СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2-97-407

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THE SYSTEM OF THREE VORTEXES OF TWO-DIMENSIONAL IDEAL HYDRODYNAMICS AS A NEW EXAMPLE OF THE (INTEGRABLE) NAMBU—POISSON MECHANICS

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1997

1. The equation of motion of a system of three vortexes can be put in the form [1, 3]

$$\begin{aligned} \dot{M}_1 &= \Gamma_1 M_1 (M_2 - M_3), \\ \dot{M}_2 &= \Gamma_2 M_2 (M_3 - M_1), \\ \dot{M}_3 &= \Gamma_3 M_3 (M_1 - M_2). \end{aligned} \tag{1}$$

Indeed, it is well known [2] that the system of N vortexes can be described by the following system of differential equations:

$$\dot{Z}_n = i \sum_{m \neq n}^{N} \frac{\Gamma_m}{Z_n^* - Z_m^*}. \tag{2}$$

Then it is easy to verify that the quantities

$$\begin{aligned} M_1 &= |Z_2 - Z_3|^2, \\ M_2 &= |Z_3 - Z_1|^2, \\ M_3 &= |Z_1 - Z_2|^2 \end{aligned} \tag{3}$$

satisfy the system (1) after changing the time parameter as follows:

$$dt = \frac{M_1 M_2 M_3}{4 S_\Delta} d\tau = (M_1 M_2 M_3)^{\frac{1}{2}} R d\tau, \tag{4}$$

where S_Δ is the area of the triangle with vertexes in the centres of the vortexes and R is the radius of the circle with the vortexes on it.

The system (1) has the integrals of motion

$$\begin{aligned} H_1 &= \sum_{i=1}^{3} \frac{M_i}{\Gamma_i}, \\ H_2 &= \sum_{i=1}^{3} \frac{ln M_i}{\Gamma_i} \end{aligned} \tag{5}$$

and can be presented in the form

$$\begin{aligned} \dot{M}_i &= \omega_{ijk} \frac{\partial H_1}{\partial M_j} \frac{\partial H_2}{\partial M_k} \\ &= \{M_i, H_1, H_2\} = \omega_{ijk} \frac{1}{\Gamma_j} \frac{1}{\Gamma_k M_k}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \omega_{ijk} &= \epsilon_{ijk} \rho, \\ \rho &= \Gamma_1 \Gamma_2 \Gamma_3 M_1 M_2 M_3 \end{aligned} \tag{7}$$

and the Nambu-Poisson bracket of the functions A, B, C on the three-dimensional phase space M^3 is

$$\{A, B, C\} = \omega_{ijk} \frac{\partial A}{\partial M_i} \frac{\partial B}{\partial M_j} \frac{\partial C}{\partial M_k}. \tag{8}$$

The fundamental bracket is

$$\{M_1, M_2, M_3\} = \omega_{ijk}. \tag{9}$$

Then we can again change the time parameter as

$$d\tau = \rho du \tag{10}$$

and obtain Nambu's mechanics [4]

$$\dot{M}_i = \epsilon_{ijk}\frac{\partial H_1}{\partial M_j}\frac{\partial H_2}{\partial M_k},$$

$$\begin{aligned} \dot{M}_1 &= \frac{M_2 - M_3}{\Gamma_2\Gamma_3 M_2 M_3}, \\ \dot{M}_2 &= \frac{M_3 - M_1}{\Gamma_3\Gamma_1 M_3 M_1}, \\ \dot{M}_3 &= \frac{M_1 - M_2}{\Gamma_1\Gamma_2 M_1 M_2}. \end{aligned} \tag{11}$$

2. The second-order, ternary, Nambu-Poisson structure (6-9) reduces to the two first-order, binary, (Nambu-)Poisson structures[3]

$$\begin{aligned} \{M_i, M_j\}_1 &= (\{M_i, M_j, H_1\} = \omega_{ijk}\frac{\partial H_1}{\partial M_k} = \omega_{ijk}\frac{1}{\Gamma_k}) = \omega^1_{ij}, \\ \{M_i, M_j\}_2 &= (\{M_i, M_j, H_2\} = \omega_{ijk}\frac{\partial H_2}{\partial M_k} = \omega_{ijk}\frac{1}{\Gamma_k M_k}) = \omega^2_{ij}. \end{aligned} \tag{12}$$

These Poisson structures are reducible, there are nontrivial functions H_1 and H_2 for which hold

$$\begin{aligned} \{A, H_1\}_1 &= 0, \\ \{A, H_2\}_2 &= 0 \end{aligned} \tag{13}$$

for any function A.

