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B.S. Getmanov[1], Paul M. Sutcliffe[2]

YANG—MILLS—HIGGS SOLITONS DYNAMICS IN (2+1)-DIMENSIONS

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[1]E-mail address: bget@get.jinr.dubna.su
[2]Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, U.K.
E-mail address: paul@cara.ma.hw.ac.uk

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1. Introduction

Soliton solutions are important aspects of many field theories in both classical and quantum physics. A useful classification of such soliton systems is provided by the notion of integrability. Integrable equations occur mainly in (1+1)-dimensions and are characterised by a number of special properties, such as possessing an infinite number of conserved quantities. Multi-soliton solutions may be constructed explicitly, using methods such as the inverse scattering transform. The generalisation of integrable system to higher dimensions has been less fruitful. There are limited examples of integrable system in (2+1)-dimensions, but these are non-relativistic.

Turning now to non-integrable systems there are many examples of higher dimensional solitons, such as skyrmions, monopoles, vortices and lumps. Such theories are relativistic and the solitons have a topological nature. The non-integrability of these systems means that numerical simulations and analytical approximations must be used to study soliton dynamics. Given these limitations a remarkable amount is known about, for example, the dynamics of BPS monopoles.[1,2,3] Such studies reveal that soliton dynamics in these systems is highly non-trivial.

One important example of a higher dimensional integrable system is the self-dual Yang-Mills (sd YM) equation. Work in this equation and its dimensional reduction has not only led to a unification of known low dimensional integrable systems[4] but has also provided new examples of higher dimensional integrable equations.[5] It is in the study of one such integrable model in (2+1)-dimensions that a recent result[6] has demonstrated that soliton dynamics can be highly non-trivial in integrable models. This suggests an area in which new phenomena may occur that are not present in lower dimensions and provides a connection between the solitons of integrable and non-integrable systems. In this paper we construct monopole-like solutions of an integrable relativistic Yang-Mills-Higgs equations. Even though the equation is integrable we find monopole dynamics can be highly non-trivial.

2. Instanton construction of the BPS monopole

In this section we give a brief summary of Manton's instanton construction of the BPS monopole.[7] Consider an SU(2) gauge theory in euclidean four-space. The spacetime coordinates are x_μ, $\mu = 0, 1, 2, 3$, with metric $\eta^{\mu\nu} = \delta^{\mu\nu}$. Let A_μ be the $su(2)$ – valued gauge potential with gauge field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu , A_\nu]$. Then the self-duality equations (sdYM) are

$$F_{\mu\nu} = \frac{1}{2} e_{\mu\nu\alpha\beta} F^{\alpha\beta} \tag{2.1}$$

where $e_{\mu\nu\alpha\beta}$ is the totally antisymmetric tensor. If we dimensionally reduce by requiring that all gauge potentials are independent of the x_3 coordinate, *ie* $\partial_3 A_\mu = 0$, and identify the residual gauge potential with the Higgs field, *ie* $\Phi = A_3$. then (2.1) becomes the Bogomolny equation for static BPS monopoles in three-space

$$D_\mu \Phi = -\frac{1}{2} e_{\mu\alpha\beta} F^{\alpha\beta} \tag{2.2}$$

where $D_\mu = \partial_\mu + [A_\mu,$ is the covariant derivative and indices range over the values 0, 1, 2.

Solutions of the sdYM equations may be obtained from the Corrigan-Fairlie-t'Hooft-Wilczek (CFtHW) ansatz[8,9,10,11]

$$A_\mu = \frac{1}{2} \sigma_{\mu\nu} \partial^\nu \log \rho \tag{2.3}$$

where $\sigma_{\mu\nu} = (e_{0\mu\nu\alpha} + \delta_{\mu\alpha}\delta_{0\nu} - \delta_{0\mu}\delta_{\nu\alpha})\sigma^\alpha$ and σ_α, $\alpha = 1, 2, 3$ are the Pauli matrices. Here $\rho(x_\mu)$ is a real function called the superpotential. By substitution of the ansatz (2.3) into (2.1) we obtain a solution of the sdYM equations if the superpotential satisfies

$$\Box \rho = 0 \tag{2.4}$$

where $\Box = \partial^\mu \partial_\mu$ is the wave operator in (4+0)-dimensions.

