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EXTENDED $N=2$ SUPERSYMMETRIC MATRIX $(1,s)$-KdV HIERARCHIES

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Расширенные $N=2$ суперсимметричные
матричные иерархии $(1,s)$-КдФ

Мы предлагаем операторы Лакса для $N=2$ суперсимметричного матричного обобщения бозонных иерархий $(1,s)$-КдФ. Простейшие примеры — $N=2$ суперсимметричные иерархии $a=4$ КдФ и $a=5/2$ Буссинеска — обсуждены подробно.

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Extended $N=2$ Supersymmetric Matrix $(1,s)$-KdV Hierarchies

We propose the Lax operators for $N=2$ supersymmetric matrix generalization of the bosonic $(1,s)$-KdV hierarchies. The simplest examples — the $N=2$ supersymmetric $a=4$ KdV and $a=5/2$ Boussinesq hierarchies — are discussed in detail.

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1. Introduction. The existence of three different infinite families of $N=2$ supersymmetric integrable hierarchies with the $N=2$ super W_s algebras as their second Hamiltonian structure is a well–established fact by now [1, 2, 3]. Their bosonic limits have been analyzed in [4], and three different families of the corresponding bosonic hierarchies and their Lax operators have been selected. Then, a complete description in terms of super Lax operators for two out of three families has been proposed in [5, 6], and the generalization to the matrix case has been derived in [6].

The last remaining family of $N=2$ hierarchies is supersymmetrization of the bosonic $(1,s)$-KdV hierarchies [4]. We call them the $N=2$ supersymmetric $(1,s)$-KdV hierarchies. As opposed to the bosonic counterparts of the former two hierarchies [4], the $(1,s)$-KdV hierarchy is irreducible (see [7] and references therein), i.e. its Lax operator cannot be decomposed into a direct sum of some more elementary components. This reduction property leads to a strong restriction of the original supersymmetric Lax operator: its bosonic limit should be irreducible. In other words, it should generate only a single operator component. This property is surely satisfied for a supersymmetric Lax operator which is a pure bosonic pseudo-differential operator with the coefficients expressed in terms of $N=2$ superfields and their fermionic derivatives in such a way that it commutes with one of the two $N=2$ fermionic derivatives. The Lax operator of this kind has in fact been observed in [4] for the $N=2$ $a=5/2$ Boussinesq hierarchy in the negative-power decomposition over bosonic derivative up to the ∂^{-5} order. Quite recently, its closed analytic representation has been obtained in [8].

The aim of the present letter is to present a new infinite class of reductions (with a finite number of fields) of $N=2$ supersymmetric matrix KP hierarchy which includes the above–mentioned family of $N=2$ $(1,s)$-KdV hierarchies in the scalar case.

2. Extended matrix $N=2$ super $(1,s)$-KdV hierarchy. The Lax operator

$$\begin{aligned} L^{red}_{KP} &= I\partial + a_0 + \omega_0 D \\ &+ \sum_{j=-\infty}^{-1} \left(a_j\partial - [Da_j]\overline{D} + \omega_j D\partial - \frac{1}{2}[D\omega_j]\left[D,\overline{D}\right]\right)\partial^{j-1}, \quad (1) \end{aligned}$$

derived by reduction $[D, L^{red}_{KP}]=0$ [9] of the $N=2$ supersymmetric matrix

KP hierarchy has been constructed in [6]. Here, a_j and ω_j at $j \geq 1$ (a_0 and ω_0) are generic (chiral) bosonic and fermionic square matrix $N=2$ superfields. The Lax operator (1) still contains an infinite number of fields. Its further reductions [6],

$$(L_{KP}^{red})^s = I\partial^s + \sum_{j=1}^{s-1}\left(J_{s-j}\partial - [DJ_{s-j}]\overline{D}\right)\partial^{j-1} - J_s - \overline{D}\partial^{-1}[DJ_s] - F\overline{F} - F\overline{D}\partial^{-1}\left[D\overline{F}\right], \quad (2)$$

are characterized by a finite number of fields and contain two out of three families of $N=2$ supersymmetric hierarchies with $N=2$ super W_s algebras as their second Hamiltonian structure in the scalar case at $F=\overline{F}=0$ (see [6] for details).

