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SPIN PROPERTIES OF FOUR-DIMENSIONAL SPACE AND CONFINEMENT

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Спиновые свойства четырехмерного пространства и конфайнмент

Выдвигаются аргументы в пользу того, что волновое уравнение для кварков может отличаться от уравнения Дирака. Предложено основное волновое уравнение для кварков и рассмотрены некоторые вытекающие отсюда следствия.

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Spin Properties of Four-Dimensional Space and Confinement

It is argued from geometrical, group-theoretical, and physical points of view that the wave equation for a quark can differ from the Dirac equation. The basic wave equation for quarks is proposed and some consequences are considered.

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1 Introduction

According to the modern standpoint, spacetime theory is the one that possesses a mathematical representation whose elements are a smooth four-dimensional manifold M and geometrical objects defined on M. The system of real local coordinates on M is defined as a topological mapping of an open region $U \subset M$ onto the Euclidean 4-dimensional space E_4. Thus, the Euclidean 4-dimensional space E_4 is a fundamental structure element of the mathematical apparatus of contemporary physics. However, it can be shown that E_4 has underlying structure that is exhibited in the existence of a group of transformations that does not coincide with the $SO(4)$ group. Consider In physical space E_3 and for comparison on physical plane E_2 consider the groups of rotations and dilatations with generators

$$D = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad M_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$

$$M_2 = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \quad M_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

and, respectively,

$$D = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad M = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Denote these groups as $D \otimes SO(3)$ and $D \otimes SO(2)$. One can show that an element of the first group can be parametrized by real numbers a, b, c, d which is suitable to consider as a quaternion $q = ai + bj + ck + d$ and for the second one we have two real parameters $w = u + iv$. It is easy to verify one-to-one correspondence between the algebra of quaternions and complex numbers and groups $D \otimes SO(3)$ and $D \otimes SO(2)$. Transformations of the groups $D \otimes SO(3)$ and $D \otimes SO(2)$ in E_3 and E_2 can be represented as follows

$$R' = qR\bar{q} \quad r' = wrw,$$

where $R = xi + yj + zk$ in the first case and $r = x + iy$ in the second. When we consider groups $D \otimes SO(3)$ and $D \otimes SO(2)$ as linear spaces, then it is not difficult to see with the help of the well known algebra that the 4d space of quaternions Q_4 and 2d space of complex numbers Q_2 give spinor representations of the groups in question, which are defined as follows

$$u' = qu, \quad z' = wz. \qquad (1)$$

A remarkable property of these transformations is that there is only one point $q = 0$, $(z = 0)$ that is stable under the transformations (1). When we fix any other point, the transformations are reduced to the identical transformation. Another important feature of transformations (1) is that the Euclidean scalar products in Q_4 and Q_2

$$(q,q) = a^2 + b^2 + c^2 + d^2, \quad (w,w) = u^2 + v^2$$

are invariant with respect to transformations (1) under the conditions $q\bar{q} = 1$, $w\bar{w} = 1$. But this does not mean that Q_4 and Q_2 are really the Euclidean spaces because (1) takes place. It should be noted that there is simple mapping from Q_2 to E_2 of the following form

$$r = z^2.$$

This mapping is known as Bohlin transformation [1]. But one can show that there is no mapping from Q_4 to the 4d Euclidean space. Thus, what we usually call the 4d Euclidean space by the analogy with the 3d physical space in reality is Q_4. The absence of such a mapping follows from the fact that there is no real Dirac matrix γ_i with the properties defined as follows

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}, \quad i,j = 1,2,3,4.$$

However, there is mapping from Q_4 to E_3 which can be defined as follows

$$R = qi\bar{q}.$$

This mapping is known as the Hopf one. So, components of the vector q are observable not in a direct way, but only through some expressions built up from these components. In a certain sense, this situation is similar to the case with a wave function of quantum mechanics. It is evident that a 3d sphere in Q_4

$$a^2 + b^2 + c^2 + d^2 = \rho^2$$

inherits the properties of the enveloping space. In view of unusual properties of a 4d space and, respectively, three-dimensional sphere S^3, consider the investigations connected with the last object.

