^* \otimes F$ of the Hopf dual H^* on F. The correspondence is defined by

$$f \triangleleft h = < h , f^{(1)} > f^{(0)} , \tag{16}$$

where a coaction is assumed to be $f \to f^{(1)} \otimes f^{(0)}$ with $h \in H$, $f^{(1)} \in H^*$, $f, f^{(0)} \in F$. For coactions, 'covariant' still means 'respecting multiplication'. This is expressed by

$$(fg) \longrightarrow f^{(1)} g^{(1)} \otimes f^{(0)} g^{(0)} . \tag{17}$$

In our case, the left A-coaction dual to (13) is

$$\text{left } A\text{-coaction:} \qquad a \longrightarrow \Delta(a) \equiv a_{(1)} \otimes a_{(2)} , \quad y \longrightarrow 1 \otimes y . \tag{18}$$

The very last relation explains why the elements of A^* are called left-invariant in this situation.

Further, $A \rtimes A^*$ is covariant under a left A^*-action and also under its dual right A-coaction. Explicit form of the A^*-action is taken to be the well-known Hopf adjoint,

$$x \overset{\text{ad}}{\triangleright} p = x_{(1)} p S(x_{(2)}) , \quad p \in A \rtimes A^* , \tag{19}$$

which is evidently covariant:

$$x \overset{\text{ad}}{\triangleright} (pq) = x_{(1)} pq S(x_{(2)}) = x_{(1)} p S(x_{(2)}) x_{(3)} q S(x_{(4)}) = (x_{(1)} \overset{\text{ad}}{\triangleright} p)(x_{(2)} \overset{\text{ad}}{\triangleright} q) .$$

Moreover, for $p = a \in A$ one shows [20, 7] that

$$\begin{aligned}
x \overset{\text{ad}}{\triangleright} a &\equiv x_{(1)} a S(x_{(2)}) = a_{(1)} < x_{(1)}, a_{(2)} > x_{(2)} S(x_{(3)}) \\
&= \varepsilon(x_{(2)}) a_{(1)} < x_{(1)}, a_{(2)} > = a_{(1)} < x, a_{(2)} > = x \triangleright a ,
\end{aligned} \tag{20}$$

i.e., we recover the left action (9) and can rewrite (19) as

$$\text{left } A^*\text{-action:} \qquad x \triangleright a = a_{(1)} < x, a_{(2)} > , \quad x \triangleright y = x_{(1)} y S(x_{(2)}) . \tag{21}$$

The corresponding dual right A-coaction is deduced from the general rule [18] analogous to (16), which relates left action $H \triangleright F$ with right coaction $F \to F \otimes H^*$:

$$h \triangleright g = < h , g^{(1)} > g^{(0)} , \quad g \to g^{(0)} \otimes g^{(1)} , \quad h \in H, \; g^{(1)} \in H^*, \; g, g^{(0)} \in F , \tag{22}$$

and is explicitly given by [7]

$$\text{right } A\text{-coaction:} \quad a \to \Delta(a) \equiv a_{(1)} \otimes a_{(2)}, \quad y \to (e^\alpha \overset{\text{ad}}{\triangleright} y) \otimes e_\alpha, \qquad (23)$$

where $\{e_\alpha\}, \{e^\alpha\}$ are dual bases in A and A^*. Note that in both (18) and (23) the coaction on the A-part of $A \rtimes A^*$ is just a coproduct.

Being the covariant (co)actions, eqs. (13),(18),(21) and (23) characterize $A \rtimes A^*$ as a left (right) (co)module algebra. It is in this sense that the cross-product algebra $A \rtimes A^*$ may be called bicovariant [7, 19]. Of course, this bicovariance is merely a reflection of the underlying Hopf algebra structure of A.

III. Woronowicz's differential complex as a Hopf algebra

Let us now recall the basic definitions of the Woronowicz noncommutative differential calculus [1, 10]. First, a basis $\{\omega^i\}$ of left-invariant 1-forms should be chosen in the bimodule Γ in (1). Any element $\rho \in \Gamma$ can be uniquely represented as $\rho = a_i\omega^i$, $a_i \in A$. Next, one specifies commutation relations between functions and differential forms,

$$\omega^i a = (f^i_j \triangleright a)\omega^j, \qquad (24)$$

the coalgebra structure of Γ,

$$\Delta(\omega^i) = 1 \otimes \omega^i + \omega^j \otimes r^i_j, \qquad (25)$$

and a differential map $d : A \longrightarrow \Gamma$:

$$da = (\chi_i \triangleright a)\omega^i. \qquad (26)$$

Here a is arbitrary element of A, $r^i_j \in A$, χ_i and f^i_j belong to A^*. The Hopf-algebra consistency (or bicovariance) conditions of the calculus are:

$$\begin{aligned}
(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \quad &\Longrightarrow \quad \Delta(r^i_j) = r^k_i \otimes r^j_k, && (27)\\
\omega(ab) = (\omega a)b \quad &\Longrightarrow \quad \Delta(f^i_j) = f^i_k \otimes f^k_j, && (28)\\
\Delta(\omega a) = \Delta(\omega)\Delta(a) \quad &\Longrightarrow \quad (f^j_i \triangleright a)r^i_k = r^j_i(a \triangleleft f^i_k), && (29)\\
d(ab) = (da)b + a\,db \quad &\Longrightarrow \quad \Delta(\chi_i) = \chi_j \otimes f^j_i + 1 \otimes \chi_i, && (30)\\
\Delta \circ d = (d \otimes 1 + 1 \otimes d) \circ \Delta \quad &\Longrightarrow \quad a \triangleleft \chi_i = (\chi_j \triangleright a)r^j_i, && (31)
\end{aligned}$$

supplemented by the formulas

$$\varepsilon(f^i_j) = \delta^i_j, \quad \varepsilon(r^i_j) = \delta^i_j, \quad S(f^j_k)f^k_i = \delta^j_i, \quad S(r^k_i)r^j_k = \delta^j_i, \qquad (32)$$

which are obtained from the properties of counit and antipode. Woronowicz's theory asserts that every set of elements $\{r^j_i, f^j_i, \chi_i\}$ obeying eqs. (27)–(32) gives us an example of a bicovariant differential calculus on the Hopf algebra A.

For illustration, let us derive (31) (cf. [21]).

$$\begin{aligned}
\Delta(da) = \Delta((\chi_i \triangleright a)\omega^i) &= <\chi_i, a_{(3)}> (a_{(1)} \otimes a_{(2)})(\omega^j \otimes r^i_j + 1 \otimes \omega^i)\\
&= a_{(1)}\omega^j \otimes (\chi_i \triangleright a_{(2)})r^i_j + a_{(1)} \otimes (\chi_i \triangleright a_{(2)})\omega^i, \qquad (33)
\end{aligned}$$

$$\begin{aligned}
da_{(1)} \otimes a_{(2)} + a_{(1)} \otimes da_{(2)} &= (\chi_i \triangleright a_{(1)})\omega^i \otimes a_{(2)} + a_{(1)} \otimes (\chi_i \triangleright a_{(2)})\omega^i\\
&= a_{(1)} < \chi_i, a_{(2)} > \omega^i \otimes a_{(3)} + a_{(1)} \otimes (\chi_i \triangleright a_{(2)})\omega^i\\
&= a_{(1)}\omega^i \otimes (a_{(2)} \triangleleft \chi_i) + a_{(1)} \otimes (\chi_i \triangleright a_{(2)})\omega^i. \qquad (34)
\end{aligned}$$

Independence of $\{\omega^i\}$ yields

$$a_{(1)} \otimes (\chi_i \triangleright a_{(2)})r^i_j = a_{(1)} \otimes (a_{(2)} \triangleleft \chi_j). \qquad (35)$$

Acting on both sides of this equation by $\varepsilon \otimes id$, we come to (31).

Consider now the graded Hopf algebra (differential complex) $\Gamma^\wedge$ given by (1),(3) jointly with its dual $(\Gamma^\wedge)^*$:

$$\begin{array}{ccccccc}
A & \xrightarrow{d} & \Gamma & \xrightarrow{d} & \Gamma^2 & \xrightarrow{d} & \ldots\\
\updownarrow & & \updownarrow & & \updownarrow & & \\
A^* & \xleftarrow{d^*} & \Gamma^* & \xleftarrow{d^*} & \Gamma^{2*} & \xleftarrow{d^*} & \ldots
\end{array} \qquad (36)$$

(vertical arrows indicate non-zero duality brackets implied by grading). Analogously to (12), an associative algebra $\mathcal{G} = \Gamma^\wedge \rtimes (\Gamma^\wedge)^*$ can be introduced using the cross-product construction (here $(\Gamma^\wedge)^* = A^* \oplus \Gamma^* \oplus \ldots$). We place $\mathcal{G}$ in the center of our approach. It means that we assume the following guiding principle:

All cross-commutation relations among functions, forms, Lie derivatives, and inner derivations are to be chosen according to the rules (12) of a cross-product algebra. In other words, given Woronowicz's calculus (and, hence, the Hopf algebra $\Gamma^\wedge$), we then have to use only standard Hopf-algebra technique $\Gamma^\wedge \Rightarrow (\Gamma^\wedge)^* \Rightarrow \Gamma^\wedge \rtimes (\Gamma^\wedge)^*$ to construct the whole algebra of these four types of elements.

