СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2-97-418

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ON THE CORRESPONDENCE BETWEEN THE DYNAMICS WITH ODD AND EVEN BRACKETS AND GENERALIZED NAMBU'S MECHANICS

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1997

1. Let us consider a system of ordinary differential equations

$$\dot{x}_n = v_n(x), \ 1 \leq n \leq N, \tag{1}$$

where $v_n(x)$ are the components of the vector which determines the flow on the state space F^N.

To the system (1) reduce the Hamiltonian equations of motion with M degrees of freedom

$$\begin{aligned} \dot{q}_m &= \frac{\partial H}{\partial p_m} = \{q_m, H\}, \\ \dot{p}_m &= -\frac{\partial H}{\partial q_m} = \{p_m, H\}, \\ & 1 \leq m \leq M, \end{aligned} \tag{2}$$

where for the functions A and B on the phase space the bracket is defined as

$$\{A, B\} = A(\overleftarrow{\partial}_{q_m} \overrightarrow{\partial}_{p_m} - \overleftarrow{\partial}_{p_m} \overrightarrow{\partial}_{q_m})B, \ \partial_{q_m} \equiv \frac{\partial}{\partial q_m} \tag{3}$$

and summation is assumed (as usual and as in the following text) under the repeated indexes. Indeed, for this it is enough to put together q's and p's in

$$x = (x_1, x_2, ..., x_{2M}) = (q_1, q_2, ..., q_M, p_1, p_2, ..., p_M)$$

of the point of the $N = 2M$-dimensional phase space. The right-hand side of the system (1) in this case has the form

$$v_n = \epsilon_{nm} \frac{\partial H}{\partial x_m}, \ \ 1 \leq n, m \leq 2M,$$

where the symplectic matrix ϵ is

$$\epsilon = \begin{pmatrix} O & I \\ -I & O \end{pmatrix},$$

I is the $M \times M-$dimensional unit matrix.

2. The system (1) can be extended to Hamiltonian one [1]. Let us consider the following Lagrangian:

$$S = \int dt p_n(\dot{x}_n - v_n(x)), \tag{4}$$

with the equations of motion

$$\begin{aligned} \dot{x}_n &= v_n(x) = \{x_n, H_2\}_2, \\ \dot{p}_n &= -p_m \frac{\partial v_m}{\partial x_n} = \{p_n, H_2\}_2, \\ & 1 \leq n, m \leq M, \end{aligned} \tag{5}$$

where the second-level Hamiltonian, H_2, and the corresponding Poisson bracket are

$$\begin{aligned} H_2 &= p_n v_n(x), \\ \{A, B\}_2 &= A(\overleftarrow{\partial}_{x_n}\overrightarrow{\partial}_{p_n} - \overleftarrow{\partial}_{p_n}\overrightarrow{\partial}_{x_n})B. \end{aligned} \tag{6}$$

Let us see that the systems (1) and (5) are equivalent [1], i.e. the general solution of the system (1) defines the general solution of the system (5). That the general solution of (5) defines the general solution of (1) is obvious. Let us take the system of equations for variations

$$\dot{y}_n = \frac{\partial v_n}{\partial x_m} y_m. \tag{7}$$

If the general solution of the system (1) is

$$x_n = f_n(t, c_1, c_2, ..., c_N), \; c_n = f_n^{-1}(t, x_1, x_2, ..., x_N), \tag{8}$$

then the general solution of the system (7) will be

$$y_n = \frac{\partial f_n}{\partial c_m} k_m, \tag{9}$$

where, $c_1, c_2, ..., c_N$ and $k_1, k_2, ..., k_N$ are arbitrary constants.