It appears that besides reductions (2), there exist other reductions of the Lax operator (1) which in the scalar case correspond to the last remaining family of $N=2$ hierarchies with the $N=2$ super W_s algebras as their second Hamiltonian structure, i.e. $N=2$ $(1,s)$-KdV hierarchies. Based on the inputs given above, we are led to the following conjecture for the expression of the matrix–valued pseudo–differential operator with a finite number of superfields representing the new reductions of the Lax operator (1):

$$L_{KP}^{red} \equiv L_s = I\partial - \left[D\mathcal{L}_s^{-1}\overline{D}\mathcal{L}_s\right], \quad \mathcal{L}_s \equiv I\partial^s + \sum_{j=0}^{s-1} J_{s-j}\partial^j + \overline{F}\partial^{-1}F. \quad (3)$$

Here, $s=0,1,2,\ldots$, $F \equiv F_{aA}(Z)$ and $\overline{F} \equiv \overline{F}_{Aa}(Z)$ ($A,B=1,\ldots,k$; $a,b=1,\ldots,n+m$) are chiral and antichiral rectangular matrix-valued $N=2$ superfields,

$$DF = 0, \quad \overline{D}\,\overline{F} = 0, \quad (4)$$

respectively, and $J_j \equiv (J_j(Z))_{AB}$ are $k \times k$ matrix–valued bosonic $N=2$ superfields with the scaling dimensions in length $[F]=[\overline{F}]=-(s+1)/2$, $[J_j]=-j$; I is the $k\times k$ unity matrix, $I \equiv \delta_{A,B}$, and the matrix product is understood. The matrix entries are bosonic superfields for $a=1,\ldots,n$ and fermionic superfields for $a=n+1,\ldots,n+m$, i.e., $F_{aA}\overline{F}_{Bb} = (-1)^{d_a\overline{d}_b}\overline{F}_{Bb}F_{aA}$, where d_a and $\overline{d}_b$ are the Grassmann parities of the matrix elements F_{aA} and $\overline{F}_{Bb}$, respectively, $d_a=1$ ($d_a=0$) for fermionic (bosonic) entries. $Z=(z,\theta,\overline{\theta})$ is a coordinate of the $N=2$ superspace, $dZ \equiv dzd\theta d\overline{\theta}$. In (3) the square brackets mean that the $N=2$ supersymmetric fermionic covariant derivatives D and $\overline{D}$,

$$D = \frac{\partial}{\partial\theta} - \frac{1}{2}\overline{\theta}\frac{\partial}{\partial z}, \quad \overline{D} = \frac{\partial}{\partial\overline{\theta}} - \frac{1}{2}\theta\frac{\partial}{\partial z},$$
$$D^2 = \overline{D}^2 = 0, \quad \left\{D,\overline{D}\right\} = -\frac{\partial}{\partial z} \equiv -\partial, \quad (5)$$

act only on the matrix superfields inside the brackets. Let us stress the property of L_s (3) to commute with the fermionic derivative D, that is $[D,L_s]=0$.

The flows are the standard ones,

$$\frac{\partial}{\partial t_p}L_s = [A_p, L_s], \quad A_p = (L_s^p)_+, \quad (6)$$

where $p=1,2,\ldots$, and the subscript $+$ means a differential part of a pseudo-differential operator. Let us remark that the Lax-pair representation (6) (with a generic matrix Lax operator of the type $[D,L]=0$) being multiplied by the projector $-D\partial^{-1}\overline{D}$ from the right, can identically be rewritten in the same form but with the operators L and A_p replaced by the operators $\mathcal{L} \equiv -DL\partial^{-1}\overline{D}$ and $\mathcal{A}_p \equiv -DA_p\partial^{-1}\overline{D}$, respectively, obeying the chirality preserving condition $D\mathcal{L} = \mathcal{L}\overline{D} = 0$ (as opposed to the former condition $[D,L]=0$ we started with). In the scalar (generic matrix) case, Lax operators of the last type have been considered in [5] ([6]). The relation of the chirality preserving scalar Lax operators of Ref. [5] with the former ones (which have been introduced in [10, 4, 6]) was observed recently [11].

