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THE DISCRETE SYMMETRIES OF THE $N = 2$ SUPERSYMMETRIC GNLS HIERARCHIES

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1. Introduction. The goal of the present Letter is to construct the mappings that act like the discrete symmetry transformations of the $N = 2$ supersymmetric (n, m) Generalized Nonlinear Schrödinger ((n, m)-GNLS) hierarchy [1]. Recently, a variety of $N = 2$ supersymmetric integrable hierarchies, derived by the junction of the Lax operators for the $N = 2$ supersymmetric $(n-1, m)$-GNLS and $a = 4$ KdV [2, 3] hierarchies, was proposed in [4]. We also explain its origin. We demonstrate that this variety is gauge related to the variety of $N = 2$ supersymmetric (n, m)-GNLS hierarchies.

Let us start with a short summary of the main facts concerning the $N = 2$ supersymmetric (n, m)-GNLS hierarchy [1] and introduce some new relations which will be useful in what follows.

The Lax operator of the $N = 2$ supersymmetric (n, m)-GNLS hierarchy has the following form[1]:

$$L = \partial - \frac{1}{2}(F_a \overline{F}_a + F_a \overline{D} \partial^{-1} \left[D \overline{F}_a\right]), \quad [D, L] = 0, \tag{1}$$

where $F_a(Z)$ and $\overline{F}_a(Z)$ $(a, b = 1, \ldots, n+m)$ are chiral and antichiral $N = 2$ superfields

$$DF_a(Z) = 0, \quad \overline{D}\,\overline{F}_a(Z) = 0, \tag{2}$$

respectively, which are bosonic for $a = 1, \ldots, n$ and fermionic for $a = n+1, \ldots, n+m$; $Z = (z, \theta, \overline{\theta})$ is a coordinate of $N = 2$ superspace and $D, \overline{D}$ are the $N = 2$ supersymmetric fermionic covariant derivatives

$$\begin{gathered} D = \frac{\partial}{\partial \theta} - \frac{1}{2}\overline{\theta}\frac{\partial}{\partial z}, \quad \overline{D} = \frac{\partial}{\partial \overline{\theta}} - \frac{1}{2}\theta\frac{\partial}{\partial z}, \\ D^2 = \overline{D}^2 = 0, \quad \left\{D, \overline{D}\right\} = -\frac{\partial}{\partial z} \equiv -\partial. \end{gathered} \tag{3}$$

For positive-integer k, such a Lax operator provides the consistent flows

$$\frac{\partial}{\partial t_k} L = [A, L], \quad A = (L^k)_{\geq 1} \tag{4}$$

[1]Hereafter, summation over repeated indices is understood and the square brackets mean that entering operators act only on superfields inside the brackets, e.g., the fermionic derivative D in the Lax operator (1) acts only on the term $\overline{F}_a$ inside the brackets.

and the infinite number of conserved currents can be obtained as follows:

$$H_k = \int dZ (L^k)_0, \tag{5}$$

where the subscripts ≥ 1 and 0 mean the sum of the purely derivative terms and the constant part of the operator, respectively. There are four additional integrals of motion

$$\widetilde{H}_1 = \int dz F_a \overline{F}_a, \tag{6}$$

where we have only space integration due to the equation of motion

$$-\frac{1}{2}\frac{\partial}{\partial t_k}(F_a \overline{F}_a) = ((L^k)_0)\,', \tag{7}$$

where the sign $'$ means the derivative with respect to z.

Equations belonging to the $N=2$ supersymmetric (n,m)-GNLS hierarchy admit the complex structure

$$\begin{aligned} &F_a^* = (-i)^{d_a-1}\mathcal{P}_{ab}\overline{F}_b, \quad \overline{F}_a^* = (-i)^{d_a-1}\mathcal{P}_{ab}F_b,\\ &\theta^* = \overline{\theta}, \quad \overline{\theta}^* = \theta, \quad t_k^* = (-1)^{k+1}t_k, \quad z^* = z, \end{aligned} \tag{8}$$

where i is the imaginary unity and d_a define the grading $F_a F_b = (-1)^{d_a d_b} F_b F_a$ with the property $d_a = 1$ ($d_a = 0$) for fermionic (bosonic) superfields; $\mathcal{P}_{ab}$ is a permutation matrix ($\mathcal{P}^2 = I$) belonging to the discrete permutation subgroup of the $GL(n|m)$ supergroup, which is the group of invariance of the Lax operator (1).