In order to dimensionally reduce to the monopole equation the gauge potentials must be independent of x_3. This can be achieved by setting

$$\rho = e^{\lambda x_3} \zeta \; (x_0, x_1, x_2) \tag{2.5}$$

where λ is a real constant. The equation for ζ is then the Helmoholtz equation

$$\nabla^2 \zeta + \lambda^2 \zeta = 0 \tag{2.6}$$

where ∇^2 is the wave operator in (3+0)-dimensions. However, solutions of (2.6) lead to singular gauge potentials, since ρ must not vanish for the gauge fields to be nonsingular. However, if the replacement $\lambda \to i\lambda$ is made then (2.6) becomes

$$\nabla^2 \zeta - \lambda^2 \zeta = 0 \tag{2.7}$$

the solutions of which lead to non-singular but complex gauge potentials. The remarkable result is the following. A radially symmetric solution to (2.7) is given by

$$\zeta = \frac{1}{r} \sinh \lambda r \tag{2.8}$$

and although the solution is complex the gauge potentials can be transformed by complex gauge transformation into the real BPS monopole solution.[7] It is this construction of the BPS monopole which motivates the construction in paper of monopoles in (2+1)-dimensions.

This construction of the BPS monopole can also be reformulated in way that avoids the complexification and in which the monopole may be interpreted as arising from an infinite instanton chain.[12] This interpretation has a two-dimensional analogue in which sine-Gordon solitons may be infinite limits of CP^1 instanton chains .[13]

3. The Yang-Mills-Higgs-Bogomolny equation

Let us now consider the sdYM equation (2.1) in (2+2)-dimensions. Write the coordinates x_μ, $\mu = 0,1,2,3$ as $x_\mu = (t, x, y, s)$ and let the metric be

$$dx_\mu dx^\mu = dt^2 - dx^2 - dy^2 + ds^2 \tag{3.1}$$

with t and s timelike coordinates and x and y spacelike. If we now dimensionally reduce by requiring that all gauge potentials be independent of the second time coordinate s, *ie* $\partial_s A_\mu = 0$, and identify the residual gauge potential with the Higgs field, *ie* $\Phi = A_s$ then the sdYM equation becomes

$$D_\mu \Phi = \frac{1}{2} e_{\mu\alpha\beta} F^{\alpha\beta} \tag{3.2}$$

where the metric is the Minkowski metric $g^{\mu\nu} = diag(1,-1,-1)$.

This is an integrable Yang-Mills-Higgs-Bogomolny (YMHB) equation in (2+1) dimensions which is a Minkowsky analogue of the Bogomolny equation (2.2) for static monopoles in euclidean three-space. Note that (3.2) has the desirable property of Lorentz invariance, i.e. it has an SO(2,1) spacetime symmetry. The Bogomolny equation (2.2) is a first order (in the gauge potentials) equation which gives a subset of solutions to a second order equation derived from a lagrangian. This lagrangian consists of a term quadratic in the gauge field and a term quadratic in the covariant derivative of a Higgs field. In the same way the YMHB equation (3.2) may also be considered as giving a subset of solutions of a second order equation derived from a lagrangian. The lagrangian has the same form as that associated with the Bogomolny equation but this time the sign between the two contributing terms is opposite. One consequence of this is that the energy is not positive definite. In fact it is easily seen that for solutions of (3.2) this energy vanishes identically. Note that a useful gauge invariant quantity to describe the system is given by the length of the Higgs field $\|\Phi\|^2 = -\det(\Phi)$, which in general will be not vanishing.

Soliton solutions of (3.2), for a gauge group $\mathcal{G}$ = SU(2), representing the dynamics of multi-lump configurations have already been constructed using twistor methods[14,15] and (using a chiral field formulations which breaks the Lorenz invariance) through the use of the Riemman method with zeros.[5] In the chiral field formulation numerical simulations have shown that solutions also exist in which the dynamics and scattering of such lump solitons is highly non-trivial and exotic,[6] even though the equation is integrable.

The equation (3.2) is integrable for any choice of gauge group, so that we could for example consider a complex gauge group such as $\mathcal{G}$ = SL(2,c). The previous studies mentioned above chose the real form $\mathcal{G}$ =SU(2), and in this paper we again choose a real form but here we consider $\mathcal{G}$ =SU(1,1). The gauge potentials and Higgs field are therefore $su(1,1)$-valued so that $A_\mu = A_\mu^\alpha T^\beta \delta_{\alpha\beta}$, $\Phi = \Phi^\alpha T^\beta \delta_{\alpha\beta}$ where T^α, $\alpha = 0,1,2$ are the generators of $su(1,1)$ given by $T^0 = \frac{i}{2}\sigma_3$, $T^1 = \frac{1}{2}\sigma_1$, $T^2 = \frac{1}{2}\sigma_2$.