For the Lax operator L_s (3) the $N=2$ and $N=1$ residues[1] vanish since it does not contain the $N=2$ fermionic derivatives acting as operators. Nevertheless, an infinite number of Hamiltonians can be obtained by using the non–standard definition of the $N=2$ residue introduced in [4] for the Lax operator of the $N=2$ $a=5/2$ Boussinesq hierarchy which coincides with the definition of the residue for bosonic pseudo-differential operators, i.e. it is the integrated coefficient of ∂^{-1}

$$H_p = \int dz\; tr(L_s^p)_{\partial^{-1}}|, \quad (7)$$

[1]Let us recall that the standard $N=2$ ($N=1$) super-residue is defined as the $N=2$ ($N=1$) superfield integral of the coefficient of the operator $[D,\overline{D}]\partial^{-1}$ ($D\partial^{-1}$ or $\overline{D}\partial^{-1}$).

where $|$ means the $(\theta, \overline{\theta}) \to 0$ limit, the integration is over the space coordinate z, and tr means the usual matrix trace. These Hamiltonians are $N=2$ supersymmetric. Indeed, the operators $tr(L_s^p)_{\partial^{-1}}|$ (for the L_s given by eqs. (3)) can be represented as

$$tr(L_s^p)_{\partial^{-1}}| = [D, \overline{D}]\mathcal{H}_p| + \text{full space derivative terms} \tag{8}$$

with local superfield functions $\mathcal{H}_p$. Consequently, the Hamiltonians H_p (7) can identically be rewritten in a manifestly supersymmetric form

$$H_p = \int dZ \mathcal{H}_p, \tag{9}$$

where the integration is over the $N=2$ superspace coordinate Z.

One can easily derive the bosonic limit of the Lax operator L_s (3) at $F = \overline{F} = 0$,

$$\begin{aligned} L_s^{bos} &= (I\partial^s + u_1\partial^{s-1} + \textstyle\sum_{j=0}^{s-2} u_{s-j}\partial^j)^{-1}(I\partial^{s+1} + u_1\partial^s \\ &+ \textstyle\sum_{j=1}^{s-1}(u_{s-j+1} - v_{s-j})\partial^j - v_s), \end{aligned} \tag{10}$$

where u_j and v_j are bosonic matrix components of the superfield matrix J_j,

$$u_j = J_j|, \quad v_j = D\overline{D}J_j|. \tag{11}$$

In the scalar case, i.e. at $k=1$, the Lax operator L_s^{bos} (10) in fact reproduces the Lax operator $L^{(1)}_{[s;\alpha]}$ [4] of the $(1,s)$-KdV hierarchy. Therefore, we conclude that the supersymmetric Lax operator L_s (3) at $F = \overline{F} = 0$ corresponds to the matrix $N=2$ supersymmetric generalization of the bosonic $(1,s)$-KdV hierarchy, while if it contains the superfield matrices F and $\overline{F}$, it generates the extended matrix $N=2$ $(1,s)$-KdV hierarchy.

3. Examples: scalar case. To better understand what kind of hierarchies we have proposed, let us discuss the first four hierarchies corresponding to the values $s = 0,1,2$ and $s=3$ in the Lax operator L_s (3) in a simpler and more studied scalar case (i.e., at $k=1$). In this case, J_j $(j = 1, \ldots, s)$ are generic scalar $N=2$ bosonic superfields with spins j, while the chiral (antichiral) $N=2$ superfields F_a $(\overline{F}_a)$ contain n bosonic and m fermionic components with spin $(s+1)/2$.

1. The $s=0$ case.

For this simplest case the Lax operator (3)

$$L_0 = \partial - \left[D\frac{1}{1+\overline{F}\partial^{-1}F}\overline{D}(\overline{F}\partial^{-1}F)\right] \tag{12}$$

does not contain any superfields J_j, and the chiral/antichiral superfields F and $\overline{F}$ have spins 1/2. The second–flow equations (6) have the following form:

$$\frac{\partial}{\partial t_2}F = -F\,'' + 2D(F\overline{F}\,\overline{D}F), \quad \frac{\partial}{\partial t_2}\overline{F} = \overline{F}\,'' + 2\overline{D}\left((D\overline{F})F\overline{F}\right) \tag{13}$$

and coincide with the corresponding flow of the $N=2$ GNLS hierarchy [4]. Therefore, the Lax operator (12) provides a new description of the $N=2$ GNLS hierarchy.