From eq. (4) with the Lax operator (1), one can easily extract the equations for the superfields F_a,

$$\frac{\partial}{\partial t_k} F_a = ((L^k)_{\geq 1} F_a)_0. \tag{9}$$

Applying the transformations (8) to eqs. (9), one can derive the corresponding equations for the superfields $\overline{F}_a$,

$$\frac{\partial}{\partial t_k}\overline{F}_a = (-1)^{k+1}((L^{*\,k})_{\geq 1}\overline{F}_a)_0, \tag{10}$$

where L^* is the complex-conjugate Lax operator

$$L^* = \partial + \frac{1}{2}(F_a\overline{F}_a - \overline{F}_a D\partial^{-1}\left[\overline{D}F_a\right]), \quad A^* = (L^{*\,k})_{\geq 1}, \quad [\overline{D}, L^*] = 0, \tag{11}$$

which also provides the consistent flows. The first nontrivial flow from (9), (10) is the second flow which reads

$$\frac{\partial}{\partial t_2}F_a = F_a\,'' + D(F_b\overline{F}_b\,\overline{D}F_a), \quad \frac{\partial}{\partial t_2}\overline{F}_a = -\overline{F}_a\,'' + \overline{D}(F_b\overline{F}_b D\overline{F}_a). \tag{12}$$

The set of equations (12) form the $N=2$ supersymmetric GNLS equations.

2. Discrete symmetries of the $N=2$ super-GNLS hierarchies. Here, we demonstrate that in addition to the transformations of the $N=2$ supersymmetry and $GL(n|m)$ supergroup, the $N=2$ supersymmetric (n,m)-GNLS hierarchy is invariant with respect to discrete mappings. In the particular cases corresponding to $n=0, m=1$ and $n=1, m=0$, such mappings were obtained in [5, 6]. Following the scheme developed in [6], we derive their generalizations for arbitrary values of the discrete parameters n and m.

Applying the gauge transformation

$$\widetilde{L} = G^{-1}LG, \quad \widetilde{A} = G^{-1}AG - G^{-1}\frac{\partial}{\partial t_k}G, \quad \frac{\partial}{\partial t_k}\widetilde{L} = [\widetilde{A}, \widetilde{L}] \tag{13}$$

with the gauge function G equal to some given bosonic superfield F_l,

$$G = F_l \tag{14}$$

(i.e., the index l is an arbitrary fixed index belonging to the range $1 \leq l \leq n$), substituting the t_k-derivative of F_l (9) into (13), introducing the new superfield basis
$\{J(Z), \widetilde{F}_j(Z), \widetilde{\overline{F}}_j(Z), j = 1,\ldots,l-1,l+1,\ldots,n,\ldots,n+m\}$

$$\begin{aligned} &\widetilde{F}_j = \frac{1}{\sqrt{2}}F_l^{-1}F_j, \quad \widetilde{\overline{F}}_j = -\frac{1}{\sqrt{2}}\overline{D}D\partial^{-1}(F_l\overline{F}_j),\\ &J = \frac{1}{2}(\frac{1}{2}F_a\overline{F}_a - (\ln F_l)\,'\,) \end{aligned} \tag{15}$$

and making obvious algebraic manipulations in the result, we obtain the following explicit expressions for the operators $\widetilde{L}$ and $\widetilde{A}$:

$$\begin{aligned} &\widetilde{L} = \partial - 2J - 2\overline{D}\partial^{-1}\left[D(J - \frac{1}{2}\widetilde{F}_j\widetilde{\overline{F}}_j)\right] - \widetilde{F}_j\overline{D}\partial^{-1}\left[D\widetilde{\overline{F}}_j\right],\\ &\widetilde{A} = (\widetilde{L}^k)_{\geq 1}, \end{aligned} \tag{16}$$

which coincide with the LA-pair considered in [4]. Thus, the integrable extension of the $N=2$ supersymmetric $a=4$ KdV hierarchy of Ref. [4] is gauge related to the $N=2$ supersymmetric (n,m)-GNLS hierarchy [1] and relations (15) establish their explicit connection. For the particular case $n=1, m=0$, relation (15) was obtained in [1].

In the new basis (15), the second flow equations (12) become

$$\frac{\partial}{\partial t_2}\widetilde{F}_j = \widetilde{F}_j\,'' + 4D(J\overline{D}\widetilde{F}_j), \quad \frac{\partial}{\partial t_2}\widetilde{\overline{F}}_j = -\widetilde{\overline{F}}_j\,'' + 4\overline{D}(JD\widetilde{\overline{F}}_j),$$
$$\frac{\partial}{\partial t_2}J = (-[D,\overline{D}]J - 2J^2 + D\widetilde{\overline{F}}_j \cdot \overline{D}\widetilde{F}_j)\,', \qquad (17)$$

and one can observe that they, as well as other equations belonging to the hierarchy, admit the complex structure

$$\widetilde{F}_j^* = (-i)^{d_j-1}\widetilde{\mathcal{P}}_{jc}\widetilde{\overline{F}}_c, \quad \widetilde{\overline{F}}_j^* = (-i)^{d_j-1}\widetilde{\mathcal{P}}_{jc}\widetilde{F}_c, \quad J^* = -J,$$
$$\theta^* = \overline{\theta}, \quad \overline{\theta}^* = \theta, \quad t_k^* = (-1)^{k+1}t_k, \quad z^* = z. \qquad (18)$$

Applying the complex-conjugation transformations (8) and (18) to (15), we observe that in addition to relation (15), there exists one more

$$\widetilde{F}_j = -\frac{1}{\sqrt{2}}iD\overline{D}\partial^{-1}(\overline{F}_l\mathcal{P}_{jc}F_c), \quad \widetilde{\overline{F}}_j = -\frac{1}{\sqrt{2}}i\overline{F}_l^{\,-1}\mathcal{P}_{jc}\overline{F}_c,$$
$$J = \frac{1}{2}(\frac{1}{2}F_a\overline{F}_a + (\ln \overline{F}_l)\,'\,), \qquad (19)$$

which connects the second flow equations (17) to (12), as well as their corresponding hierarchies. Denoting the superfields F_a and $\overline{F}_a$ in (15) by the new letters $\overleftarrow{F}_a$ and $\overleftarrow{\overline{F}}_a$, respectively, and equating the corresponding superfields $\widetilde{F}_j$, $\widetilde{\overline{F}}_j$ and J belonging to the relations (15) and (19), we derive the mapping

$$D\overline{D}\partial^{-1}(\overline{F}_lF_j) = i\overleftarrow{F}_l^{\,-1}\mathcal{P}_{jc}\overleftarrow{F}_c, \quad \overline{D}D\partial^{-1}(\overleftarrow{F}_l\overleftarrow{\overline{F}}_j) = i\overline{F}_l^{\,-1}\mathcal{P}_{jc}\overline{F}_c,$$
$$\frac{1}{2}(\overleftarrow{F}_l\overleftarrow{\overline{F}}_l + \overleftarrow{F}_j\overleftarrow{\overline{F}}_j - F_l\overline{F}_l - F_j\overline{F}_j) = (\ln(\overleftarrow{F}_l\overline{F}_l))\,' \qquad (20)$$