In the following selection we shall construct monopole-like solution of (3.2) through the use of a CFtHW-like ansatz. Solutions which describe static deformed monopoles and exotic dynamics are also given.

4. Monopole Solutions

We now introduce a Lorentz covariant analogue of the CFtHW ansatz for the case of an SU(1,1) gauge group after reduction to (2+1)-dimensions. We define an orthonormalised basis in (2+1)-dimensional Minkowski spacetime M^3 given by the three vectors $n_\mu{}^\alpha$, $\mu, \nu \ldots = 0, 1, 2$ in M^3 and $\alpha, \beta \ldots = 0, 1, 2$ in internal space. The vectors are defined by the properties $g_{\alpha\beta}\, n_\mu{}^\alpha\, n_\nu{}^\beta = g_{\mu\nu}$, $g^{\mu\nu}\, n_\mu{}^\alpha\, n_\nu{}^\beta = g^{\alpha\beta}$, $e_{\alpha\beta\gamma}\, n_\mu{}^\alpha\, n_\nu{}^\beta\, n_\sigma{}^\gamma = e_{\mu\nu\sigma}$. We look for solutions to (3.2) in the form

$$\begin{aligned} A_\nu^\alpha &= n_\mu^\alpha\, e^\mu_{\nu\sigma}\, \partial^\sigma \log\rho \\ \Phi^\alpha &= n_\mu^\alpha\, \partial^\mu \log\rho \end{aligned} \tag{4.1}$$

where $\rho(t, x, y)$ is the real-valued superpotential. Substitution of the ansatz (4.1) into (3.2) gives a solution if the superpotential satisfies

$$\Box\rho = 0 \tag{4.2}$$

where $\Box = \partial_t^2 - \partial_x^2 - \partial_y^2$ is the wave operator in (2+1)-dimensions. With this ansatz the length of the Higgs field has a simple expression in terms of the superpotential, namely

$$\|\Phi\|^2 = -\frac{1}{4}\Box\log\rho\,. \tag{4.3}$$

We now construct our real monopole-like solutions. Following [16] for notational convenience denote the timelike vector $n^0_\mu = n_\mu$, and the spacelike vector $n^i_\mu = k^i_\mu, i = 1, 2$, and introduce the variable $\omega = \sqrt{(k^i \tilde{x})^2} = \sqrt{(n\tilde{x})^2 - \tilde{x}^2_\mu}$ where $\tilde{x}_\mu = x_\mu - x^{(0)}_\mu$, with constants $x^{(0)}_\mu$, and (ab) denotes the spacetime inner product $a_\mu b^\mu$. Also let $\tau = (n\tilde{x})$. We choose the non-singular solution of (4.2)

$$\rho = e^{\delta\lambda\tau} I_0(\lambda\omega) \tag{4.4}$$

where λ is real constant, δ = ±1 and I_p denotes the modified Bessel function of order p (see [17] for a summary of the properties of I_p); it produces the monopole-like solution in an arbitrary reference frame

$$A^a_\nu = \lambda n^a_\mu e^\mu_{\nu\sigma} \left(\frac{I_1(\lambda\omega)}{I_0(\lambda\omega)} \partial^\sigma \omega + \delta n^\sigma \right)$$

(4.5)

$$\Phi^a = \lambda n^a_\sigma \left(\frac{I_1(\lambda\omega)}{I_0(\lambda\omega)} \partial^\sigma \omega + \delta n^\sigma \right).$$

The Higgs density for this solution is given by

$$\|\Phi\|^2 = \frac{\lambda^2}{4} \left(1 - \left(\frac{I_1(\lambda\omega)}{I_0(\lambda\omega)} \right)^2 \right). \tag{4.6}$$