From the geometrical point of view a 3d sphere is a space of constant positive curvature. The Kepler-Coulomb problem in this space has a long history and was first investigated by Schrödinger [2]. The Schrödinger equation for the Kepler-Coulomb problem in S^3 was recently analyzed by Pogosyan and Sissakian [3]. Path Integral Formulation of the Smorodinsky-Winternitz Potentials on the 3d sphere was presented in [4]. On the other hand, S^3 is the configuration space for the Top. The quantum-mechanical problem for free motion of a Top was investigated shortly after creation of quantum mechanics in its modern form (see for example [5]). The solution of general quantum-mechanical problem of a non-symmetric Top were expounded by Smorodinsky and Lukáč [6]. It is evident that the connection between these two directions of investigation is very important. Moreover, in the 30's it has been emphasized by Casimir [7] that from a physical point of view the notion of a rigid bogy is as fundamental as the notion of a material point. At last we would like to emphasize that QCD is conceptually a simple theory and its structure is solely determined by the symmetry principles. However, there is no connection between such important phenomena as confinement and quark-lepton symmetry, on the one hand, and the first principles QCD, on the other hand. Despite prolonged and complicated experiments, free quarks have not been observed though it is commonly accepted that quarks are true elementary particles like electrons. Experimentalists gradually came to the conclusion that the matter is not in the details of experiments but rather in the fundamental properties of the matter, search for which was made under many different assumptions. For instance, it is hypothesized that the quark confinement can be explained by topological methods that have recently found still a wider use in physics. Nevertheless, the most natural and reliable approach to the problem of confinement should be looked for in the possibility to modify the Dirac equation in view of unusual quark properties.

Summarizing all the facts considered above we put forward the conjecture that the configuration space of quarks is a three-dimensional space of constant positive curvature. In the framework of this conjecture we shall derive below the corresponding basic wave equation for a quark and consider some its properties.

2 Wave equation

Here we will define a homogeneous spacetime manifold that differs from the Minkowski spacetime by geometrical and topological properties and show that a spacetime manifold of that kind obeys all the required conditions and is of definite interest for the physics of quarks.

In the five-dimensional Minkowski spacetime $M^5_{1,4}$ with Cartesian coordinates x^A (indices denoted by capital letters run over five values $0,1,2,3,4$) we will consider the one-sheet hyperboloid H^4

$$\eta_{AB}x^Ax^B = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2 = -a^2, \qquad (2)$$

where a is the radius of H^4, and prove that it is a homogeneous spacetime.

We will use the scalar product $(X,Y) = \eta_{AB}U^AV^B$ for any vector fields $X = U^A\partial_A$ and $Y = V^A\partial_A$ on $M^5_{1,4}$. The vector fields

$$P_A = \delta^C_A\partial_C, \quad M_{AB} = (x_A\delta^C_B - x_B\delta^C_A)\partial_C,$$

where $x_A = \eta_{AB}x^B$, are generators of the Poincare group of the five-dimensional Minkowski spacetime. All vector fields M_{AB} are orthogonal to the radius-vector $R = x^C\partial_C$, whereas for vector fields P_A this is not the case. Expanding P_A in the direction of the radius-vector R and the one orthogonal to it, we obtain the vector fields

$$M_A = aP_A + \frac{1}{a}(R,P_A)R = (a\delta^C_A + \frac{1}{a}x_Ax^C)\partial_C,$$

tangent to H^4, since from (2) it follows that $(R,M_A) = 0$ at every point of H^4. The vector fields M_A and M_{AB} are generators of the group of conformal transformations of H^4 because

$$[M_A, M_B] = -M_{AB}, \quad [M_A, M_{BC}] = \eta_{AB}M_C - \eta_{AC}M_B. \qquad (3)$$

Let us now introduce the vector fields

$$D_0 = M_0, \quad D_1 = M_{14} + M_{23}, \quad D_2 = M_{24} + M_{31}, \quad D_3 = M_{34} + M_{12} \qquad (4)$$

with components

$$\begin{aligned} D_0 &= (a + \tfrac{x_0^2}{a}, & &\tfrac{x_0x^1}{a}, & &\tfrac{x_0x^2}{a}, & &\tfrac{x_0x^3}{a}, & &\tfrac{x_0x^4}{a}), \\ D_1 &= (0, & &-x_4, & &-x_3, & &x_2, & &x_1), \\ D_2 &= (0, & &x_3, & &-x_4, & &-x_1, & &x_2, \\ D_3 &= (0, & &-x_2, & &x_1, & &-x_4, & &x_3). \end{aligned}$$