that acts like the discrete symmetry transformation of the $N=2$ supersymmetric (n,m)-GNLS hierarchy. Acting by the fermionic covariant derivatives D and $\overline{D}$ on the first line of eqs. (20), these relations can be rewritten in a slightly different but equivalent local form

$$\overline{D}\,(\overline{F}_lF_j + i\overleftarrow{F}_l^{\,-1}\mathcal{P}_{jc}\overleftarrow{F}_c) = 0, \quad D\,(\overleftarrow{F}_l\overleftarrow{\overline{F}}_j + i\overline{F}_l^{\,-1}\mathcal{P}_{jc}\overline{F}_c) = 0, \qquad (21)$$

which can be more convenient for applications. Actually, it is easy to understand that up to an arbitrary permutation $\mathcal{P}_{jc}$, relation (20) gives us n different discrete symmetry-mappings if one remembers that the index l enters (20) like a discrete parameter[2] taking n values $l=1,\dots,n$.

Now consider gauge transformation (13) with the gauge function

$$G = (D\overline{F}_f)^{-1}, \qquad (22)$$

where $\overline{F}_f$ is some given fermionic superfield (i.e., the index f is an arbitrary fixed index belonging to the range $n+1 \le f \le n+m$). After introducing the new superfield basis $\{J(Z), \widetilde{F}_j(Z), \widetilde{\overline{F}}_j(Z), j=1,\dots,n,\dots,f-1,f+1,\dots,n+m\}$ according to formulae

$$\widetilde{F}_j = \frac{1}{\sqrt{2}}(D\overline{F}_f)F_j, \quad \widetilde{\overline{F}}_j = -\frac{1}{\sqrt{2}}\overline{D}D((D\overline{F}_f)^{-1}\overline{F}_j),$$
$$J = \frac{1}{2}(\frac{1}{2}F_a\overline{F}_a + (\ln D\overline{F}_f)\,'\,) \qquad (23)$$

we obtain the following expression for the Lax operator $\widetilde{L}$

$$\widetilde{L} = \partial - 2J + 2\left[D(J - \frac{1}{2}\widetilde{F}_j\partial^{-1}\widetilde{\overline{F}}_j)\right]\overline{D}\partial^{-1} - \widetilde{F}_j\overline{D}\partial^{-1}\left[D\partial^{-1}\widetilde{\overline{F}}_j\right]. \qquad (24)$$

We do not present the explicit expression for the operator $\widetilde{A}$ here, because what we actually need for our purpose is only the transformation law (23) in the new basis. For the particular case $n=0, m=1$, relation (23) has been discussed in [7, 3].

In the new basis (23), the second flow equations (12) become

$$\frac{\partial}{\partial t_2}\widetilde{F}_j = \widetilde{F}_j\,'' + 4D\overline{D}(J\widetilde{F}_j), \quad \frac{\partial}{\partial t_2}\widetilde{\overline{F}}_j = -\widetilde{\overline{F}}_j\,'' + 4\overline{D}D(J\widetilde{\overline{F}}_j),$$
$$\frac{\partial}{\partial t_2}J = ([D,\overline{D}]J - 2J^2 - \widetilde{F}_j\widetilde{\overline{F}}_j)\,', \qquad (25)$$

and one can see that they admit the complex structure

$$\widetilde{F}_j^* = (-i)^{d_j}\widetilde{\mathcal{P}}_{jc}\widetilde{\overline{F}}_c, \quad \widetilde{\overline{F}}_j^* = (-i)^{d_j}\widetilde{\mathcal{P}}_{jc}\widetilde{F}_c, \quad J^* = -J,$$
$$\theta^* = \overline{\theta}, \quad \overline{\theta}^* = \theta, \quad t_2^* = -t_2, \quad z^* = z. \qquad (26)$$

[2] Let us remember that in (20), there is no summation over repeated indices l.