The velocity of the monopole is given by $\boldsymbol{v} = \boldsymbol{n} / n_0$ where $\boldsymbol{n} = (n_1, n_2)$. We use the standard parametrisation $n = (\cosh\beta, \sinh\beta\cos\phi, \sinh\beta\sin\phi)$, in terms of the rapidity β and angle ϕ. The static monopole solution is obtained by choosing the rest frame $n_\mu = (1,0,0)$, upon which $(k^i k^j) = -\delta^{ij}$, $i, j = 1, 2$, and $\omega = \sqrt{(x_1 - x^{(0)}_1)^2 + (x_2 - x^{(0)}_2)^2} = |r - r_0|$. In **fig. 1a** we plot the Higgs density (4.6), for a static monopole with $\lambda = 1$ and $x^{(0)}_\mu = 0$. (In all plots we present in this paper the x and y axes with both have the range [-40, 40]). We see that $\|\Phi\|^2 \to 0$ as $r \to \infty$. In fact the asymptotic behaviour for the static monopole solution is

$$\|\Phi\|^2 \sim \frac{\lambda}{4r} \quad \text{as } r \to \infty. \tag{4.7}$$

We call such a solution a monopole with scale λ. Note that this monopole does not have the BPS limit boundary condition ($\|\Phi\|^2 \to 1$ as $r \to \infty$) and is not a topological soliton.

Let us now consider the solution that can be obtained through a linear superposition in ρ of monopole solutions. One may at first think that this would generate multimonopole solutions, however this is not the case.

The general "N - soliton - like" superpotential is

$$\rho = \sum_{a=1}^{N} e^{\delta_a \lambda_a \tau_a} I_0(\lambda_a \omega_a)$$

where (4.8)

$$\omega_a = \sqrt{(n_a[x - x^{(0)}_a])^2 - [x_\mu - x^{(0)}_{\mu a}]^2}$$
$$\tau_a = (n_a[x - x^{(0)}_a]).$$

A set of vectors $n^a_{\mu a}$, is introduced for each soliton like component $a = 1, \ldots, N$. This solution has $6N$ real parameters giving the scale, speed, angle of motion and spacetime location (at $t = 0$) for each soliton-like component, and the set of parameters $\delta_a = \pm 1$. Again the solution is non-vanishing so that the gauge potential are non-singular.

Here we analyse the case $N = 2$ in detail. Denote $\delta_1 / \delta_2 = \delta$. We first consider the case $\boldsymbol{v}_1 = \boldsymbol{v}_2 = 0, \lambda_1 = \lambda_2 = \lambda, \delta = 1$ for which the solution (4.8) gives static gauge potentials. If $x^{(0)}_1 = x^{(0)}_2 = x^{(0)}$ and $y^{(0)}_1 = y^{(0)}_2 = y^{(0)}$ then the solution represents a monopole with scale λ and position $((x^{(0)}, y^{(0)})$ (note that multiplication of ρ by a constant has no effect on the gauge potentials or Higgs field). If we now allow $y^{(0)}_1 \neq y^{(0)}_2$ we find that the solution represents a static deformed monopole. The Higgs density is always localised in a single region (never in two distinct regions) and the maximum of the Higgs density is located at a point on the line $x = x^{(0)}$ between the points $y = y^{(0)}_1$ and $y = y^{(0)}_2$. The width of the monopole is decreased in the y-direction and increased in the x-direction.

Fig.1b shows a plot of the Higgs density for the parameter values $\lambda=1, x^{(0)}=0$ and $y_1^{(0)}=-y_2^{(0)}=5$.

Let us now consider time dependent solution. From now on we shall set $\delta_1=1$ (the case $\delta_1=-1$ can be obtained by time reversal). To begin with we shall investigate the radially symmetric case. This requires the parameter choice $x_1^{(0)}=x_2^{(0)}=y_1^{(0)}=y_2^{(0)}=\mathbf{v}_1=\mathbf{v}_2=0$. For $\delta=1$ (which then requires $\lambda_1\neq\lambda_2$ for the gauge potential to be time-dependent) we find that the Higgs density represents an undeformed monopole for all time, with the scale of the monopole being a monotonically increasing function of t. The monopole scale $\lambda(t)$ as a function of time has the asymptotic limits $\lambda(t=-\infty)=\min\{\lambda_1,\lambda_2\}$ and $\lambda(t=+\infty)=\max\{\lambda_1,\lambda_2\}$. Clearly the arguments of the exponential factors in the two terms of (4.8) are responsible for this behaviour. So such a solution describes the dynamical evolution of size changing monopole. But this is not all; there is also a time asymmetric phenomenon. For negative times the monopole is accompanied by a radial ring which contracts as time increases and vanishes for positive times. It would be nice to interpret this ring as radiation, but this is not the case, since the speed of the rings contraction is not equal to unity. In **fig.2** we plot the Higgs density at increasing time for the solution with parameter values $\lambda_1=0.4$ and $\lambda_2=1.0$. We also show the first plot (t=-20) magnified so that the ring is clearly visible. If we now consider the radially symmetric case with $\delta=-1$ (see **fig.3**), we find that as $|t|\to\infty$ we again obtain a similar result to the $\delta=1$ case, with one or other of the superposed monopole solutions dominating. However, as we would expect by examining the arguments of the exponentials in (4.8), we now find that as $t\to-\infty$ a monopole of scale λ_1 is obtained and as $t\to+\infty$ a monopole of scale λ_2 is obtained. Again there is also a radial ring which contracts as time increases and vanishes for positive times. This ring differs from the $\delta=1$ case in that it is a ring of negative Higgs density and has a much greater magnitude.

