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ON SOME MODIFICATIONS OF NONLINEAR *n*-FIELD MODEL AND THEIR EXACT PARTICLE-LIKE SOLUTIONS IN *d*-DIMENSIONS

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1. One of the most interesting nonlinear models of the high energy physics [1] and condensed matter physics [2], the so-called n-field model, is given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu n_a \partial^\mu n_a, \quad n_a n_a = n_1^2 + n_2^2 + n_3^2 = 1, \tag{1}$$

where

$$\mu = 0, 1, 2, ...d, \; a = 1, 2, 3; \; \partial_\mu = \frac{\partial}{\partial x^\mu}, \tag{2}$$

and we assume summation over repeated indices.

In this paper we consider the following modifications of the model (1) [3, 4], [2], [5, 6]:

$$\mathcal{L}_1 = \frac{1}{2}\partial_\mu n_a \partial^\mu n_a + J_a n_a, \tag{3}$$

$$\mathcal{L}_2 = \frac{1}{2}\partial_\mu n_a \partial^\mu n_a - J(1 - n_3), \tag{4}$$

$$\mathcal{L}_3 = \frac{1}{2}\partial_\mu n_a \partial^\mu n_a - \frac{m^2}{2}(1 - n_3^2), \tag{5}$$

$$1 = n_1^2 + n_2^2 + n_3^2,$$

where J and m are external source and mass parameters respectively.

It is convenient to introduce a new field variable

$$z = \frac{n_1 + in_2}{1 + n_3}, \tag{6}$$

$$n_1 = \frac{z^* + z}{1 + |z|^2}, \quad n_2 = \frac{i(z^* - z)}{1 + |z|^2}, \quad n_3 = \frac{1 - |z|^2}{1 + |z|^2}. \tag{7}$$

Then, the n-field given by expression (7) is automatically on the unit sphere. In terms of the new variables the Lagrangians take the following form:

$$\mathcal{L}_1 = \frac{2|\partial_\mu z|^2 + J(1 - |z|^4)}{(1 + |z|^2)^2}, \tag{8}$$

$$\mathcal{L}_2 = \frac{|\partial_\mu z|^2 - J(1 + |z|^2)|z|^2}{(1 + |z|^2)^2}, \tag{9}$$

$$\mathcal{L}_3 = 2\frac{|\partial_\mu z|^2 - m^2|z|^2}{(1 + |z|^2)^2}. \tag{10}$$

For expression (8), in (3) we make the choice

$$J_a = \delta_{a3} J. \tag{11}$$

The equations of motion are

$$(1 + |z|^2)z_\mu{}^\mu - 2z_\mu z^\mu z^* - J(1 + |z|^2)z = 0, \tag{12}$$

$$(1 + |z|^2)z_\mu{}^\mu - 2z_\mu z^\mu z^* + m^2(1 - |z|^2)z = 0. \tag{13}$$

2. For radially symmetric solutions, eqs. (12) and (13) reduce to

$$(1 + |z|^2)(\partial_\tau^2 - \partial_\eta^2)z - 2z^*((\partial_\tau z)^2 - (\partial_\eta z)^2) - J(1 + |z|^2)zr^{2(d-1)} = 0, \tag{14}$$

$$(1 + |z|^2)(\partial_\tau^2 - \partial_\eta^2)z - 2z^*((\partial_\tau z)^2 - (\partial_\eta z)^2) + m^2(1 - |z|^2)zr^{2(d-1)} = 0, \tag{15}$$

where we used the following transformations:

$$z_\mu{}^\mu = (\partial_t^2 - \Delta)z = \partial_t^2 z - \frac{1}{r^{d-1}}\partial_r(r^{d-1}\partial_r z) = r^{2(1-d)}(\partial_t^2(r^{2(d-1)}z) - (r^{d-1}\partial_r)^2 z) = r^{2(1-d)}(\partial_\tau^2 - \partial_\eta^2)z, \tag{16}$$

$$z_\mu z^\mu = (\partial_t z)^2 - (\partial_r z)^2 = r^{2(1-d)}((\partial_\tau z)^2 - (\partial_\eta z)^2),$$

where

$$\tau = \frac{t}{r^{d-1}}, \; \eta = \frac{r^{2-d} - r_0^{2-d}}{2 - d}, \; d \neq 2 \tag{17}$$

$$\tau = \frac{t}{r}, \; \eta = ln\frac{r}{r_o}, \; d = 2.$$

The main idea of this paper is that if we consider variable source and mass parameters, so that

$$J(r)r^{2(d-1)} = J_0 r_0^{2(d-1)} = const, \tag{18}$$

$$m(r)r^{d-1} = m_0 r_0^{d-1} = const, \tag{19}$$

the dependence of eqs. (14) and (15) on the dimension of the space becomes implicit through the variables τ and η. So, the d-dimensional problem reduces, for example, to one-dimensional one.