Following the above-discussed scheme, we apply the complex-conjugation transformations (8) and (26) to (23) and obtain one more mapping,

$$\widetilde{F}_j = -\frac{i}{\sqrt{2}} D\overline{D}((\overline{D}F_f)^{-1}\mathcal{P}_{jc}F_c), \quad \widetilde{\overline{F}}_j = \frac{i}{\sqrt{2}}(\overline{D}F_f)\mathcal{P}_{jc}\overline{F}_c,$$
$$J = \frac{1}{2}(\frac{1}{2}F_a\overline{F}_a - (\ln \overline{D}F_f)\,'\,), \tag{27}$$

connecting the second flow equations (25) to (12), and, therefore, the mapping

$$iD\overline{D}((\overline{D}F_f)^{-1}F_j) = -(D\overleftarrow{\overline{F}}_f)\mathcal{P}_{jc}\overleftarrow{F}_c,$$
$$i\overline{D}D((D\overleftarrow{\overline{F}}_f)^{-1}\overleftarrow{\overline{F}}_j) = (\overline{D}F_f)\mathcal{P}_{jc}\overline{F}_c,$$
$$\frac{1}{2}(F_f\overline{F}_f + F_j\overline{F}_j - \overleftarrow{F}_f\overleftarrow{\overline{F}}_f - \overleftarrow{F}_j\overleftarrow{\overline{F}}_j) = (\ln(\overline{D}F_f\cdot D\overleftarrow{\overline{F}}_f))\,' \tag{28}$$

acts like the discrete symmetry transformation of the $N=2$ supersymmetric (n,m)-GNLS hierarchy. Acting by the fermionic covariant derivatives D and $\overline{D}$ on the first line of eqs. (28), these relations can be represented in the following equivalent form:

$$\overline{D}\;(-i((\overline{D}F_f)^{-1}F_j)\,' + (D\overleftarrow{\overline{F}}_f)\mathcal{P}_{jc}\overleftarrow{F}_c) = 0,$$
$$D\;(i((D\overleftarrow{\overline{F}}_f)^{-1}\overleftarrow{\overline{F}}_j)\,' + (\overline{D}F_f)\mathcal{P}_{jc}\overline{F}_c) = 0. \tag{29}$$

Modulo an arbitrary permutation $\mathcal{P}_{jc}$, relation (28) gives us m different discrete symmetry-mappings because the index f takes m different values $l = n+1,\dots,n+m$.

Let us note that one can rewrite equations (25) in a form similar to (17),

$$-\tfrac{\partial}{\partial t_2}\Psi_j = \Psi_j\,'' + 4D(\widetilde{J}\,\overline{D}\Psi_j), \quad -\tfrac{\partial}{\partial t_2}\overline{\Psi}_j = -\overline{\Psi}_j\,'' + 4\overline{D}(\widetilde{J}D\overline{\Psi}_j),$$
$$-\tfrac{\partial}{\partial t_2}\widetilde{J} = (-[D,\overline{D}]\widetilde{J} - 2\widetilde{J}^2 + D\overline{\Psi}_j\cdot\overline{D}\Psi_j)\,', \tag{30}$$

if one introduces the new superfields $\widetilde{J}$, Ψ_j and $\overline{\Psi}_j$ by the following invertible relations[3]

$$\widetilde{J} = -J, \quad \Psi_j = i\partial^{-1}D\widetilde{\overline{F}}_j, \quad \overline{\Psi}_j = i\partial^{-1}\overline{D}\widetilde{F}_j;$$
$$J = -\widetilde{J}, \quad \widetilde{F}_j = iD\overline{\Psi}_j, \quad \widetilde{\overline{F}}_j = i\overline{D}\Psi_j. \tag{31}$$

[3]Transformations of such kind have been discussed in [4].

However the system (30) does not completely coincide with (17): in comparison with (17), its time direction is reversed. Due to this crucial difference, we can not equate the corresponding superfields entering (17) and (30) to produce new discrete symmetry-mappings for the $N=2$ supersymmetric (n,m)-GNLS hierarchy. Nevertheless, the system (30) is equivalent to (17), and relations (23), (27), and (31) establish an explicit connection of the integrable hierarchy of Ref. [4] to the $N=2$ supersymmetric (n,m)-GNLS hierarchy.

