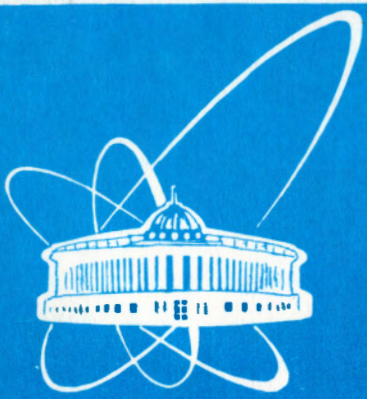

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PERTURBATIVE FRAMEWORK FOR THE $\pi^+\pi^-$ ATOM

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Пертурбативный подход к описанию $\pi^+\pi^-$-атома

На основе теоретико-полевого уравнения Бете–Солпитера развит пертурбативный подход для вычисления характеристик $\pi^+\pi^-$-атома. Получено замкнутое выражение для поправки первого порядка к времени жизни $\pi^+\pi^-$-атома.

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Perturbative Framework for the $\pi^+\pi^-$ Atom

The perturbative framework is developed for the calculation of the $\pi^+\pi^-$ atom characteristics on the basis of the field-theoretical Bethe–Salpeter approach. A closed expression for the first-order correction to the $\pi^+\pi^-$ atom lifetime has been obtained.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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At present time the experiments on the study of hadronic atoms $\pi\pi$ [1], πp, πd [2] have being carried out. Namely, the first estimate of the $\pi^+\pi^-$ atom lifetime was given in Ref. [1]. Now the DIRAC collaboration works out an experiment at CERN on the high precision measurement of the lifetime of $\pi^+\pi^-$ atoms. This experiment might provide a decisive improvement in the direct determination of the difference of the S-wave $\pi\pi$ scattering lengths and thus serve as a valuable test for the predictions of Chiral Perturbation Theory [3]. In the view of these experiments there arises a need in the theoretical framework which would enable one to calculate the characteristics of such atoms with a high accuracy based on the ideas of standard model.

The theoretical study of hadronic atoms starts from Refs. [4]-[6] where the non-relativistic relations of the energy level displacement of the hadronic atom due to strong interactions and its lifetime with the strong scattering lengths are established. The expression for the width Γ_0 of the $\pi^+\pi^-$ atom in the ground state is

$$\Gamma_0 = \frac{16\pi}{9}\sqrt{\frac{2\Delta m_\pi}{m_\pi}}\,(a_0^0 - a_0^2)^2\,\phi_0^2 \qquad (1)$$

where Δm_π is the $m_{\pi^\pm} - m_{\pi^0}$ mass difference, and ϕ_0 is the value of the Coulomb wave function (w.f.) of the pionium at the origin.

The approach to the study of the problem of hadronic atoms, developed in Ref. [4], makes use of the general characteristic feature of the hadronic atoms – the factorization of strong and electromagnetic interactions. The formula (1) demonstrates this factorization property explicitly, expressing the atom lifetime as a product of two factors – the Coulomb w.f. at the origin and the strong interaction factor, completely concentrated in the $\pi\pi$ strong scattering lengths.

The problem of evaluation of the electromagnetic and strong corrections to the basic formula (1) within different approaches is addressed in Refs. [7]-[14]. For a brief review see Ref. [12]. In this paper within the Bethe-Salpeter (BS) approach we have derived the relativistic analogue of the formula (1) taking into account the correction due to the displacement of the bound state pole position by strong interactions (strong correction) in the first order. This correction was found to be of the relative order 10^{-3}. It should be stressed that the field-theoretical approaches

[10, 12, 13, 14] to the problem, unlike the potential treatment [7, 11], do not refer to a concept of the phenomenological strong interaction $\pi\pi$ potential, which is a source of an additional ambiguity in the calculations of hadronic atom characteristics. In the former approaches these characteristics are expressed directly in terms of the underlying strong interaction (chiral) Lagrangian, and the results can be compared to the experiment, providing the consistent test of the predictions of chiral theory.

In the present work we suggest a relativistic perturbative framework for the calculation of the energy levels and lifetime of hadronic atoms. The main purpose of this work is to demonstrate a possibility (not only in the potential scattering theory, but in the BS treatment as well) of the clear-cut factorization of strong and electromagnetic interactions in the observable characteristics of hadronic atoms, avoiding the double-counting problem in the calculation of these quantities. One should note that the suggested approach allows to calculate strong and electromagnetic corrections in all orders of the perturbation theory. At the present stage we apply the general formalism to the calculation of the first-order strong and electromagnetic corrections to the pionium lifetime. The results for strong corrections obtained in Ref. [12] are reproduced in these calculations.