The resulting algebra is bicovariant by construction. Its bicovariance in the sense of Woronowicz's left and right covariance [1] is implied by the Hopf-algebra nature of $\Gamma^\wedge$ [13], whereas its bicovariance in the sense of Schupp, Watts and Zumino, expressed by eqs. (13),(18),(21) and (23), proves to be an inherent feature of the cross-product (see Sect. 2), and stems, at the very end, from the same Hopf structure of $\Gamma^\wedge$.

IV. Explicit form of commutational relations

It only remains to put all the relevant objects in the corresponding 'boxes'. We already know that functions and 1-forms are situated in A and Γ, respectively. Owing to (18), one may consider A^* (acting on A from the left) as an algebra of left-invariant (and $A \rtimes A^*$ – of general) vector fields on a quantum group A. It is generally accepted [3, 7, 11, 22] that Lie derivatives $\mathcal{L}_h$ along a (left-invariant) vector field $h \in A^*$ must be related with its action on arbitrary elements of $\mathcal{G}$:

$$\mathcal{L}_h \doteq h \overset{\text{ad}}{\triangleright} \tag{37}$$

which, due to (20), reduces to ordinary left action (9) $h \triangleright \rho$ for $\rho \in \Gamma^\wedge$.

It seems also natural to relate inner derivations with elements of Γ^* [14]. We propose the following definition [23]. Let $\gamma_i \in \Gamma^*$ be determined by fixing its duality bracket with a general element of Γ,

$$<\gamma_i, a\omega^j>= \varepsilon(a)\,\delta_i^j\,, \tag{38}$$

and $<\gamma_i, \rho>= 0$ for $\rho \in A, \Gamma^2, \Gamma^3, \ldots$. Then we can define a basis of inner derivations $\{\imath_i\}$ as follows:

$$\imath_i \doteq \gamma_i \overset{\text{ad}}{\triangleright} \tag{39}$$

(the same comment as for eq. (37) applies). Here we make no attempt to associate some $\imath_h \in \Gamma^*$ with any $h \in A^*$, for it looks unnatural in the context of our approach (see, however, [11, 24] for a discussion of such a possibility).

The cross-product algebra we are seeking for, i.e., an algebra which includes four types of differential-geometric objects, $a, \omega^i, \mathcal{L}_h$ and $\imath_i$, is implicitly contained in the above definitions. In order to make it more transparent, we employ these definitions for obtaining a set of helpful relations.

To begin with, the dual differential map d^* is introduced by

$$< d^*\theta, \rho >\doteq< \theta, d\rho >\,, \qquad \rho \in \Gamma^\wedge\,,\ \theta \in \Gamma^{\wedge *}\,. \tag{40}$$

It commutes with elements of A^*,

$$d^* \circ h = h \circ d^*\,, \quad \text{i.e.,}\ d^*(h\theta) = h\, d^*\theta\,, \quad h \in A^*\,, \tag{41}$$

and transforms γ_i to χ_i:

$$\chi_i = d^*\gamma_i\,. \tag{42}$$

Both formulas are derived via duality:

$$< d^*(h\theta), \rho >=< h\theta, d\rho >=< h \otimes \theta, \Delta(d\rho) >=< h, \rho_{(1)} >< \theta, d\rho_{(2)} >$$

$$=< h, \rho_{(1)} >< d^*\theta, \rho_{(2)} >=< h \otimes d^*\theta, \Delta(\rho) >=< h\, d^*\theta, \rho > \tag{43}$$

(we used $< h, d\rho_{(1)} >= 0$), and

$$< d^*\gamma_i, a >=< \gamma_i, da >=< \gamma_i, (\chi_j \triangleright a)\omega^j >= \varepsilon(\chi_i \triangleright a)$$

$$= \varepsilon(a_{(1)}) < \chi_i, a_{(2)} >=< \chi_i, a >\,. \tag{44}$$

Further, to verify that the coproduct of γ_i is given by

$$\Delta(\gamma_i) = 1 \otimes \gamma_i + \gamma_j \otimes f_i^j \tag{45}$$

it suffices to compute its bracket with a general element in $A \otimes \Gamma + \Gamma \otimes A$:

$$< \Delta(\gamma_i) - 1 \otimes \gamma_i - \gamma_j \otimes f_i^j\,,\ a \otimes b\omega^k + c\omega^k \otimes e >$$

$$=< \gamma_i, ab\omega^k + c\omega^k e > -\varepsilon(a) < \gamma_i, b\omega^k > - < \gamma_j, c\omega^k >< f_i^j, e >$$

$$=< \gamma_i, c(f_m^k \triangleright e)\omega^m > -\varepsilon(c) < f_i^k, e >= \varepsilon(c)\,\varepsilon(f_i^k \triangleright e) - \varepsilon(c) < f_i^k, e >= 0\,,$$

where $a, b, c, e \in A$. A comparison of (45) with (25) displays a 'left appearance' of $\Delta(\gamma_i)$. Nevertheless, unlike the ω^i-case (25), we prefer not to use the words 'left invariance' here, to avoid confusion with the left invariance under A- coaction (18) appropriate to any object in $\Gamma^{\wedge *}$. However, a similarity of (45) and (25) enables one to show in a way quite analogous to [1] that any element $\theta \in \Gamma^*$ is uniquely represented in the form $\theta = h^i\gamma_i$, $h^i \in A^*$.

Now we are in a position to derive 10 commutation relations among $a \in A$, $\omega^i \in \Gamma$, $\chi_i \in A^*$ and $\gamma_i \in \Gamma^*$. Three of them are already present in the original Woronowicz theory. They are: internal multiplication rule inside the algebra A, eq. (24), and the recipe how to multiply ω^i. The latter is unambiguously fixed in the framework of Woronowicz's scheme [1] but generally cannot be written down in a closed form (see [2, 10, 12]). Another four,

$$\chi_i a - a\chi_i = (\chi_j \triangleright a) f^j_i \,, \tag{46}$$

$$\gamma_i a - a\gamma_i = 0\,, \tag{47}$$

$$\chi_i\omega^j - \omega^j\chi_i = C^j_{lk}\omega^l f^k_i \,, \quad C^j_{lk} \doteq < \chi_k, r^j_l >\,, \tag{48}$$

$$\gamma_i\omega^j + \omega^j\gamma_i = f^j_i \,. \tag{49}$$

are immediately obtained by the application of the cross-product rule (12) to $\Gamma^\wedge \rtimes (\Gamma^\wedge)^*$. The remaining commutation relations require the use of the duality arguments. Let us first derive a formula

$$\gamma_i h = (r^j_i \triangleright h)\gamma_j \equiv < h_{(2)}, r^j_i > h_{(1)}\gamma_j \,. \tag{50}$$

We have

$$< \gamma_i h\,, a\omega^k > = < \gamma_i \otimes h\,, a_{(1)} \otimes a_{(2)}\omega^k + a_{(1)}\omega^j \otimes a_{(2)} r^k_j >$$

$$= < \gamma_i\,, a_{(1)}\omega^j >< h\,, a_{(2)} r^k_j > = \delta^j_i \varepsilon(a_{(1)}) < h_{(1)}, a_{(2)} >< h_{(2)}, r^k_j >$$

$$= < r^k_i \triangleright h\,, a > = < r^j_i \triangleright h\,, a_{(1)} > \varepsilon(a_{(2)})\delta^k_j$$

$$= < r^j_i \triangleright h\,, a_{(1)} >< \gamma_j, a_{(2)}\omega^k > = < (r^j_i \triangleright h)\gamma_j\,, a\omega^k > \,. \tag{51}$$

Using (41) and (42), we come to analogous formula for χ_i:

$$\chi_i h = (r^j_i \triangleright h)\chi_j \equiv < h_{(2)}, r^j_i > h_{(1)}\chi_j \,. \tag{52}$$

This can be also proved by a direct calculation:

$$< \chi_i h\,, a > = < \chi_i, a_{(1)} >< h\,, a_{(2)} > = < h\,,\ a \triangleleft \chi_i > = < h\,,\ (\chi_j \triangleright a) r^j_i >$$

$$= < h_{(1)}, \chi_j \triangleright a >< h_{(2)}, r^j_i > = < h_{(1)}, a_{(1)} >< \chi_j, a_{(2)} >< h_{(2)}, r^j_i >$$

$$= < r^j_i \triangleright h\,, a_{(1)} >< \chi_j, a_{(2)} > = < (r^j_i \triangleright h)\chi_j, a > \,. \tag{53}$$