3. It is easy to find that eq. (13) with variable mass (19) has a solution of the form

$$z(\eta, \tau) = z(\eta)e^{i\omega\tau}, \tag{20}$$

where

$$z(\eta) = exp(\pm\sqrt{m_0^2 r_0^{2(d-1)} - \omega^2}\eta), \tag{21}$$

$$\tau = \begin{cases} \frac{t}{r}, & d = 2, \\ \frac{t}{r^{d-1}}, & d \neq 2, \end{cases}$$

$$\eta = \begin{cases} ln\frac{r}{r_0}, & d = 2, \\ \frac{r^{2-d} - r_0^{2-d}}{2-d}, & d \neq 2. \end{cases}$$

Another eq., (12), under the condition (18) has a sphaleron solution and will be considered elsewhere. Note that d-dependence in the solution (20) is analytical, so we can consider not only natural values for d, but also noninteger, fractal values.

The energy and charge (functionals) of the model (10)

$$\mathcal{H} = 2\int d^dx \frac{|\partial_t z|^2 + |\nabla z|^2 + m^2|z|^2}{(1+|z|^2)^2}, \tag{22}$$

$$\mathcal{J}_\mu = 2\int d^dx \frac{ie(z^*\partial_\mu z - z\partial_\mu z^*)}{(1+|z|^2)^2}, \tag{23}$$

calculated from the solution (20):

$$z(t,r) = exp(i\omega\frac{t}{r})(\frac{r}{r_0})^{\pm\sqrt{m_0^2r_0^2-\omega^2}},\ d=2, \tag{24}$$

$$z(t,r) = exp(i\omega\frac{t}{r^{d-1}})exp(\pm\sqrt{m_0^2r_0^{2(d-1)} - \omega^2}(r^{2-d} - r_0^{2-d})/(2-d)),\ d\neq 2, \tag{25}$$

take the following values:

$$d=1,\ z(t,x) = exp(i\omega t)exp(\pm\sqrt{m_0^2-\omega^2}(x-x_0)), \tag{26}$$

$$E_1 = \frac{2m_0^2}{\sqrt{m_0^2-\omega^2}},\ Q_1 = -\frac{2e\omega}{\sqrt{m_0^2-\omega^2}};$$

$$d=2,\ \omega=0,\ E_2 = 4\pi m_0 r_0,\ Q_2 = 0,$$

$$d=2,\ 0\leq\omega^2\leq m_0^2r_0^2 - \frac{1}{4}, \tag{27}$$

$$Q_2 = -8\pi e\omega r_0\int_0^\infty \frac{dx}{(x^{\sqrt{m_0^2r_0^2-\omega^2}} + x^{-\sqrt{m_0^2r_0^2-\omega^2}})^2}, \tag{28}$$

$$d\neq 2,\ Q_d = -e\omega r_0\Omega_d\int_0^\infty \frac{dr}{ch^2(\sqrt{m_0^2r_0^{2(d-1)} - \omega^2}(r^{2-d} - r_0^{2-d})/(2-d))}, \tag{29}$$

$$d\neq 2,\ \omega=0,\ E_d = m_0^2r_0^d\Omega_d\int_0^\infty \frac{dx}{x^{d-1}ch^2(m_0r_0\frac{x^{2-d}-1}{2-d})}, \tag{30}$$

$$\Omega_d = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

For $d\neq 1$ and $\omega\neq 0$, the energy contains the time-dependent contribution:

$$E_d = E_{1d} + E_{2d}, \tag{31}$$

$$E_{1d} = \omega^2t^2(d-1)^2\Omega_d/2\int_0^\infty \frac{dr}{r^{d+1}ch^2(\sqrt{m_0^2r_0^{2(d-1)} - \omega^2}(r^{2-d} - r_0^{2-d})/(2-d))},$$

$$E_{2d} = m_0^2r_0^{2(d-1)}\Omega_d\int_0^\infty \frac{dr}{r^{d-1}ch^2(\sqrt{m_0^2r_0^{2(d-1)} - \omega^2}(r^{2-d} - r_0^{2-d})/(2-d))}. \tag{32}$$

For $d>2$, the time-dependent contribution is finite.