Our framework is based on the perturbative expansion which is performed around the solution of the BS equation with the Coulomb kernel similar to that introduced in Ref. [15]

$$V_C(\mathbf{p},\mathbf{q}) = \sqrt{w(\mathbf{p})}\frac{4im_\pi e^2}{(\mathbf{p}-\mathbf{q})^2}\sqrt{w(\mathbf{q})}, \qquad w(\mathbf{p}) = \sqrt{m_\pi^2+\mathbf{p}^2} \tag{2}$$

The factor $\sqrt{w(\mathbf{p})w(\mathbf{q})}$ introduced in the kernel (2) enables one to reduce the BS equation with such a kernel to the exactly solvable Schrödinger equation with the Coulomb potential. Then, the exact solution of the BS equation with this kernel is written in the form

$$\psi_C(p) = iG_0(M^\star;p)4\sqrt{w(\mathbf{p})}\,\frac{4\pi\alpha m_\pi\phi_0}{\mathbf{p}^2+\gamma^2}, \quad \bar\psi_C(p) = \psi_C(p), \tag{3}$$

where $\gamma = m_\pi\alpha/2$ and $M^{\star 2} = m_\pi^2(4-\alpha^2)$ is the eigenvalue corresponding to the unperturbed ground-state solution. G_0 denotes the free Green's function of the $\pi^+\pi^-$-pair. The exact Green's function corresponding to the Coulomb kernel (2)

is given by the well-known expression

$$G_C(P^\star;p,q) = (2\pi)^4\delta^{(4)}(p-q)G_0(P^\star;p) + G_0(P^\star;p)T_C(E^\star;\mathbf{p},\mathbf{q})G_0(P^\star;q). \tag{4}$$

Here T_C is given by

$$T_C(E^\star;\mathbf{p},\mathbf{q}) = 16i\pi m_\pi\alpha\sqrt{w(\mathbf{p})w(\mathbf{q})}\left[\frac{1}{(\mathbf{p}-\mathbf{q})^2}+\int_0^1\frac{\nu d\rho\rho^{-\nu}}{D(\rho;\mathbf{p},\mathbf{q})}\right] \tag{5}$$

$$D(\rho;\mathbf{p},\mathbf{q}) = (\mathbf{p}-\mathbf{q})^2\rho - \frac{m_\pi}{4E^\star}\left(E^\star-\frac{\mathbf{p}^2}{m_\pi}\right)\left(E^\star-\frac{\mathbf{q}^2}{m_\pi}\right)(1-\rho)^2,$$

where $\nu = \alpha\sqrt{m_\pi/(-4E^\star)}$ and $E^\star = (P^{\star 2}-4m_\pi^2)/(4m_\pi)$.

The full BS equation for the $\pi^+\pi^-$ atom w.f. $\chi(p)$ is written as

$$G_0^{-1}(P;p)\chi(p) = \int\frac{d^4k}{(2\pi)^4}V(P;p,q)\chi(q), \tag{6}$$

where $V(P;p,q)$ denotes the full BS kernel which is constructed from the underlying (effective) Lagrangian according to the general rules and includes all strong and electromagnetic two-charged-pion irreducible diagrams. In particular, it contains the diagrams with two neutral pions in the intermediate state which determine the decay the $\pi^+\pi^-$ atom into $\pi^0\pi^0$. Note that in addition $V(P;p,q)$ contains the charged pion self-energy diagrams attached to the outgoing pionic legs (with the relative momentum q), which are two-particle reducible. These diagrams arise in the definition of the kernel $V(P;p,q)$ because the free two-particle Green's function is used in the l.h.s. of Eq. (6) instead of the dressed one. The c.m. momentum squared P^2 of the atom has the complex value, corresponding to the fact that the atom is an unstable system. According to the conventional parametrization, we can write $P^2 = \bar M^2 = M^2 - iM\Gamma$ where M denotes the "mass" of the atom, and Γ is the atom decay width.

The full four-point Green's function $G(P)$ for the kernel V has a pole in the complex P^2 plane at the bound-state energy. The relation between the exact w.f. $\chi(p)$ and the Coulomb w.f. ψ_C is given by [12]

$$<\chi| = C<\psi_C|\,G_C^{-1}(P^\star)G(P), \quad P^{\star 2}\to M^{\star 2},\ P^2\to\bar M^2 \tag{7}$$

where C is the normalization constant. In what follows we assume that the limiting procedure is performed with the use of the prescription [12] $P^{\star 2} = M^{\star 2}+\lambda$, $P^2 =$

$\bar{M}^2 + \lambda$, $\lambda \to 0$. The validity of Eq. (7) can be trivially checked, extracting the bound-state pole in $G(P)$ and using the BS equation for ψ_C.