It is worth mentioning that the same technique leads to a helpful formula

$$f^j_i h = (r^k_i \triangleright h \triangleleft S^{-1}(r^j_m)) f^m_k \equiv < h_{(1)}, S^{-1}(r^j_m) > h_{(2)} f^m_k < h_{(3)}, r^k_i > \tag{54}$$

which can be used, in conjunction with (52), to deduce the structure relations of bicovariant differential calculus in the form given in [10, 25]:

$$\chi_i\chi_j - \sigma^{lk}_{ij}\chi_l\chi_k = C^k_{ij}\chi_k\,, \quad \sigma^{kl}_{ij} \doteq < f^k_j\,, r^l_i >\,, \tag{55}$$

$$\sigma^{mn}_{ij} f^i_k f^j_l = \sigma^{ij}_{kl} f^m_i f^n_j \,, \tag{56}$$

$$\chi_k f^n_l = \sigma^{ij}_{kl} f^n_i \chi_j \,, \tag{57}$$

$$C^i_{mn} f^m_j f^n_k + f^i_j \chi_k = \sigma^{mn}_{jk} \chi_m f^i_n + C^m_{jk} f^i_m \,. \tag{58}$$

Now we can list the remaining three commutational relations. One of them is (55), and the other two are as follows:

$$\gamma_i\chi_j - \sigma^{lk}_{ij}\chi_l\gamma_k = C^k_{ij}\gamma_k\,, \tag{59}$$

$$< \gamma_i\gamma_j\,, a\,\omega^m\omega^n > = \varepsilon(a)(\sigma^{mn}_{ij} - \delta^m_i\delta^n_j)\,. \tag{60}$$

Eq. (59) stems from (50), whereas (60) is verified by a straightforward calculation.

Thus, we have completed the explicit construction of the cross-product algebra generated by a, ω^i, χ_i and γ_i.

V. Cartan identity

Remarkably, this quantum algebra exhibits some features which exactly correspond to the well-known classical relations. First, the Lie derivatives commute with exterior differentiation:

$$\mathcal{L}_h \circ d = d \circ \mathcal{L}_h\,, \qquad h \in A^*\,. \tag{61}$$

Really, prove it for $a \in A$:

$$h \triangleright da = da_{(1)} < h\,, a_{(2)} > = d(h \triangleright a)\,. \tag{62}$$

Then, from

$$h \triangleright (db_1 \ldots db_n) = (h_{(1)} \triangleright db_1) \ldots (h_{(n)} \triangleright db_n) = d(h_{(1)} \triangleright b_1) \ldots d(h_{(n)} \triangleright b_n) \tag{63}$$

and the Leibniz rule it follows that

$$h \triangleright (d(a\,db_1 \ldots db_n)) = d(h \triangleright (a\,db_1 \ldots db_n))\,, \tag{64}$$

which is exactly (61).

Furthermore, the Cartan identity in the classical form can be shown to be valid:

$$\mathcal{L}_{\chi_i} = d \circ \imath_i + \imath_i \circ d\,. \tag{65}$$

One needs to verify that

$$\chi_i \triangleright \rho = d(\gamma_i \triangleright \rho) + \gamma_i \triangleright (d\rho)\,, \qquad \rho \in \Gamma^\wedge\,. \tag{66}$$

For $\rho = a \in A$ eq. (66) is almost trivial and follows from

$$\gamma_i \triangleright a = 0\,, \qquad \gamma_i \triangleright da = \chi_i \triangleright a\,. \tag{67}$$

Let $\rho = a\,db$ $(a, b \in A)$. To show that

$$\chi_i \triangleright (a\,db) = d(\gamma_i \triangleright a\,db) + \gamma_i \triangleright (da\,db) \tag{68}$$

we calculate each term separately,

$$\chi_i \triangleright (a\,db) = a_{(1)} db_{(1)} < 1 \otimes \chi_i + \chi_j \otimes f^j_i\,,\ a_{(2)} \otimes b_{(2)} >$$

$$= a(\chi_i \triangleright db) + (\chi_j \triangleright a)(f^j_i \triangleright db)\,, \tag{69}$$

$$d(\gamma_i \triangleright (a\,db)) = da\,(\chi_i \triangleright b) + a\,d(\chi_i \triangleright b)\,, \tag{70}$$

$$\gamma_i \triangleright (da\,db) = -da\,(\chi_i \triangleright b) + (\chi_j \triangleright a)(f^j_i \triangleright db)\,, \tag{71}$$

and then use (62).