3. Bosonic limit of the mappings. Let us briefly discuss the bosonic limit of the mappings (20) and (28) in order to generate the discrete symmetries for the bosonic GNLS and modified GNLS (mGNLS) hierarchies.

To do this, we set all fermionic components of the superfields F_a and $\overline{F}_a$ equal to zero and define the bosonic components as [1]

$$b_\alpha = \frac{1}{\sqrt{2}}F_\alpha|, \quad \overline{b}_\beta = \frac{1}{\sqrt{2}}\overline{F}_\beta|, \quad 1\le\alpha,\beta\le n,$$
$$g_s = \frac{1}{\sqrt{2}}\overline{D}F_{s+n}|\,\exp(-\partial^{-1}(b_\beta\overline{b}_\beta)),$$
$$\overline{g}_p = \frac{1}{\sqrt{2}}D\overline{F}_{p+n}|\,\exp(\partial^{-1}(b_\beta\overline{b}_\beta)), \quad 1\le s,p\le m, \tag{32}$$

where $|$ means the $(\theta,\bar\theta)\to 0$ limit. In terms of such components, eqs. (12) for the fields $b_\alpha,\overline{b}_\alpha$ and $g_s,\overline{g}_s$ are completely decoupled:

$$\tfrac{\partial}{\partial t_2}b_\alpha = b_\alpha\,'' - 2b_\beta\overline{b}_\beta b_\alpha\,', \quad \tfrac{\partial}{\partial t_2}\overline{b}_\alpha = -\overline{b}_\alpha\,'' - 2b_\beta\overline{b}_\beta\overline{b}_\alpha\,', \tag{33}$$

$$\tfrac{\partial}{\partial t_2}g_s = g_s\,'' - 2g_p\overline{g}_p g_s, \quad \tfrac{\partial}{\partial t_2}\overline{g}_s = -\overline{g}_s\,'' + 2g_p\overline{g}_p\overline{g}_s. \tag{34}$$

The set of equations (34) form the bosonic GNLS equations [8]. Concerning the set of equations (33), we call them mGNLS equations, reflecting the name of its first representative—modified NLS equation [9] corresponding to the case of $n=1$.

The bosonic limit of the mapping (20) ((28)), acting like a discrete symmetry transformation of equations (33) and (34), also splits into two independent mappings, which one can see from the explicit expressions

$$
\begin{aligned}
&\overline{b}_l b_\alpha{}' + i(\overleftarrow{b}_l^{-1} \mathcal{P}_{\alpha\beta} \overleftarrow{b}_\beta)' = 0, \quad \overleftarrow{b}_l \overleftarrow{\overline{b}}_\alpha{}' + i(\overline{b}_l^{-1} \mathcal{P}_{\alpha\beta} \overline{b}_\beta)' = 0, \quad \alpha \neq l, \\
&\overleftarrow{b}_l \overleftarrow{\overline{b}}_l + \overleftarrow{b}_\alpha \overleftarrow{\overline{b}}_\alpha - b_l \overline{b}_l - b_\alpha \overline{b}_\alpha = (\ln(\overleftarrow{b}_l \overline{b}_l))', \\
&\overleftarrow{b}_l \overleftarrow{\overline{b}}_l{}' + \overleftarrow{b}_\alpha \overleftarrow{\overline{b}}_\alpha{}' - b_l \overline{b}_l{}' - b_\alpha \overline{b}_\alpha{}' = (\ln \overline{b}_l)'',
\end{aligned} \tag{35}
$$

$$
\overleftarrow{g}_s = \mathcal{P}_{sp} g_p, \quad \overleftarrow{\overline{g}}_s = \mathcal{P}_{sp} \overline{g}_p \tag{36}
$$