We now remove the restriction of radial symmetry and consider the general solution (4.8). For $\delta=-1$ the generic situation is that for large positive times the solution describes a monopole with parameter values associated with the second term of (4.8). For large negative times the solution represents a monopole with parameter values determined by the first term of (4.8). Near t=0 the solution can be quite complicated and not resemble a monopole at all. Note that such a solution can therefore describe the dynamical evolution of a transmuting monopole, which can change scale, position, speed and direction. In **fig.4** we plot the Higgs density for increasing times for a solutions with parameter values $x_1^{(0)}=x_2^{(0)}=y_1^{(0)}=y_2^{(0)}=0$, $\lambda_1=\lambda_2=1$, $\upsilon_1=0.1$, $\phi_1=0$, $\upsilon_2=0.2$, $\phi_2=\pi/2$ and $\delta=-1$. This describes a monopole which moves along the x-axis and transforms into a monopole moving along the y-axis with double the speed.

Turning to the general case with $\delta=1$ we find that the situation is much more complicated. In particular the solution never (unless $\upsilon_1=\upsilon_2=0$)describes undeformed monopoles, even in the limit $|t|\to\infty$. The $\delta=1$ case is therefore very difficult to interpret. In **fig.5** we show a plot for one of the simplest cases, in order to demonstrate the exotic nature of the solution. The parameters are $x_1^{(0)}=x_2^{(0)}=y_1^{(0)}=y_2^{(0)}=\phi_1=\phi_2=0$, $\lambda_1=\lambda_2=1$, $\upsilon_1=-\upsilon_2=0.9$ and $\delta=1$. In this plot (unlike the previous plots) the scale of the z-axis is not the same for each time plot. In fact there is a dramatic increase in scale of the structures with increasing time. For the Higgs density consists of a plane wave which grows taller and thinner with time. For large negative times the situation is even more exotic. The only comment we make is that there are (after magnification) two visible structures which are located at the points where one may naively expect to find two monopoles given the solution (4.8). Whether this is relevant or not we are unable to say.

5. Twistor construction of the planar monopole

In this section we show how our monopole solution has a surprisingly simple form in terms of the twistor construction. Not only is the simplicity of the solution interesting, but providing a twistor formulation may prove to be a useful first step in the construction of multi-monopole solution.

The well known twistor correspondence for self-dual Yang-Mills fields in 4-dimensional spacetime is that they correspond to certain holomorphic vector bundles over the standard complex 3-dimensional twistor space.[18] Now since the YHMB equation (3.2) is a reduction of the self-duality equations there is a reduced version of the standard twistor correspondence to 3-dimensional spacetime; namely that gauge fields satisfying (3.2) correspond to certain holomorphic vector bundles over a mini-twistor space[19,14,20] Π , which is a 2-dimensional complex manifold isomorphic to the holomorphic tangent bundle to the Riemann sphere, i.e. $\Pi \cong T\,\mathbb{C}P^1$.

Here this correspondence will be briefly described and then used explicitly in order to obtain the monopole solution to (3.2).

The space Π is a fibre bundle over $\mathbb{C}P^1$ with each fibre being a copy of $\mathbb{C}$. Let ξ be the standard coordinates on the base space $\mathbb{C}P^1$. Cover this base space with the two coordinate patches $U = (\xi : |\xi| \leq 1)$ and $\hat{U} = (\xi : |\xi| \geq 1)$. The fibre coordinates $\xi\ \gamma$ over U and $\hat{\gamma}$ over $\hat{U}$ are patched by $\hat{\gamma} = \xi^{-2}\gamma$. A reality structure is introduced by defining an anti-holomorphic involution on the base space $\sigma(\xi) = \xi^{-1}$, which may then be extended to Π by defining $\sigma : (\xi, \gamma) \to (\bar{\xi}^{-1}, \bar{\gamma})$. The real sections (i.e. those preserved by the involution) are then given by

$$\gamma = -\frac{i}{2} z\,\xi + t + \frac{i}{2}\bar{z}\,\xi^{-1} \qquad (5.1)$$

where $z = x + iy \in \mathbb{C}$, $t \in \mathbb{R}$.