At last, consider the general case $\rho = a\,db\,B\,,\ B = dc_1 \ldots dc_n$, where $a, b, \ldots, c_i \in A$:

$$\chi_i \triangleright (a\,db\,B) = a\,db\,(\chi_i \triangleright B) + a(\chi_j \triangleright db)(f^j_i \triangleright B) + (\chi_k \triangleright a)(f^k_j \triangleright db)(f^j_i \triangleright B)\,, \tag{72}$$

$$\gamma_i \triangleright (a\,db\,B) = -a\,db\,(\gamma_i \triangleright B) + a(\chi_j \triangleright b)(f^j_i \triangleright B)\,, \tag{73}$$

$$d(\gamma_i \triangleright (a\,db\,B)) = -da\,db\,(\gamma_i \triangleright B) + a(\chi_j \triangleright db)(f^j_i \triangleright B)$$

$$+\, a\,db\,d(\gamma_i \triangleright B) + da\,(\chi_j \triangleright b)(f^j_i \triangleright B) + a(\chi_j \triangleright b)\,d(f^j_i \triangleright B)\,, \tag{74}$$

$$\gamma_i \triangleright (da\,db\,B) = da\,db\,(\gamma_i \triangleright B) - da\,(\chi_j \triangleright b)(f^j_i \triangleright B)$$

$$+\, (\chi_k \triangleright a)(f^k_j \triangleright db)(f^j_i \triangleright B)\,. \tag{75}$$

After summing this up, it remains to prove that

$$a\,db\,(\chi_i \triangleright B) = a\,db\,d(\gamma_i \triangleright B)\,, \tag{76}$$

or

$$\chi_i \triangleright (dc_1 \ldots dc_n) = d(\gamma_i \triangleright (dc_1 \ldots dc_n))\,, \tag{77}$$

that is the same problem at a lower level. Thus, the proof is completed by induction.

To conclude this section, we compare the duality $< \Gamma^*, \Gamma >$ used above ('vertical' duality in (36) between 1-forms and inner derivations) with a duality $<< A^*, \Gamma >>$ between vector fields $\in A^*$ and differential 1-forms $\in \Gamma$. The latter is a natural generalization of ordinary classical duality, and is assumed as a basis of an alternative construction of bicovariant differential calculus on the Hopf algebras in [22]. It is easily seen that the dual differential map d^* establishes a direct relation between these two dualities in the following way:

$$<< d^*\theta, \rho >> = < \theta, \rho >\,, \qquad \theta \in \Gamma^*,\ \rho \in \Gamma\,. \tag{78}$$

VI. Comparison with other approaches

Now the above results (mostly, the commutation relations (46)–(49), (55), (59) and (60)) are to be compared with other approaches known in the literature [3, 11, 14, 12]. To achieve this, it is convenient to chose another set of generators for the $\Gamma^{\wedge *}$-part of our cross-product algebra. We switch from χ_i, γ_i to $\tilde{\chi}_i, \tilde{\gamma}_i$ defined by

$$da = \omega^i(\tilde{\chi}_i \triangleright a)\,, \tag{79}$$

$$< \tilde{\gamma}_i, \omega^j a >= \varepsilon(a)\delta^j_i\,, \quad < \tilde{\gamma}_i, \rho >= 0\,, \qquad \rho \in A, \Gamma^2, \Gamma^3, \cdots\ . \tag{80}$$

Introducing also $\varphi^i_j \in A^*$ via

$$a\omega^j = \omega^i(\varphi^j_i \triangleright a) \tag{81}$$

and proceeding by complete analogy with Sect. 3 and 4, we obtain

$$\varphi^j_i = S^{-1}(f^j_i)\,, \quad \tilde{\gamma}_i = \sigma^{mj}_{mk}\gamma_j\varphi^k_i\,, \quad \tilde{\chi}_i = \varphi^j_i\chi_j = d^*\tilde{\gamma}_i\,, \tag{82}$$

$$\Delta(\varphi^i_j) = \varphi^k_j \otimes \varphi^i_k\,, \quad \Delta(\tilde{\gamma}_i) = \tilde{\gamma}_i \otimes 1 + \varphi^j_i \otimes \tilde{\gamma}_j\,, \quad \Delta(\tilde{\chi}_i) = \tilde{\chi}_i \otimes 1 + \varphi^j_i \otimes \tilde{\chi}_j\,. \tag{83}$$