In order to perform the perturbative expansion of the bound-state characteristics M and Γ around the unperturbed values we, as in Ref. [12], split the full BS kernel V into two parts as $V = V_C + V'$ and consider V' as a perturbation. It can be shown that Eq. (7) is equivalent to

$$< \chi| = -C^{-1} < \psi_C|\left[1 + (\Delta G_0^{-1} - V')G_R Q\right]^{-1}, \quad \Delta G_0^{-1} = G_0^{-1}(P) - G_0^{-1}(P^*). \quad (8)$$

With the use of Eq. (8) the following identity is easily obtained

$$< \psi_C|\left[1 + (\Delta G_0^{-1} - V')G_R Q\right]^{-1}(\Delta G_0^{-1} - V')|\psi_C >= 0, \quad (9)$$

which is an exact relation and serves as a basic equation for performing the perturbative expansion for the bound-state energy.

In the Eqs. (8) and (9) $G_R Q$ stands for the regular (pole subtracted) part of the Coulomb Green's function (4), projected onto the subspace, orthogonal to the ground-state unperturbed solution. This quantity can be further split into two pieces, according to $G_R Q = G_0(M^*) + \delta G$. Here the function δG corresponds to the ladder of the exchanged Coulomb photons and thereby contains explicit powers of α. It is given by the following expression:

$$\begin{aligned}
\delta G &= i\sqrt{w(\mathbf{p})w(\mathbf{q})}\left[\Phi(\mathbf{p},\mathbf{q}) - S(\mathbf{p})S(\mathbf{q})\frac{8}{M^*}\frac{\partial}{\partial M^*}\right]G_0(M^*,p)G_0(M^*,q) \\
\Phi(\mathbf{p},\mathbf{q}) &= 16\pi m_\pi \alpha\left[\frac{1}{(\mathbf{p}-\mathbf{q})^2} + I_R(\mathbf{p},\mathbf{q})\right] + (m_\pi\alpha)^{-2}S(\mathbf{p})S(\mathbf{q})R(\mathbf{p},\mathbf{q}) \qquad (10) \\
S(\mathbf{p}) &= 4\pi m_\pi \alpha \phi_0(\mathbf{p}^2+\gamma^2)^{-1}, \; R(\mathbf{p},\mathbf{q}) = 25 - \sqrt{\frac{8}{\pi m_\pi \alpha}}[S(\mathbf{p}) + S(\mathbf{q})] + \cdots
\end{aligned}$$

where the ellipses stand for the higher-order terms in α. The integral $I_R(\mathbf{p},\mathbf{q})$ is given by

$$I_R(\mathbf{p},\mathbf{q}) = \int_0^1 \frac{d\rho}{\rho}\,[D^{-1}(\rho;\mathbf{p},\mathbf{q}) - D^{-1}(0;\mathbf{p},\mathbf{q})], \;\; E^* = -\frac{1}{4}m_\pi\alpha^2. \quad (11)$$

The equation (8) expresses the exact BS w.f. of the atom in terms of the unperturbed w.f. via the perturbative expansion in the perturbation potential V'. This potential consists of the following pieces:

1. The purely strong part, which is isotopically invariant. This part survives when the electromagnetic interactions are "turned off" in the Lagrangian.

2. The part, containing the diagrams with the finite mass insertions which are responsible for the $m_{\pi^\pm} - m_{\pi^0}$ electromagnetic mass difference.

3. The part, containing the exchanges of one, two, ... virtual photons and an arbitrary number of strong interaction vertices.

Note that the terms 1 and 2 are more important due to the following reasons. The first term includes strong interactions responsible for the decay of the pionium. The second term makes this decay kinematically allowed due to finite difference of charged and neutral pion masses. Consequently, it seems to be natural to consider together the pieces 1 and 2. We refer to the corresponding potential as V_{12}. The T-matrix corresponding to the potential V_{12} is defined by $T_{12}(P) = V_{12}(P) + V_{12}(P)G_0(P)T_{12}(P)$. The rest of the potential V' is referred as $V_3 = V' - V_{12}$. In what follows we restrict ourselves to the first order in the fine structure constant α, i.e. consider the diagrams with only one virtual photon contained in V_3.

Returning to the basic equation (9), we expand it in the perturbative series considering V_3 and δG as perturbations. Meanwhile we expand ΔG_0^{-1} in the Taylor series in $\delta M = M - M^*$ and substitute $M = M^* + \Delta E^{(1)} + \Delta E^{(2)} - i/2\,\Gamma^{(1)} - i/2\,\Gamma^{(2)} + (8M^*)^{-1}\Gamma^{(1)2} + \cdots$.