The system conjugated with (7)

$$\dot{p}_n = -p_m \frac{\partial v_m}{\partial x_n} \tag{10}$$

has the following integrals of motion:

$$h_n = p_m \frac{\partial f_m}{\partial c_n}, \; 1 \leq n \leq N. \tag{11}$$

Indeed, according to the systems (7) and (10), the quantity

$$h = p_n y_n \tag{12}$$

is conserved. Inserting the solution (9) into (12), we obtain

$$h = p_m \frac{\partial f_m}{\partial c_n} k_n = (p_m \frac{\partial f_m}{\partial c_n}) k_n, \tag{13}$$

so, as the constants k_n are arbitrary, the quantities in the bracket are also constants. Now from the system (11), when the matrix

$$(A_{mn}) = \left(\frac{\partial f_m}{\partial c_n}\right) \tag{14}$$

is invertible, i.e. in the case of the general position, the quantities p_n are uniquely defined via the integrals h_n and general solution of the system (1):

$$p_n = h_m A_{mn}^{-1}. \tag{15}$$

3. Let us pose an inverse problem: define the (non-linear) system (1) via (linear sub-)system (5),(10). The quantities (11) represent N independent integrals of the (sub-)system (10). Let us proof that these integrals uniquely define the system (1). Indeed, we have

$$0 = \dot{h}_n = \frac{\partial h_n}{\partial t} + \frac{\partial h_n}{\partial p_m}\dot{p}_m + \frac{\partial h_n}{\partial x_m}\dot{x}_m, \; 1 \leq n \leq N, \tag{16}$$

In the case of general position, when the integrals h_n are (functionally) independent, or in simpler terms, the matrix

$$(B_{nm}) = \left(\frac{\partial h_n}{\partial x_m}\right) \tag{17}$$

is invertible, i.e. the Jakobian $J(h/x)$ of the vector h with respect to the vector x is nonzero:

$$J(h/x) = det B \neq 0, \tag{18}$$

from the system (16) we uniquely define $\dot{x}_n$:

$$\begin{aligned} \dot{x}_n &= -B_{nm}^{-1}(\frac{\partial h_m}{\partial t} + \frac{\partial h_n}{\partial p_m}\dot{p}_m) \\ &= \frac{J(h_1, h_2, ..., h_{m-1}, x_n, h_{m+1}, ..., h_n)}{J(h_1, h_2, ..., h_n)}(\frac{\partial h_n}{\partial p_m} h_l A_{lk}^{-1} \frac{\partial v_k}{\partial x_m} - \frac{\partial h_m}{\partial t}), \end{aligned} \tag{19}$$

and consequently define the system (1). If we take another equivalent system of integrals, e.g. when h_1 takes the form

$$h_1 = t(x_1, x_2, ..., x_N) - t, \tag{20}$$

and the remaining integrals do not depend explicitly on t, the system (19) reduces to the system of equations of Nambu's mechanics [2]

$$\begin{aligned} \dot{x}_n &= \varepsilon_{nn_2n_3...n_N} \frac{\partial h_2}{\partial x_{n_2}} \frac{\partial h_3}{\partial x_{n_3}} ... \frac{\partial h_N}{\partial x_{n_N}} \\ &= \{x_n, h_2, h_3, ..., h_N\}, \end{aligned} \tag{21}$$

where N-nar bracket [3] is defined as

$$\begin{aligned} \{A_1, A_2, ..., A_N\} &= \varepsilon_{n_1n_2...n_N} \frac{\partial A_1}{\partial x_{n_1}} \frac{\partial A_2}{\partial x_{n_2}} ... \frac{\partial A_N}{\partial x_{n_N}} \\ &= J(A/x). \end{aligned} \tag{22}$$

Let us consider a simple example:

$$\dot{x} = x^2. \tag{23}$$

The general solution of the equation (23) is

$$\begin{aligned} x &= \frac{x_0}{1 - x_0 t}, \\ x_0 &= \frac{x}{1 + xt}. \end{aligned} \tag{24}$$