As for commutational relations, in the $\{a, \omega\}$-sector they remain unchanged, those between a, ω and χ, γ follow directly from (12),

$$\tilde{\chi}_i a - (\varphi^j_i \triangleright a)\tilde{\chi}_j = \tilde{\chi}_i \triangleright a\,, \tag{84}$$

$$\tilde{\gamma}_i a - (\varphi^j_i \triangleright a)\tilde{\gamma}_j = 0\,, \tag{85}$$

$$\tilde{\chi}_i \omega^j - \tilde{\sigma}^{jl}_{ik}\omega^k \tilde{\chi}_l = \tilde{C}^j_{ki}\omega^k , \tag{86}$$

$$\tilde{\gamma}_i \omega^j + \tilde{\sigma}^{jl}_{ik}\omega^k \tilde{\gamma}_l = \delta^j_i , \tag{87}$$

where

$$\tilde{\sigma}^{ki}_{jl} \doteq < \varphi^i_j , r^k_l > = (\sigma^{-1})^{ki}_{jl} , \quad \tilde{C}^i_{jk} \doteq < \tilde{\chi}_k , r^i_j > = C^i_{sl}(\sigma^{-1})^{sl}_{kj} , \tag{88}$$

and those inside $\{\chi, \gamma\}$ look like

$$\tilde{\gamma}_i \tilde{\chi}_j - \tilde{\sigma}^{kl}_{ji}\tilde{\chi}_l \tilde{\gamma}_k = \tilde{C}^k_{ij}\tilde{\gamma}_k , \tag{89}$$

$$\tilde{\chi}_i \tilde{\chi}_j - \tilde{\sigma}^{kl}_{ji}\tilde{\chi}_l \tilde{\chi}_k = \tilde{C}^k_{ij}\tilde{\chi}_k , \tag{90}$$

$$< \tilde{\gamma}_i \tilde{\gamma}_j , \omega^m \omega^n a > = \varepsilon(a)(-\tilde{\sigma}^{mn}_{ji} + \delta^m_j \delta^n_i) . \tag{91}$$

Formulas (89),(90) are obtained with the use of

$$\tilde{\chi}_i h = (h \triangleleft r^j_i)\tilde{\chi}_j , \quad \tilde{\gamma}_i h = (h \triangleleft r^j_i)\tilde{\gamma}_j , \tag{92}$$

that can be derived similarly to (50),(52). The resulting cross-commutation algebra conforms to Schupp's paper [14].

VII. R-matrix formulation of differential calculus on $GL_q(N)$

To compare our formulas with analogous relations in [3], we consider a specific realization [9] of Woronowicz's differential calculus in case of the quantum group $GL_q(N)$, and use the matrix representations for all generators. Here A, A^* will be the dual Hopf algebras [26] described by the relations

$$R_{12}T_1T_2 = T_2T_1R_{12} , \quad \Delta(T) = T \otimes T , \quad \varepsilon(T) = \mathbf{1} , \tag{93}$$

$$R_{12}L^{\pm}_2 L^{\pm}_1 = L^{\pm}_1 L^{\pm}_2 R_{12} , \quad R_{12}L^{+}_2 L^{-}_1 = L^{-}_1 L^{+}_2 R_{12} , \tag{94}$$

$$\Delta(L^{\pm}) = L^{\pm} \otimes L^{\pm} , \quad \varepsilon(L^{\pm}) = \mathbf{1} , \tag{95}$$

$$< T_1 , L^{+}_2 > = R_{12} , \quad < T_1 , L^{-}_2 > = R^{-1}_{21} , \tag{96}$$

$$< T_1 , S(L^{+}_2) > = R^{-1}_{12} , \quad < T_1 , S(L^{-}_2) > = R_{21} \tag{97}$$

(generators $t^j_i \in A$ and $l^{\pm j}_i \in A^*$ form matrices T and $L^{\pm}$, respectively), where R is a special numerical matrix related to $GL_q(N)$ [26] which obeys the Yang-Baxter

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{98}$$

and Hecke

$$R^{ij}_{pq} = (R^{-1})^{ji}_{qp} + \lambda \delta^i_q \delta^j_p \qquad (\lambda = q - q^{-1}) \tag{99}$$

conditions. Let us also introduce a numerical matrix D by

$$D^i_j \doteq \tilde{R}^{mi}_{jm} , \quad R^{mj}_{pn}\tilde{R}^{in}_{mq} = \tilde{R}^{mj}_{pn}R^{in}_{mq} = \delta^i_p \delta^j_q , \tag{100}$$

and fix the differential map $d : A \to \Gamma$ via

$$dT = T\,\Omega \tag{101}$$

in terms of left-invariant Maurer-Cartan forms Ω. Then the Woronowicz bicovariant differential calculus on $GL_q(N)$ is produced by the following choice [9, 10, 12] of the elements r, f, χ:

$$r^{li}_{kj} = S(t^i_k)t^l_j , \quad f^{il}_{jk} = l^{-i}_{\ k} S(l^{+l}_{\ j}) , \quad \chi^l_k = \frac{1}{\lambda}[\,(D^{-1})^l_k - (D^{-1})^j_i f^{il}_{jk}\,] , \tag{102}$$

which serve to define the Hopf and differential structure of the calculus as follows (note doubling the indices due to the matrix format used):

$$\Delta(\Omega^i_j) = 1 \otimes \Omega^i_j + \Omega^k_l \otimes r^{li}_{kj} , \tag{103}$$

$$\Omega^i_j t^m_n = (f^{il}_{jk} \triangleright t^m_n)\Omega^k_l , \tag{104}$$

$$dt^m_n = (\chi^p_q \triangleright t^m_n)\Omega^q_p = t^m_k \Omega^k_n \tag{105}$$

(the last equation implies $< \chi^p_q , t^m_n > = \delta^p_n \delta^m_q$). From (82) and (102) we get

$$\varphi^{il}_{jk} = l^{+l}_{\ j} S^{-1}(l^{-i}_{\ k}) , \quad \tilde{\chi}^l_k = \frac{1}{\lambda}[\,(D^{-1})^i_j \varphi^{jl}_{ik} - (D^{-1})^l_k\,] \tag{106}$$

and can now write down all commutational relations in the matrix form.

If, before doing so, we perform one more redefinition,

$$J = -\tilde{\gamma}D , \quad X = -\tilde{\chi}D , \quad Y = 1 - \lambda X , \tag{107}$$

so that

$$Y^i_j = l^{+i}_{\ k} S(l^{-k}_{\ j}) , \tag{108}$$

we end up with a complete set of commutation relations in terms of matrices T, Ω, Y and J:

$$R_{12}T_1T_2 = T_2T_1R_{12}\,, \tag{109}$$
$$\Omega_1T_2 = T_2R_{12}^{-1}\Omega_1R_{21}^{-1}\,, \tag{110}$$
$$\Omega_1R_{21}^{-1}\Omega_2R_{21} = -R_{21}^{-1}\Omega_2R_{12}^{-1}\Omega_1\,, \tag{111}$$
$$Y_1T_2 = T_2R_{21}Y_1R_{12}\,, \tag{112}$$
$$\Omega_1R_{12}Y_2R_{21} = R_{12}Y_2R_{21}\Omega_1\,, \tag{113}$$
$$J_1T_2 = T_2R_{21}J_1R_{12}\,, \tag{114}$$
$$\Omega_1R_{12}J_2R_{21} + R_{12}J_2R_{21}\Omega_1 = \frac{1}{\lambda}(1 - R_{12}R_{21})\,, \tag{115}$$
$$Y_1R_{12}Y_2R_{21} = R_{12}Y_2R_{21}Y_1\,, \tag{116}$$
$$J_1R_{12}Y_2R_{21} = R_{12}Y_2R_{21}J_1\,, \tag{117}$$
$$J_1R_{12}J_2R_{21} = -R_{21}^{-1}J_2R_{21}J_1\,. \tag{118}$$

Several comments are in order. In this specific realization of the Woronowicz calculus, it proves possible to present multiplication relations for Ω in a closed form (111). The commutation rule (116) for Y is often called the *reflection equation* [27, 28, 29], and the related formula for X

$$X_1R_{12}X_2R_{21} - R_{12}X_2R_{21}X_1 = \lambda^{-1}(X_1R_{12}R_{21} - R_{12}R_{21}X_1) \tag{119}$$

– the *quantum Lie algebra* [1, 30, 7, 31, 32], because it generalizes classical commutator in the Lie algebra of left-invariant vector fields. In terms of T and Y, the left and right A-coactions in (18) and (23) take the form

$$\text{left:} \qquad t^i_j \longrightarrow t^i_k \otimes t^k_j\,, \quad Y^i_j \longrightarrow \mathbf{1} \otimes Y^i_j\,, \tag{120}$$
$$\text{right:} \qquad t^i_j \longrightarrow t^i_k \otimes t^k_j\,, \quad Y^i_j \longrightarrow Y^k_l \otimes S(t^i_k)t^l_j\,. \tag{121}$$

This shows explicitly that the algebra A^* proves to be left-invariant and right-coadjoint-covariant.

Algebra (109)–(118) is exactly the $GL_q(N)$ bicovariant differential algebra found in [3] and discussed further in [12]. We have shown that it is produced just by application of the cross-product recipe to the original Woronowicz differential complex, whose Hopf-algebra properties account for bicovariance of the algebra.

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