It is not difficult to see that the vector fields D_0, D_1, D_2, D_3 are continuous and do not vanish at any point of H^4. As $(D_a, D_b) = 0$ for $a \neq b$, $a,b = 0,1,2,3$ and

$$(D_0,D_0) = \quad -(D_1,D_1) = \quad -(D_2,D_2) = \quad -(D_3,D_3) = \quad a^2 + x_0^2,$$

the vector fields D_0, D_1, D_2, D_3 are linearly independent at every point of H^4. From (3) it follows that

$$[D_0, D_i] = 0, \quad [D_i, D_j] = 2e_{ijk}D_k, \quad i,j,k = 1,2,3,$$

where e_{ijk} is the completely antisymmetric Levi-Civita symbol with $e_{123} = 1$. In this way, we have proved that the one-sheet hyperboloid (2) admits a simply transitive group of transformations [8] with the generators (4) having only the following nonzero structure constants

$$f^1_{23} = f^2_{31} = f^3_{12} = 2. \qquad (5)$$

Hence H^4 is really homogeneous manifold.

In accordance with the Dirac equation we write the wave equation in the homogeneous spacetime in the form

$$\gamma^cP_c\psi = \mu\psi, \qquad (6)$$

where

$$\gamma^a\gamma^b + \gamma^b\gamma^a = -2\eta^{ab},$$

$$P_c = D_c + \frac{iqa}{\hbar c}A_c - \frac{1}{2}f_c, \quad f_c = f^a_{ac},$$

q is the charge of a particle and A_c are components of the vector potential of the electromagnetic field in the basis D_a. Besides, in H^4

$$\mu = \frac{mca}{\hbar}.$$

For the present, we do not concretize the values of structure constants of a simply transitive group of transformations of the spacetime H^4. As in general $[D_a, D_b] = f^c_{ab}D_c$, we have

$$[P_a, P_b] = f^c_{ab}P_c + \frac{iqa}{\hbar c}F_{ab},$$

where

$$F_{ab} = D_a A_b - D_b A_a - f^c_{ab} A_c \tag{7}$$

are components of the strength tensor of the electromagnetic field in the basis D_a. When the wave equation is established, it is not difficult to write equations of the electromagnetic field. The Jacobi identity $[P_a[P_b, P_c]] + [P_b[P_c, P_a]] + [P_c[P_a, P_b]] = 0$ results in the first four Maxwell equations

$$D_a F_{bc} + D_b F_{ca} + D_c F_{ab} + f^d_{ab} F_{cd} + f^d_{bc} F_{ad} + f^d_{ca} F_{bd} = 0. \tag{8}$$

By analogy, from (8) it follows that the remaining Maxwell equations are of the form

$$D_a F^{ab} + f_a F^{ab} + \frac{1}{2} f^b_{ad} F^{ad} = \frac{4\pi a}{c} j^b, \tag{9}$$

where j^b are components of the current vector in the basis D_a.

Now we will write the Maxwell equations in the three-dimensional vector form. As usual, we put

$$j^a = (c\rho, \vec{\mathrm{j}}), \quad A_a = (\varphi, \quad -\vec{\mathrm{A}}),$$

$$E_i = F_{0i}, \quad H_i = \frac{1}{2} e_{ijk} F^{jk}, \quad i,j,k = 1,2,3.$$

Then from (5) and (7) we obtain

$$\vec{\mathrm{E}} = -\nabla_0 \vec{\mathrm{A}} - \nabla\varphi, \quad \vec{\mathrm{H}} = \mathrm{rot}\vec{\mathrm{A}} = \nabla \times \vec{\mathrm{A}} - 2\vec{\mathrm{A}}, \tag{10}$$

where

$$\nabla = (\nabla_1, \quad \nabla_2, \quad \nabla_3), \quad \nabla_0 = D_0, \quad \nabla_i = D_i, \quad i = 1,2,3.$$

Considering that $\mathrm{div}\vec{A} = \sum_{i=1}^{3} \nabla_i A_i$, we can write the Maxwell equations (8) and (9) in the familiar vector form

$$-\nabla_0 \vec{\mathrm{H}} = \mathrm{rot}\vec{\mathrm{E}}, \quad \mathrm{div}\vec{\mathrm{H}} = 0, \quad \mathrm{rot}\vec{\mathrm{H}} = \nabla_0 \vec{\mathrm{E}} + \frac{4\pi \mathrm{a}}{\mathrm{c}} \vec{\mathrm{j}}, \quad \mathrm{div}\vec{\mathrm{E}} = 4\pi \mathrm{a}\rho. \tag{11}$$