Solutions of (3.2) then correspond to rank two holomorphic vector bundles E over Π satisfying the condition that E is trivial when restricted to real sections. E is also required to have a reality structure, as described below, in order to ensure that the gauge fields are $su(1,1)$-valued.

Let F be the 2×2 patching matrix which patches $E|_U$ to $E|_{\hat{U}}$. Then the required reality structure is that F must satisfy

$$F^{\dagger} = -1 \qquad (5.2)$$
$$\det F = 1$$

where $F^{\dagger}(\xi, \gamma) = F(\xi^{-1}, \bar{\gamma})^{*}$, and $*$ denotes complex conjugate transpose.

The gauge potentials and Higgs field are extracted from F by splitting it into the form

$$F = \hat{H} H^{-1} \qquad (5.3)$$

where H is holomorphic in U and $\hat{H}$ is holomorphic in $\hat{U}$. The choice of H and $\hat{H}$ is not unique and corresponds to a choice of gauge.

For the purpose of constructing the monopole solution the patching matrix may be taken to have the form of the Atiyah-Ward ansatz A_n[21]. Namely

$$F(\xi, \gamma) = \begin{pmatrix} \xi^{n} & \Gamma(\xi,\gamma) \\ 0 & \xi^{-n} \end{pmatrix}, \qquad (5.4)$$

where n is positive integer and Γ is an element of the cohomology group $H^1(\Pi, \mathcal{O}(-2n))$.

This patching matrix does not satisfy the reality condition (5.2), but for some Γ it may be equivalent to one which does. Namely, there may exist a unimodular matrix K which is holomorphic on $\hat{U}$, such that $\tilde{F} = KF$ satisfies (5.2). Multiplication by K simply amounts to a change of coordinates in the bundle and leaves the gauge fields unaffected. In particular if Γ is real, in the sense that $\Gamma^{\dagger} = \Gamma$, then $\tilde{F} = i\sigma_1 F$ satisfies (5.2).

For the ansatz A_1 the Higgs field has the simple form

$$\|\Phi\|^2 = -\frac{1}{4}\,\Box \log \Delta \qquad (5.5)$$

where Δ is given by

$$\Delta = \frac{1}{2\pi i}\oint \frac{\Gamma(\xi,\gamma)}{\xi}\, d\xi \qquad (5.6)$$

and the contour of integration is $|\xi| = 1$. Note that Δ automatically satisfies the wave equation, $\Box\Delta = 0$, and corresponds (in some gauge) to the superpotential.

Having set up the twistor machinery we find that the static monopole solution with scale λ is obtained from the surprisingly simple function

$$\Gamma = e^{\lambda\gamma}. \qquad (5.7)$$

Note that this function is real, so that the gauge potentials are $su(1,1)$-valued in some gauge. The contour integration (5.6) can be performed by making use of the generating function for the modified Bessel function,

$$G(u,k) = \exp\left(\frac{u}{2}(k + k^{-1})\right) = \sum_{m=-\infty}^{\infty} I_m(u) k^m \qquad (5.8)$$

to show that

$$e^{\lambda\gamma} = e^{\lambda t} \sum_{m=-\infty}^{\infty} I_m(\lambda\ r)(-ie^{i\theta}\xi)^m \qquad (5.9)$$

where $z = re^{i\theta}$. Hence we obtain the monopole solution

$$\Delta = e^{\lambda t} I_0(\lambda\ \ r)\,. \tag{5.10}$$

In the twistor formalism a BPS monopole of charge n can be obtained[22,23,24] from the ansatz A_n. We are encouraged by the simplicity of the twistor form (5.7) of the monopole solution to hope that, in a similar manner, by considering higher ansätze we may obtain multi-monopole solutions. This issue is currently under investigation.

6. Conclusion

We have seen that monopole-like solutions exist to a relativistic integrable Yang-Mills-Higgs system in (2+1)-dimensions. Furthermore, we have seen that introducing an ansatz reduces the equation to a linear equation for a superpotential. Solutions representing static deformed monopoles and also the dynamics of deforming monopoles can be constructed through a simple superposition procedure. It is interesting that exact solutions can be found which describe exotic soliton dynamics in an integrable system and supports other recent work which suggests that the almost trivial dynamics of solitons in integrable systems in (1+1)-dimensions may not survive the generalisation of integrable soliton systems to higher dimensions.