The equations for variations (7) reduce to

$$\dot{y} = 2xy, \tag{25}$$

the conjugated system (10) reduces to

$$\dot{p} = -2xp, \tag{26}$$

the general solution (9) reduces to

$$y = \frac{\partial x}{\partial x_0} k = \frac{k}{(1 - x_0)^2} = (1 + xt)^2 k, \tag{27}$$

the integrals (11) reduce to

$$h = h(t, p, x) = p(1 + xt)^2, \tag{28}$$

and equations (16) reduce to

$$0 = \dot{h} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial p}\dot{p} + \frac{\partial h}{\partial x}\dot{x} = 2tp(1 + xt)(\dot{x} - x^2) = 0. \tag{29}$$

So the linear equation (26) and the integral (28) define the non-linear equation (23).

4. Let us consider [1, 4] a general first-order system with the Lagrangian

$$L = A_n(q)\dot{q}_n - H(q), \tag{30}$$

and equations of motion

$$M_{nm}\dot{q}_m = \frac{\partial H}{\partial q_n}, \tag{31}$$

where

$$M_{nm} = \left(\frac{\partial A_m}{\partial q_n} - \frac{\partial A_n}{\partial q_m}\right). \tag{32}$$

When the matrix M is invertible, from (31) we obtain

$$\begin{aligned} \dot{q}_n &= M^{-1}_{nm}\frac{\partial H}{\partial q_m}, \\ &= \{q_n, H\}, \end{aligned} \tag{33}$$

where the bracket is

$$\{A, B\} = A \overleftarrow{\partial}_{q_n} M^{-1}_{nm} \overrightarrow{\partial}_{q_m} B. \tag{34}$$

In this case it is known [5] that Dirac's approach [6] to the first-order system (32) (with constraints) gives the same results.

Let us see that our system (4-5) belongs to the class of models (30), with invertible matrix M (32). Indeed, unifying the phase space coordinates

$$(x_1, ..., x_N, p_1, ..., p_N) = (q_1, ..., q_{2N}) = q,$$

and comparing expressions (4) and (30), we get

$$\begin{aligned} A_n &= q_{n+N},\ A_{n+N} = 0,\ 1 \le n \le N, \\ H &= q_{n+N}v_n(q_1, ..., q_N). \end{aligned} \tag{35}$$

Now we see that our models (4) belong to the first class systems (30); our equations of motion (5) are of the type (33); the (symplectic) matrix M has the (only nonzero) elements

$$M_{n,n+N} = 1 = -M_{n+N,n},\ 1 \le n \le N \tag{36}$$

and is invertible, $M^{-1} = -M$.

5. Note that Hamiltonization of the system (1) in the form (5) gives the basis (for the variational formalism) of the theory of optimal processes [7]. Then, in the supersymmetric models [8] some of the variables x are real, others are grassmann. When the grassmann parities of the variable x_n and coniugated with them variable p_n are the same, the Hamiltonian (6) is even and the bracket (6) is the even Poisson-Martin bracket [9]. When the grassmann parities of x_n and p_n are different, the Hamiltonian (6) is odd and the bracket (6) is the Buttin odd bracket [10].

In this paper, we solve in general form the problem of the correspondence between the descriptions of Hamiltonian (supersymmetric) models (2) (first level) and extended models (4 - 6)(second level). We showed that the (integrals (11) and) linear (sub)system (5),(10) determine the initial (non-linear) system (1) in the form of (generalized) Nambu's equations (19), (21).

Note also that the paper [11] gives the description of the hydrodynamic invariants by a supersymmetric extension of the phase space and odd bracket. In the paper [12] a correspondence between descriptions by even and odd brackets of a model of supersymmetric quantum mechanics [13] is shown. The particular case of the equation (19), when only the integral h_N is time-dependent, has been considered in [14]. The latter paper contains also several examples of the illustrations.

It is a pleasure to thank Professor R.M. Muradyan for ref.[3], Doctor I.V. Amirkhanov for suggesting the example (23) and the members of the seminar $\odot N M \pi$ for stimulating discussions.

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Received by Publishing Department
on December 31, 1997.