4. Let us make some field-theoretical motivations for the variable "mass" (19):

$$m(r) \sim \frac{1}{r^{d-1}}. \tag{33}$$

If we take the massless equation for a scalar field ϕ with a point-like source

$$\Delta\phi = g\delta^d(x), \tag{34}$$

then

$$\phi = g\Delta^{-1}\delta^d(x) = \frac{g}{(2\pi)^d}\int d^dp\frac{e^{ipx}}{p^2} \sim \frac{g}{r^{d-2}}. \tag{35}$$

For massless Dirac equation:

$$\gamma\partial\psi = g\delta^d(x). \tag{36}$$

we have

$$\psi \sim \frac{g}{r^{d-1}}. \tag{37}$$

Now take the following Lagrangian:

$$\mathcal{L} = \frac{|\partial_\mu z|^2 - h|z|^2\bar\psi\psi}{(1+|z|^2)^2} + \bar\psi\gamma\partial\psi + \bar\eta\psi + \bar\psi\eta. \tag{38}$$

where h is a coupling constant and η is a source of the fermion field. In the $O(h^0)$ approximation and point-like sources, we have

$$\psi_0 \sim \frac{g}{r^{d-1}},\ \ |z_0| = 1, \tag{39}$$

as, by dimensional consideration, it is easy to see that the model (1) has not nontrivial localized solutions at $d\neq 1$ and stable solutions at $d=1$. If we take into account the interaction (but not back-reaction of the "hadron" field z on the "quark" field ψ), we will have our situation, (10),(19). So, we have some argument for spinor "quarks" (36-37) in preference to the scalar one (34).

Let us consider gauged versions of the models. To do this, note that the Lagrangians (8-10), (38) are invariant under the global gauge transformations:

$$z \to z' = e^{i\alpha}z,\ \alpha = const. \tag{40}$$

Now we localize the transformation (40) by considering variable α and the extended Lagrangian including the vector gauge field A_μ:

$$\mathcal{L} = \frac{2|D_\mu z|^2}{(1+|z|^2)^2} - \frac{1}{2}F_{\mu\nu}^2, \tag{41}$$

where

$$D_\mu = \partial_\mu - ieA_\mu,\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and e is a gauge coupling constant.

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$$\begin{aligned} z_\mu{}^\mu &= (\partial_t^2 - \Delta)z = \partial_t^2 z - \frac{1}{r^{d-1}}\partial_r(r^{d-1}\partial_r z) \\ &= r^{2(1-d)}(\partial_t^2(r^{2(d-1)}z) - (r^{d-1}\partial_r)^2 z) = r^{2(1-d)}(\partial_\tau^2 - \partial_\eta^2)z, \\ z_\mu z^\mu &= (\partial_t z)^2 - (\partial_r z)^2 = r^{2(1-d)}((\partial_\tau z)^2 - (\partial_\eta z)^2), \end{aligned} \tag{16}$$

where

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If we assume that there is a condensate of the vector field $A_\mu = const \neq 0$, then the Lagrangian (41) reduces to the Lagrangian (10) with constant mass $m^2 = -e^2 A_\mu A^\mu$. To the variable mass parameter corresponds the variable vector field (condensate). If we take the "hadron" condensate, $z = z_0 = const$, the Lagrangian (41) reduces to the massive vector field Lagrangian

$$\mathcal{L} = -\frac{1}{2}F_{\mu\nu}^2 + \frac{1}{2}A^2. \tag{42}$$

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Махалдиани Н. E2-97-408

О некоторых модификациях модели n-поля
и точных частицеподобных решениях в d-мерном пространстве

Рассматриваются несколько модификаций модели n-поля. Для одной из них, с переменным параметром массы, найдено частицеподобное решение в аналитическом виде в d-мерном пространстве. Приведены теоретикополевые аргументы для объяснения переменной массы.

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On Some Modifications of Nonlinear n-Field Model
and Their Exact Particle-Like Solutions in d-Dimensions

We consider several modifications of the n-field model. For one of them, with variable mass parameter, we find an explicit particle-like solution in the d-dimensional case. We give field-theoretic arguments for the proposed variable mass.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna, 1997