Making use of the commutation relations $[\nabla_i, \nabla_j] = 2e_{ijk}\nabla_k$, $i,j,k = 1,2,3$, it is not difficult to verify the identities

$$\mathrm{div}\,\mathrm{rot} = 0, \quad \mathrm{rot}\,\mathrm{grad} = 0.$$

Besides,

$$\mathrm{div}\mathrm{grad} = \Delta,$$

where Δ is the Laplacian on a three-dimensional sphere. Torsion, i.e. non-Abelian character of a simply transitive group of transformations of H^4 manifests itself not only in the definition of the operator rot, but also in the identities

$$(\mathrm{rot} + 1)^2 = -\Delta + 1 + \mathrm{grad}\,\mathrm{div}.$$

Since the space section of H^4 is a three-dimensional sphere, it is interesting to show that the Dirac equation is connected with the Schrödinger equation for the spherical Top. To verify this, we will derive eigenvalues E of the Dirac Hamiltonian in question when there is no electromagnetic field, i.e. $F_{ab} = 0$. Squaring equation (6) and using (5), we obtain the following equation for E

$$E^2\psi = m^2c^4\psi - \frac{c^2\hbar^2}{a^2}(\Delta + P)\psi, \tag{12}$$

where

$$P = \Sigma_1\nabla_1 + \Sigma_2\nabla_2 + \Sigma_3\nabla_3$$

and $\Sigma_i = \frac{1}{2}e_{ijk}\gamma^j\gamma^k$. The operator P has properties analogous to those of the operator rot. In particular,

$$(P+1)^2 = -\Delta + 1. \tag{13}$$

Since $\Delta + P = -P(P+1)$, then

$$E^2 = m^2c^4 + p(p+1)\frac{c^2\hbar^2}{a^2},$$

where p is an eigenvalue of the operator P.

To determine eigenvalues of the operator P, consider Hermitean operators acting in the space of solutions to the wave equation (6). Generators of the group mutual to the simply transitive group of transformations of the spacetime H^4 are of the form

$$E_0 = D_0, \quad E_1 = M_{14} - M_{23}, \quad E_2 = M_{24} - M_{31}, \quad E_3 = M_{34} - M_{12},$$

which gives the three Hermitean operators

$$N_i = -\frac{i}{2} E_i$$

analogous to the momentum operators. The other three operators

$$M_i = -\frac{i}{2}(\nabla_i - \Sigma_i) = -\frac{i}{2}(D_i - \Sigma_i) \qquad (14)$$

are analogs of operators of the angular momentum of an electron. From (14) it follows that the spin of a particle in question equals $\hbar/2$. We have

$$\vec{\mathrm{M}} \times \vec{\mathrm{M}} = \mathrm{i}\vec{\mathrm{M}}, \quad \vec{\mathrm{N}} \times \vec{\mathrm{N}} = \mathrm{i}\vec{\mathrm{N}}$$

and, besides,

$$2(M^2 + N^2) = (P + \frac{3}{2})^2 - \frac{3}{4}, \quad 2(M^2 - N^2) = P + \frac{3}{2}. \qquad (15)$$

Hence, we have the operator equation

$$2(M^2 + N^2) + \frac{3}{4} = 4(M^2 - N^2)^2.$$

Since $M^2 = l(l+1)$ and $N^2 = k(k+1)$, then from the operator equation we derive the equation for l and k

$$2[l(l+1) + k(k+1)] + \frac{3}{4} = 4[l(l+1) - k(k+1)]^2.$$

This equation has two solutions: $l = k + \frac{1}{2}$ and $k = l + \frac{1}{2}$. Then, it follows that $p = 2[l(l+1) - k(k+1)] - \frac{3}{2} = -2k - 3$. Since S^3 has a metric invariant with respect to the isometric reflection, then $p = 2k + 3$ is eigenvalue too. Thus, for the energy we have the following expression

$$E^2 = m^2c^4 + n(n+1)\frac{c^2\hbar^2}{a^2} = m^2c^4(1 + n(n+1)\frac{\lambda^2}{a^2}), \qquad (16)$$

where $n = 2, 3, ...$ and $\lambda = \hbar/mc$. If formula (16) gives the quantum-mechanical value of the energy of the relativistic spherical Top, then at large a the moment of inertia $I = ma^2$ is also large and the angular velocity

is small. So, the nonrelativistic limit can be found from the condition $a \gg \lambda$. In the limit of large a it follows from (16) that

$$E = mc^2 + \frac{L^2}{2I},$$

where $L^2 = n(n+1)\hbar^2$ is the angular momentum of the spherical top and I is its moment of inertia. The latter relation is consistent with the classical formula

$$E = \frac{L^2}{2I}$$

for the energy of a Top. Thus, formula (16) gives the energy of rotation.