There are still many issues related to our monopole solutions which require investigations. The most obvious of which is the construction of multi-monopole solutions, both static and time-dependent. There is also an interesting similarity between the solutions described in this paper and monopole-like solutions of the complex self-duality equations in (3+1)-dimensions.[25]

ACKNOWLEDGEMENTS

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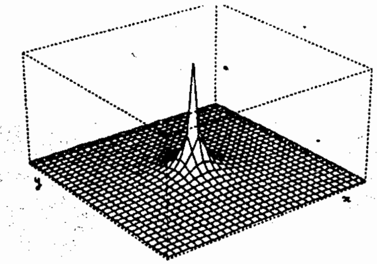

Fig.1a The Higgs density for a radially symmetric static monopole with scale $\lambda = 1$ and position $x_\mu^{(0)} = 0$.

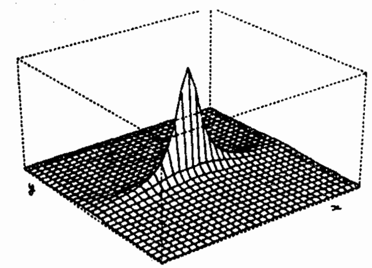

Fig.1b The Higgs density for a static deformed monopole with parameters $\lambda = 1$, $x_1^{(0)} = x_2^{(0)} = 0$ and $y_1^{(0)} = -y_2^{(0)} = 5$.

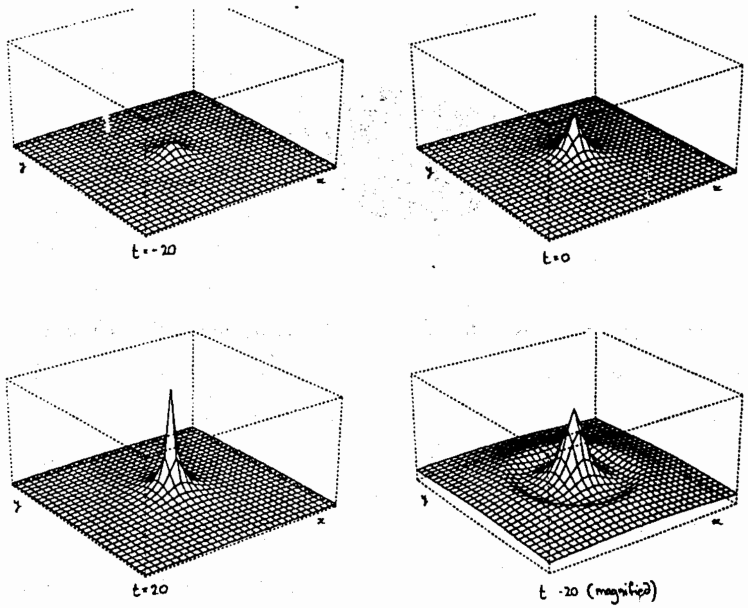

Fig.2 The Higgs density at times t=-20, 0, 20 for a radially symmetric monopole with parameters $\delta = 1, \lambda_1 = 0.4$ and $\lambda_2 = 1.0$. The plot for t=-20 after magnification is also shown.

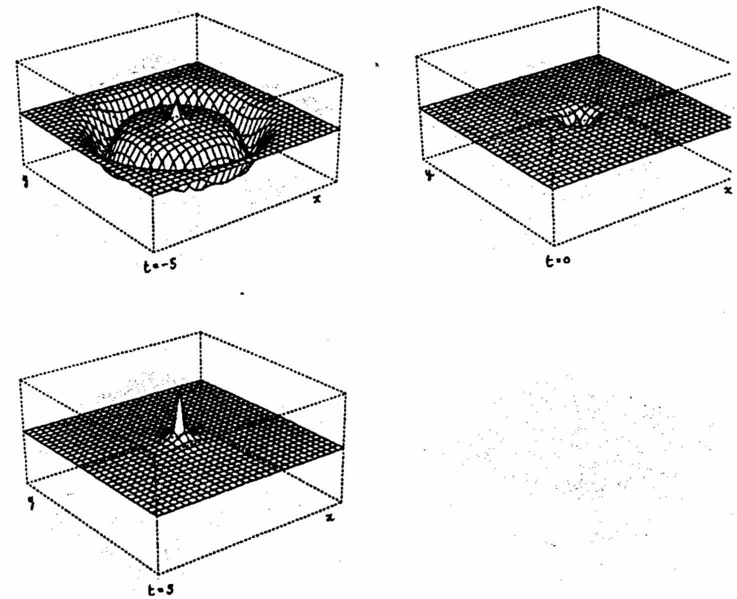

Fig.3 The Higgs density at times t=-5, 0, 5 for a radially symmetric monopole with parameters $\delta = -1, \lambda_1 = 1.0$ and $\lambda_2 = 1.5$.