Restricting ourselves to the first order of the perturbative expansion, we arrive at the following relations

$$\Delta E^{(1)} = \mathrm{Re}\left(\frac{i}{2M^*}\frac{T_{12}}{m_\pi}\phi_0^2\right), \quad -\frac{1}{2}\Gamma^{(1)} = \mathrm{Im}\left(\frac{i}{2M^*}\frac{T_{12}}{m_\pi}\phi_0^2\right). \quad (12)$$

Hereafter we use the local approximation for T_{12}, assuming that it does not depend on the relative momenta. The Eqs. (12) coincide with the well-known Deser-type formulae for the energy-level displacement and lifetime [4]. Note that on the mass shell

$$\mathrm{Re}(iT_{12}) \sim T(\pi^+\pi^- \to \pi^+\pi^-), \;\; \mathrm{Im}(iT_{12}) \sim \sqrt{\Delta m_\pi}|T(\pi^+\pi^- \to \pi^0\pi^0)|^2. \quad (13)$$

If we assume $V_3 = \delta G = 0$, we arrive at the result

$$\frac{\Gamma^{(2)}}{\Gamma^{(1)}} = -\frac{9}{8}\frac{\Delta E^{(1)}}{E_1} - 0.763\alpha, \qquad E_1 = -\frac{1}{4}m_\pi\alpha^2. \quad (14)$$

The first term of this expression called "strong correction" was obtained in our previous paper [12]. However, as opposed to the present derivation, in Ref. [12] we have used the Born approximation for the calculation of $\Delta E^{(1)}$, i.e. in Eq. (12) T_{12} was substituted by V_{12}. The last term comes from the relativistic normalization factor $\sqrt{w(\mathbf{p})w(\mathbf{q})}$ in the kernel (2) and corresponds to relativistic modification of the pionium Coulomb w.f. $|\int d^4p/(2\pi)^4\psi_C(p)|^2 = \phi_0^2(1-0.381\alpha)^2/m_\pi$. Since this correction comes from the Coulomb w.f. of the atom, it does not depend on the parameters of the strong $\pi\pi$ interaction, and for this reason it was neglected in Ref. [12].

Inclusion of δG introduces the correction due to the exchange of the infinite number of Coulomb photons in the lifetime. The integrals emerging in the calculation of this correction are ultraviolet convergent, containing, however (in a complete analogy with a well-known result from nonrelativistic scattering theory), an infrared enhancement $\alpha\ln\alpha$ which stems from the one-photon exchange piece in Eq. (5). Collecting all terms together and using Eqs. (12) for relating $\mathrm{Im}T_{12}$ to $\Delta E^{(1)}$, we finally arrive at the first-order correction to the pionium rate

$$\Gamma = \Gamma_0\Bigg(1+\underbrace{\left(-\frac{9}{8}\frac{\Delta E^{(1)}}{E_1}\right)}_{\text{strong}}+\underbrace{(-0.763\alpha)}_{\text{relativistic w.f.}}+\underbrace{(1/2+2.694-\ln\alpha)\frac{\Delta E^{(1)}}{E_1}}_{\text{Coulomb photon exchanges}}+$$

$$+\ \delta_M-\left(M^*\Gamma^{(1)}\right)^{-1}\mathrm{Re}<\psi_C|(1+T_{12}G_0(M^*))V_3(1+G_0(M^*)T_{12})|\psi_C>\Bigg) \quad (15)$$

where δ_M stands for the mass shift correction [14] and the last term collects the radiative corrections [14, 13] (including retardation correction [10], correction due to vacuum polarization [8], etc.). In the Eq. (15) all first-order strong and electromagnetic corrections are given in a closed form avoiding any difficulties connected with double counting problem. The kernel which appears in the last term: $(1+T_{12}G_0(M^*))V_3(1+G_0(M^*)T_{12})$, is constructed from the underlying Lagrangian with the use of the conventional Feynman diagrammatic technique. The detailed reexamination of the above mentioned corrections within BS approach will be addressed in our forthcoming publications.

In order to estimate the size of the calculated corrections to the pionium lifetime (Eq. (15) we have used the following value $m_\pi(2a_0^0+a_0^2)=0.49$ of the singlet scattering length corresponding to the value $\Delta E^{(1)}/E_1 = 0.24\%$. The first, second

and third terms then contribute, respectively, -0.26%, -0.55% and $+1.85\%$, and the total contribution amounts up to $\sim 1\%$ to the decay width (apart from the mass shift and radiative corrections). The largest contribution comes from the $\alpha\ln\alpha$ term in Eq. (15).

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