2. The $s=1$ case.

The Lax operator (3) has the following form:

$$L_1 = \partial - \left[D\frac{1}{\partial + J_1 + \overline{F}\partial^{-1}F}\overline{D}(J_1 + \overline{F}\partial^{-1}F)\right], \tag{14}$$

and the first two non-trivial flows (6) read as

$$\begin{aligned} &\frac{\partial}{\partial t_2}J_1 = ([D,\overline{D}\,]J_1 - J_1^2 + 2\overline{F}F)\,', \\ &\frac{\partial}{\partial t_2}F = -F\,'' + 2FD\overline{D}J_1, \quad \frac{\partial}{\partial t_2}\overline{F} = \overline{F}\,'' + 2(\overline{D}DJ_1)\overline{F}, \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{\partial}{\partial t_3}J_1 &= J_1\,''' - 3\Big[J_1[D,\overline{D}]J_1 - (\overline{D}J_1)DJ_1 - J_1^3 + J_1\overline{F}F \\ &+ (\overline{D}F)D\overline{F} + \overline{F}F\,' - \overline{F}\,'F\Big]', \\ \frac{\partial}{\partial t_3}F &= F\,''' - 3D\left[\left((\overline{D}J)F\right)' + J(\overline{D}J)F - F\overline{F}\,\overline{D}F\right], \\ \frac{\partial}{\partial t_3}\overline{F} &= \overline{F}\,''' + 3\overline{D}\left[\left((DJ)\overline{F}\right)' - J(DJ)\overline{F} + (D\overline{F})F\overline{F}\right]. \end{aligned} \tag{16}$$

From these expressions we can easily recognize that after rescaling $J_1 \to -2J_1, t_n \to -t_n$ they coincide with the corresponding flows of the $N=2$ $a=4$ KdV hierarchy [1] at $F = \overline{F} = 0$. With the non-zero superfields F and $\overline{F}$ we obtain a new extension of the $N=2$ $a=4$ KdV hierarchy. Thus, our family of $N=2$ hierarchies includes the well-known $N=2$

$a = 4$ KdV hierarchy and possesses the Lax-pair representation with the new Lax operator[2] L_1 (14).

3. The $s = 2$ case.

This case is rather interesting because it corresponds to the $N = 2$ $a = 5/2$ Boussinesq hierarchy which has been a puzzle for a long time. The Lax operator (3)

$$L_2 = \partial - \left[D\frac{1}{\partial^2 + J_1\partial + J_2 + \overline{F}\partial^{-1}F}\overline{D}(J_1\partial + J_2 + \overline{F}\partial^{-1}F)\right] \tag{17}$$

gives rise to the following second-flow equations

$$\begin{aligned} &\tfrac{\partial}{\partial t_2}J_1 = (2J_2 - J_1{}' + 2[D,\overline{D}\,]J_1 - J_1^2)\,', \\ &\tfrac{\partial}{\partial t_2}J_2 = (J_2 + 2D\overline{D}J_1)\,'' + 2(\overline{F}F)' - 2J_2J_1{}' + 2J_1D\overline{D}J_1{}', \\ &\tfrac{\partial}{\partial t_2}F = -F\,'' + 2FD\overline{D}J_1, \quad \tfrac{\partial}{\partial t_2}\overline{F} = \overline{F}\,'' + 2(\overline{D}DJ_1)\overline{F}. \end{aligned} \tag{18}$$

In the new basis[3],

$$t_2 \to -\frac{t_2}{3}, \; J_1 \to \frac{1}{3}J_1, \; J_2 \to -J_2 + \frac{1}{2}J_1{}' - \frac{1}{6}[D,\overline{D}]J_1 + \frac{2}{9}J_1{}^2, \tag{19}$$

at $F = \overline{F} = 0$, eqs. (18) coincide with the $N = 2$ $a = 5/2$ Boussinesq equation [2]. Thus, we conclude that the $N = 2$ $a = 5/2$ Boussinesq hierarchy also belongs to the family of $N = 2$ super $(1,s)$-KdV hierarchies with the Lax operator (3).

4. The $s = 3$ case.

As the last example, we present the second-flow equations

$$\begin{aligned} &\tfrac{\partial}{\partial t_2}J_1 = (2J_2 - 2J_1{}' + 3[D,\overline{D}\,]J_1 - J_1^2)\,', \\ &\tfrac{\partial}{\partial t_2}J_2 = (2J_3 + J_2{}' + 6D\overline{D}J_1{}')\,' - 2J_2J_1{}' + 4J_1D\overline{D}J_1{}', \\ &\tfrac{\partial}{\partial t_2}J_3 = (J_3 + 2D\overline{D}J_1{}')\,'' + 2J_1D\overline{D}J_1{}'' + 2J_2D\overline{D}J_1{}' \end{aligned} \tag{20}$$

of the $N = 2$ super (1,3)-KdV hierarchy which possesses the $N = 2$ W_4 algebra as the second Hamiltonian structure. This hierarchy contains the

[2]For alternative Lax-pair representations of the $N = 2$ $a = 4$ KdV hierarchy, see Refs. [1, 12, 10, 5].