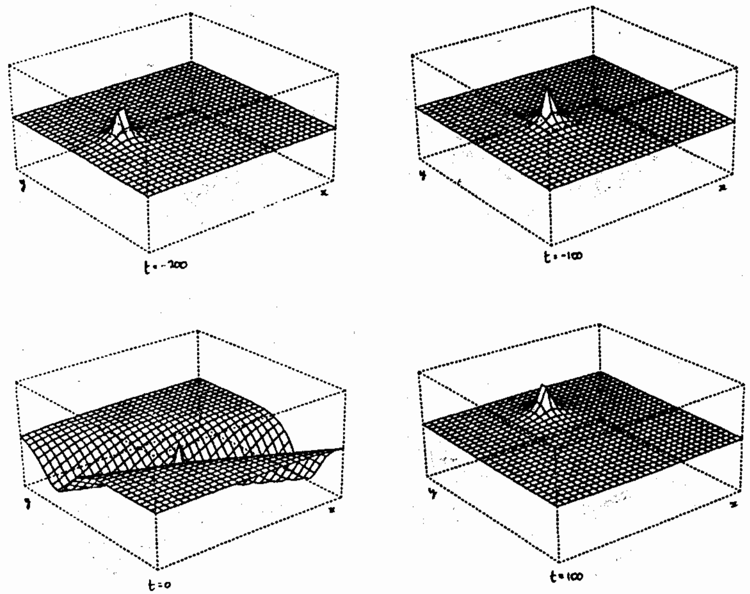

Fig.4 The Higgs density at times t=-200, -100, 0, 100 for a transmuting monopole with parameters $x_1^{(0)} = x_2^{(0)} = y_1^{(0)} = y_2^{(0)} = 0, \lambda_1 = \lambda_2 = 1, \mathbf{v}_1 = 0.1, \phi_1 = 0, \mathbf{v}_2 = 0.2,$ $\phi_2 = \pi/2$ and $\delta = -1$.

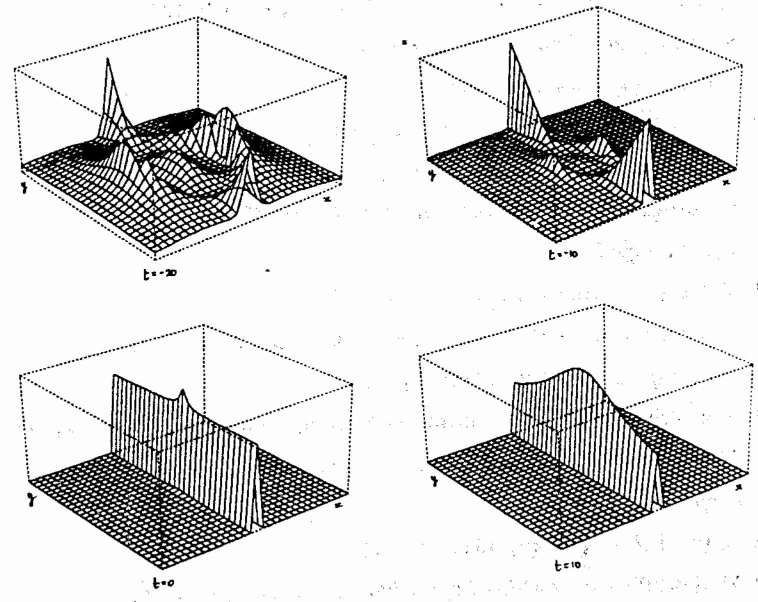

Fig.5 The (scaled) Higgs density at times t=-20, -10, 10 for parameter values $x_1^{(0)} = x_2^{(0)} = y_1^{(0)} = y_2^{(0)} = \phi_1 = \phi_2 = 0, \lambda_1 = \lambda_2 = 1, \mathbf{v}_1 = -\mathbf{v}_2 = 0.9$ and $\delta = 1$.

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