for the mapping (20), and

$$
\begin{aligned}
&\overleftarrow{b}_\alpha \mathcal{P}_{\alpha\beta} \overleftarrow{\overline{b}}_\beta{}' + b_\alpha \mathcal{P}_{\alpha\beta} \overline{b}_\beta{}' = 0, \\
&\overleftarrow{b}_\beta \overleftarrow{\overline{b}}_\beta - b_\beta \overline{b}_\beta = -(\ln \overline{b}_\alpha{}')' + (\ln(\mathcal{P}_{\alpha\beta} \overleftarrow{\overline{b}}_\beta))',
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
&-i(g_f^{-1} g_s)' + \overleftarrow{\overline{g}}_f \mathcal{P}_{sp} \overleftarrow{g}_p = 0, \quad i(\overleftarrow{\overline{g}}_f^{-1} \overleftarrow{\overline{g}}_s)' + g_f \mathcal{P}_{sp} \overline{g}_p = 0, \quad s \neq f, \\
&\overleftarrow{\overline{g}}_f = C_f g_f^{-1}, \quad \overleftarrow{g}_f \overleftarrow{\overline{g}}_f + \overleftarrow{g}_s \overleftarrow{\overline{g}}_s - g_f \overline{g}_f - g_s \overline{g}_s = -(\ln \overline{g}_f)''
\end{aligned} \tag{38}
$$

for the mapping (28), where there is no summation in eq. (37) over repeated indices α and C_f in eq. (38) is an arbitrary constant. In the derivation of these expressions, obvious simplifying transformations, as well as the integration of some intermediate equations, have been done.

The mapping (36) forms the discrete permutation subgroup of the $GL(m)$ group, which is a group of covariance for the GNLS equations (34). The mapping (38) coincides with the mapping which can be easily derived using the Darboux-Bäcklund transformations of the GNLS Lax operators constructed in [10]. Regarding the symmetry mappings (35) and (37) for the mGNLS hierarchy (33), to our knowledge, they are presented for the first time.

In addition to mappings (35) and (37), there are other symmetries of the mGNLS equations. One can produce them if one remembers that the GNLS and mGNLS equations are related by the following transformations [1]:

$$
g_s = b_s{}' \exp(-\partial^{-1}(b_p \overline{b}_p)), \quad \overline{g}_s = \overline{b}_s \exp(\partial^{-1}(b_p \overline{b}_p)). \tag{39}
$$

Applying the complex-conjugation operation (8) to relations (39) for the bosonic components (32), one can obtain one more relation,

$$
g_s = i b_s \exp(-\partial^{-1}(b_p \overline{b}_p)), \quad \overline{g}_s = i \overline{b}_s{}' \exp(\partial^{-1}(b_p \overline{b}_p)). \tag{40}
$$

Therefore, we can introduce two different relations for the fields with the arrow:

$$
\begin{aligned}
&\overleftarrow{g}_s = \overleftarrow{b}_s{}' \exp(-\partial^{-1}(\overleftarrow{b}_p \overleftarrow{\overline{b}}_p)), \quad \overleftarrow{\overline{g}}_s = \overleftarrow{\overline{b}}_s \exp(\partial^{-1}(\overleftarrow{b}_p \overleftarrow{\overline{b}}_p)), \\
&\overleftarrow{g}_s = i \overleftarrow{b}_s \exp(-\partial^{-1}(\overleftarrow{b}_p \overleftarrow{\overline{b}}_p)), \quad \overleftarrow{\overline{g}}_s = i \overleftarrow{\overline{b}}_s{}' \exp(\partial^{-1}(\overleftarrow{b}_p \overleftarrow{\overline{b}}_p)).
\end{aligned} \tag{41}
$$

If one takes some fixed combination of the fields without the arrow, $g_s, \overline{g}_s$, and the fields with the arrow $\overleftarrow{g}_s, \overleftarrow{\overline{g}}_s$ from the set of relations (39)–(41), and substitutes it into the mappings (36) and (38), one can generate new mappings for the mGNLS hierarchies, with different combinations generating different mappings. Let us only mention that one such mapping coincides with the mapping (37), and, in this way, it is possible to reproduce the mapping considered in [11] for the modified NLS equation and to obtain new mappings. It is a simple exercise to derive their explicit forms, and we do not present them here.