[3]The complexity of these transformations is the price we have to pay for the simplicity of the Lax operator (17).

$N = 2$ superfields J_1, J_2 and J_3 with the spins 1, 2 and 3, respectively, and its Lax operator looks like

$$L_3 = \partial - \left[D\frac{1}{\partial^3 + J_1\partial^2 + J_2\partial + J_3}\overline{D}(J_1\partial^2 + J_2\partial + J_3)\right]. \tag{21}$$

The extension of this system by the superfields F and $\overline{F}$ can be straightforwardly derived from the Lax-pair representation (3), (6), and we do not present it here.

4. Examples: matrix case. The construction of flows (6) in the matrix case goes without any new peculiarities. The only difference with respect to the scalar case is the appearance of some new terms in the flow equations and their ordering. To demonstrate the difference, we present the Hamiltonian densities[4] and the flow equations for the systems considered in the previous section without comments.

1. The $s = 0$ case.

$$\mathcal{H}_2 = tr(\overline{F}F' + \overline{F}F\overline{F}F)\,, \tag{22}$$

$$\tfrac{\partial}{\partial t_2}F = -F\,'' + 2D(F\overline{F}\,\overline{D}F), \quad \tfrac{\partial}{\partial t_2}\overline{F} = \overline{F}\,'' + 2\overline{D}\left((D\overline{F})F\overline{F}\right). \tag{23}$$

These equations reproduce the second-flow equations of the $N = 2$ supersymmetric matrix GNLS hierarchy [6].

2. The $s = 1$ case.

$$\begin{aligned} &\mathcal{H}_1 = tr(J_1), \quad \mathcal{H}_2 = tr(\tfrac{1}{2}J_1^2 - \overline{F}F), \\ &\mathcal{H}_3 = tr(\tfrac{1}{3}J_1^3 - J_1D\overline{D}J_1 - \overline{F}F\,' - \overline{F}FJ_1), \end{aligned} \tag{24}$$

$$\begin{aligned} &\tfrac{\partial}{\partial t_2}J_1 = ([D,\overline{D}\,]J_1 - J_1^2 + 2\overline{F}F)\,' + [J_1,[D,\overline{D}\,]J_1], \\ &\tfrac{\partial}{\partial t_2}F = -F\,'' + 2FD\overline{D}J_1, \quad \tfrac{\partial}{\partial t_2}\overline{F} = \overline{F}\,'' + 2(\overline{D}DJ_1)\overline{F}, \end{aligned} \tag{25}$$

[4]Let us recall that Hamiltonian densities are defined up to terms which are fermionic or bosonic total derivatives of arbitrary nonsingular, local functions of the superfield matrices.

$$
\begin{aligned}
\frac{\partial}{\partial t_3}J_1 &= J_1''' + 3\Big[\Big((\overline{D}DJ_1)J_1 - D(J_1\overline{D}J_1) + \overline{F}\,'F - \overline{F}F\,' \\
&+ D\overline{D}(\overline{F}F)\Big)\,' - J_1D(J_1\overline{D}J_1) - (\overline{D}(DJ_1)J_1)J_1 \\
&+ \big\{J_1, D\overline{D}(\overline{F}F)\big\} + \overline{F}FD\overline{D}J_1 + (\overline{D}DJ_1)\overline{F}F \\
&- J_1\overline{F}F\,' - \overline{F}\,'FJ_1\Big], \\
\frac{\partial}{\partial t_3}F &= F''' + 3\big[D(F\overline{F}\,\overline{D}F) - (FD\overline{D}J_1)\,' - FD(J_1\overline{D}J_1)\big], \\
\frac{\partial}{\partial t_3}\overline{F} &= \overline{F}\,''' + 3\overline{D}\big[(D\overline{F})F\overline{F} + ((DJ_1)\overline{F})\,' - (DJ_1)J_1\overline{F}\big], \qquad (26)
\end{aligned}
$$

where the brackets $(\{,\})$ $[,]$ represent the (anti)commutator.