Now consider the Coulomb law. As it is known, the Coulomb potential can be derived as a solution of the equations of electrostatics invariant under the group of Euclidean motions including rotations and translations. We will look for the Coulomb potential in the considered case in an analogous manner. From (10) and (11) it follows that for a constant electric field $\mathrm{div}\vec{\mathrm{E}} = 4\pi a\rho, \quad \vec{\mathrm{E}} = -\nabla\varphi$, and consequently, φ obeys the equation

$$\Delta\varphi = -4\pi a^2 \rho. \qquad (17)$$

An invariant of the group of rotations $O(4)$ on a three-dimensional sphere is either the arc length or the angle between radius-vectors,

$$X = (x^1, x^2, x^3, x^4), \quad Y = (y^1, y^2, y^3, y^4)$$
$$\cos\theta = \frac{1}{a^2}(x^1y^1 + x^2y^2 + x^3y^3 + x^4y^4).$$

Since

$$M_{ij}\cos\theta = (x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i})\cos\theta = \frac{1}{a^2}(x^iy^j - x^jy^i),$$

setting in (17) $\rho = 0, \quad \varphi = \varphi(z)$, where $z = \cos\theta$, we obtain the following equation for $\varphi(z)$

$$(1 - z^2)\frac{d^2\varphi}{dz^2} - 3z\frac{d\varphi}{dz} = 0.$$

The general solution to this equation is of the form

$$\varphi(z) = c_1\frac{z}{\sqrt{1 - z^2}} + c_2 = c_1 \cot\theta + c_2,$$

where c_1 and c_2 are arbitrary constants. If one particle has the coordinates $(0,0,0,a)$, then from the last formula we obtain the well-known expression [3]. A more interesting result follows from the mapping S^3 onto E_3.

Introduce the frame of reference with respect to which one of the charged quarks is at rest and has coordinates $(0,0,0,-a)$. In this system consider a stereographic projection of the three-dimensional sphere $(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2 = a^2$ from point $(0,0,0,a)$ onto the hyperplane $x^4 = 0$ with Cartesian coordinates x, y, z. We have

$$x^1 = x\frac{2a^2}{r^2+a^2}, \quad x^2 = y\frac{2a^2}{r^2+a^2},$$
$$x^3 = z\frac{2a^2}{r^2+a^2}, \quad x^4 = a\frac{r^2-a^2}{r^2+a^2},$$

where $r^2 = x^2+y^2+z^2$. It may be verified that in the coordinates x, y, z

$$\cot\theta = \frac{a}{2r} - \frac{r}{2a}$$

and, consequently, the Coulomb potential can be written in the form

$$\varphi(r) = q \quad (\frac{1}{2r} - \frac{r^2}{2a} - \frac{1}{a}), \tag{18}$$

where q is the charge. As the potential (18) does coincide with the known Cornel potential [9], we can conclude that our consideration should be developed so as to explain a successful application of the latter for describing the charmonium [10].

3 Conclusion

As the basic wave equation describing the dynamics of quarks, we have suggested the modified Dirac equation (6) written here in homogeneous coordinates. The conclusion that quarks are described by the wave equation different from the conventional wave equation for electrons is quite natural. In fact, it would be strange if the description of such different particles were based on the same equation.

The physical meaning of the phenomenon called the confinement consists in that quarks possess properties of a quantum-mechanical spherical Top. This means, in particular, that the Cornel potential expresses the fundamental physical law.

At large a, when $a \to \infty$ from the theory of quarks we derive the theory of electrons but with electrons evidently deconfined, because in this case the region of confinement is the Euclidean space. Thus, the symmetry between quakrs and leptons has a natural explanation.

As the kinematics of quarks differ from the kinematics of electrons, there are possible such decays of hadrons and nuclei in which the energy conservation law is fulfilled but the momentum is not conserved.

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