4. Conclusion. In this Letter, we constructed mappings (20) and (28), which act like a discrete symmetry transformations of the $N = 2$ supersymmetric (n, m)-GNLS hierarchy (1), (4), and produced their bosonic counterparts (35)–(38). We also established explicit relations (15), (19), (23), and (27) connecting the integrable hierarchy, obtained by the junction of the Lax operators for the $N = 2$ supersymmetric $a = 4$ KdV and $(n-1, m)$-GNLS hierarchies, to the $N = 2$ supersymmetric (n, m)-GNLS hierarchy.

Symmetry mappings contain valuable information about the integrable hierarchies corresponding to them [12, 13, 5, 6]. In addition to this, there is one more reason that stimulates interest in such mappings—they are usually integrable themselves, i.e., every new mapping may give us a new example of a one-dimensional integrable system. Thus, for the $N = 2$ supersymmetric f-Toda chain [5] corresponding to the case of $n = 0, m = 1$, the integrability under appropriate boundary conditions has been proven in [14]. It is interesting to generalize this investigation for the case of arbitrary values of the discrete parameters n and m. In this context, it is important to have Darboux-Bäcklund transformations of the $N = 2$ supersymmetric (n, m)-GNLS Lax operators which should generate our mappings and contain important information about their integrability properties and solutions. We hope to analyze this complicated problem in future publications.

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References

[1] L. Bonora, S. Krivonos and A. Sorin, Nucl. Phys. B 477 (1996) 835.

[2] C.A. Laberge and P. Mathieu, Phys. Lett. B 215 (1988) 718;
P. Labelle and P. Mathieu, J. Math. Phys. 32 (1991) 923.

[3] S. Krivonos, A. Sorin and F. Toppan, Phys. Lett. A 206 (1995) 146.

[4] E. Ivanov and S. Krivonos, *New integrable extensions of $N = 2$ KdV and Boussinesq hierarchies*, Preprint JINR E2-96-344, Dubna 1996; hep-th/9609191.

[5] V.B. Derjagin, A.N. Leznov and A.S. Sorin, *$N = 2$ superintegrable f-Toda mapping and super-NLS hierarchy in the* $(1|2)$ *superspace*, Preprint JINR E2-96-410, Dubna 1996; hep-th/9611108.

[6] A. Sorin, *The discrete symmetry of the $N = 2$ supersymmetric modified NLS hierarchy*, Preprint JINR E2-96-428, Dubna 1996; hep-th/9611148; Phys. Lett. B (in press).

[7] S. Krivonos and A. Sorin, Phys. Lett. B 357 (1995) 94.

[8] A.P. Fordy and P.P. Kulish, Commun. Math. Phys. 89 (1983) 427.

[9] H.H. Chen, Y.C. Lee and C.S. Lin, Phys. Scr. 20 (1979) 490.

[10] H. Aratyn, L.A. Ferreira and A.H. Zimerman, Phys. Lett. B 327 (1994) 266;
H. Aratyn, E. Nissimov and S. Pacheva, Phys. Lett. A 201 (1995) 293.

[11] A.N. Leznov, *Bäcklund transformation for integrable systems*, Preprint IHEP 92-87, Protvino 1992, hep-th/9404032.

[12] D.B. Fairlie and A.N. Leznov, Phys. Lett. A 199 (1995) 360; A 221 (1996) 306;
A.N. Leznov, Physica D 87 (1995) 48.

[13] A.N. Leznov and A.S. Sorin, Phys. Lett. **B 389 (1996) p.494.**

[14] A.N. Leznov and A. Sorin, *The solution of the $N = 2$ supersymmetric f-Toda chain with fixed ends*, Preprint JINR E2-97-21, Dubna 1997; solv-int/9701019.

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