3. The $s=2$ case.

$$
\mathcal{H}_1 = tr(J_1), \quad \mathcal{H}_2 = tr(\frac{1}{2}J_1^2 - J_2), \qquad (27)
$$

$$
\begin{aligned}
\frac{\partial}{\partial t_2}J_1 &= (2J_2 - J_1\,' + 2[D,\overline{D}\,]J_1 - J_1^2)\,' + [J_1,[D,\overline{D}\,]J_1], \\
\frac{\partial}{\partial t_2}J_2 &= (J_2 + 2D\overline{D}J_1)\,'' + 2(\overline{F}F)' - \{J_2, J_1\,'\} + 2J_1D\overline{D}J_1\,' \\
&+ [J_2,[D,\overline{D}\,]J_1], \\
\frac{\partial}{\partial t_2}F &= -F\,'' + 2FD\overline{D}J_1, \quad \frac{\partial}{\partial t_2}\overline{F} = \overline{F}\,'' + 2(\overline{D}DJ_1)\overline{F}. \qquad (28)
\end{aligned}
$$

4. The $s=3$ case.

$$
\mathcal{H}_1 = tr(J_1), \quad \mathcal{H}_2 = tr(\frac{1}{2}J_1^2 - J_2), \qquad (29)
$$

$$
\begin{aligned}
\frac{\partial}{\partial t_2}J_1 &= (2J_2 - 2J_1\,' + 3[D,\overline{D}\,]J_1 - J_1^2)\,' + [J_1,[D,\overline{D}\,]J_1], \\
\frac{\partial}{\partial t_2}J_2 &= (2J_3 + J_2\,' + 6D\overline{D}J_1\,')\,' - \{J_2, J_1\,'\} + 4J_1D\overline{D}J_1\,' \\
&+ [J_2,[D,\overline{D}\,]J_1], \\
\frac{\partial}{\partial t_2}J_3 &= (J_3 + 2D\overline{D}J_1\,')\,'' + 2J_1D\overline{D}J_1\,'' + 2J_2D\overline{D}J_1\,' \\
&+ 2[J_3, D\overline{D}J_1]. \qquad (30)
\end{aligned}
$$

5. Involution properties. Equations (23) and (25)-(26) admit the involution

$$
\begin{aligned}
&F^* = i^{s-1}\mathcal{I}\overline{F}^T, \quad \overline{F}^* = i^{s-1}F^T\mathcal{I}, \quad J_j^* = (-1)^j J_j^T, \\
&\theta^* = \overline{\theta}, \quad \overline{\theta}^* = \theta, \quad t_p^* = (-1)^{p+1}t_p, \quad z^* = z, \quad i^* = -i, \qquad (31)
\end{aligned}
$$

for $s=0$ and $s=1$, respectively. Here, i is the imaginary unit, the symbol T means the operation of matrix transposition, and the matrix $\mathcal{I}$ is

$$
\mathcal{I} \equiv (-i)^{d_a}\delta_{ab}, \quad \mathcal{I}\mathcal{I}^* = I, \quad \mathcal{I}^3 = \mathcal{I}^*, \quad \mathcal{I}^2 = (-1)^{d_a}\delta_{ab}. \qquad (32)
$$

The same involution property is not straightforwardly satisfied for eqs. (28) ($s=2$) and eqs. (30) ($s=3$): it is satisfied in a new basis with the superfields J_2 and J_2, J_3 being replaced by

$$
J_2 \Rightarrow J_2 - \frac{1}{2}J_1\,', \qquad (33)
$$

and

$$
J_2 \Rightarrow J_2 - J_1\,', \quad J_3 \Rightarrow J_3 - \frac{1}{2}J_2\,', \qquad (34)
$$

respectively, while all the other superfields are unchanged. It seems reasonable to conjecture the existence of a basis in the space of superfield matrices where the involution (31) is admitted for any given value of the parameter s that parametrizes the Lax operator L_s (3).

6. Conclusion. In this letter, we constructed a new infinite variety of matrix $N=2$ supersymmetric hierarchies by exhibiting the corresponding super Lax operators. Their involution properties are analyzed. As a byproduct, we solved the problem of a Lax-pair description for the last remaining family of $N=2$ hierarchies with the $N=2$ super W_s algebras as their second Hamiltonian structure and derived new extensions of such familiar hierarchies as the $N=2$ supersymmetric $a=4$ KdV and $a=5/2$ Boussinesq hierarchies. New bosonic hierarchies can be obtained from the constructed supersymmetric counterparts in the bosonic limit.

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