3. The variables M_n, $n = 1, 2, 3$, are non-negative (semi-bounded), so it is convenient to replace them with free variables x_n

$$x_n = lnM_n, \; n = 1, 2, 3. \tag{14}$$

The equation of motion (1), integrals (5) and Nambu-Poisson structures (8-12) take the following form:

$$\begin{aligned} \dot{x}_1 &= \Gamma_1(e^{x_2} - e^{x_3}), \\ \dot{x}_2 &= \Gamma_2(e^{x_3} - e^{x_1}), \\ \dot{x}_3 &= \Gamma_3(e^{x_1} - e^{x_2}), \end{aligned} \tag{15}$$

$$\begin{aligned} H_1 &= \sum_{i=1}^{3}\frac{e^{x_i}}{\Gamma_i}. \\ H_2 &= \sum_{i=1}^{3}\frac{x_i}{\Gamma_i}. \end{aligned} \tag{16}$$

$$\begin{aligned} \dot{x}_i &= \Gamma_1\Gamma_2\Gamma_3\epsilon_{ijk}\frac{\partial H_1}{\partial x_j}\frac{\partial H_2}{\partial x_k} \\ &= \{x_i, H_1, H_2\} = \{x_i, H_1\}_2 = \{x_i, H_2\}_1. \end{aligned} \tag{17}$$

$$\begin{aligned} \{A, B, C\} &= \Gamma_1\Gamma_2\Gamma_3\epsilon_{ijk}\frac{\partial A}{\partial x_i}\frac{\partial B}{\partial x_j}\frac{\partial C}{\partial x_k}, \\ \{A, B\}_1 &= \omega^1_{ij}\frac{\partial A}{\partial x_i}\frac{\partial B}{\partial x_k}, \\ \{A, B\}_2 &= \omega^2_{ij}\frac{\partial A}{\partial x_i}\frac{\partial B}{\partial x_j}, \end{aligned} \tag{18}$$

$$\begin{aligned} \omega^1_{ij} &= \Gamma_1\Gamma_2\Gamma_3\epsilon_{ijk}\frac{\partial H_1}{\partial x_j} \\ &= \Gamma_1\Gamma_2\Gamma_3\epsilon_{ijk}\frac{1}{\Gamma_j x_j}, \\ \omega^2_{ij} &= \Gamma_1\Gamma_2\Gamma_3\epsilon_{ijk}\frac{\partial H_2}{\partial x_k} \\ &= \Gamma_1\Gamma_2\Gamma_3\epsilon_{ijk}\frac{1}{\Gamma_k}. \end{aligned} \tag{19}$$

4. For the system of three equations (15) we have two integrals of motion (16), so the system (15) is integrable by quadratures [5, 6]. From H_2 it follows that

$$x_3 = \Gamma_3(H_2 - \frac{x_1}{\Gamma_1} - \frac{x_2}{\Gamma_2}). \tag{20}$$

Inserting (20) into the expression of H_1, we find that

$$\frac{\exp x_2}{\Gamma_2} + \frac{\exp(-\frac{\Gamma_3}{\Gamma_2}x_2)}{\Gamma_3}\exp(\Gamma_3(H_2 - \frac{x_1}{\Gamma_1})) = H_1 - \frac{\exp x_1}{\Gamma_1}. \tag{21}$$

Now we see that x_2 can be found from (20) as an elementary function of x_1, when

$$\begin{aligned} \Gamma_3 &= -\Gamma_2, \\ &= -2\Gamma_2, \\ &= -3\Gamma_2, \\ &= -4\Gamma_2, \\ &= \Gamma_2, \\ &= \frac{1}{2}\Gamma_2, \\ &= \frac{1}{3}\Gamma_2. \end{aligned} \tag{22}$$

For the general case, equation (21) defines x_2 as a new trancendental function $n_1(x_1)$. Then the equation for x_1 takes the form

$$\dot{x}_1 = \Gamma_1(e^{n_1(x_1)}(1+\frac{\Gamma_3}{\Gamma_2}) - \Gamma_3 H_1) - \Gamma_3 e^{x_1} \equiv n_2(x_1) \tag{23}$$

and x_1 is defined by the following quadrature:

$$N(x_1) \equiv \int_{x_{10}}^{x_1} \frac{d\tau}{n_2(x_1)} = \tau - \tau_0. \tag{24}$$

Elsewhere we consider general methods of non-linear (Nambu-)Poisson algebras [7] analysis for our model as well as detailed analysis of the formal solution.

It is a pleasure to thank Dr. S.A. Gogilidze and other members of the seminar $\odot NM\pi$ for stimulating discussion.

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Received by Publishing Department
on December 30, 1997.

Махалдиани Н. E2-97-407

Система трех вихрей идеальной двумерной гидродинамики — новый пример интегрируемой механики Намбу—Пуассона

Дается формулировка Намбу—Пуассона системы из трех обыкновенных дифференциальных уравнений, описывающей систему трех вихрей идеальной двумерной гидродинамики. Рассматриваемые уравнения проинтегрированы в квадратурах.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна, 1997

Makhaldiani N. E2-97-407

The System of Three Vortexes of Two-Dimensional Ideal Hydrodynamics as a New Example of the (Integrable) Nambu—Poisson Mechanics

A Nambu—Poisson formulation of the system of three ordinary differential equations, describing the dynamics of three vortexes of the ideal two-dimensional hydrodynamics, is given. The system is integrated by quadratures.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna, 1997