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DIRECT MODE SUMMATION FOR THE CASIMIR ENERGY OF A SOLID BALL

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Помодовое суммирование для энергии Казимира диэлектрического шара

Энергия Казимира материального шара, помещенного в бесконечную среду, рассчитана путем суммирования собственных частот с использованием контурного интегрирования. Сначала предполагается, что диэлектрическая и магнитная проницаемости шара и окружающего диэлектрика связаны условием $\varepsilon_1\mu_1 = \varepsilon_2\mu_2$. Затем проведен расчет для случая $(\varepsilon_1 - \varepsilon_2)^2/(\varepsilon_1 + \varepsilon_2)^2 \ll 1$. При этом энергия Казимира положительна и увеличивается с уменьшением радиуса шара. Такой результат полностью исключает возможность того, что эффект Казимира является причиной сонолюминесценции пузырьков в жидкости.

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Direct Mode Summation for the Casimir Energy of a Solid Ball

The Casimir energy of a solid ball placed in an infinite medium is calculated by a direct frequency summation using the contour integration. First it is assumed that the permittivity and permeability of the ball and medium satisfy the condition $\varepsilon_1\mu_1 = \varepsilon_2\mu_2$. Then the calculations are extended to the dilute dielectric ball $(\varepsilon_1 - \varepsilon_2)^2/(\varepsilon_1 + \varepsilon_2)^2 \ll 1$. The Casimir energy for the last configuration turns out to be positive, it being increased when the radius of the ball decreases. The latter eliminates completely the possibility of explaining, via the Casimir effect, the sonoluminescence for bubbles in a liquid.

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1 Introduction

The Casimir energy, determined by the first quantum correction to the ground state of a quantum field system with allowance for nontrivial boundary conditions, proves to be essential in many problems of the elementary particle theory, in quantum cosmology, and in physics of condensed matter. However, up to now there is no universal method for calculating the Casimir effect for arbitrary boundary conditions. This has been done only for simple field configurations of high symmetry: gap between two plates, sphere, cylinder, wedge and so on. The curvature of the boundary and account of the dielectric and magnetic properties of the medium lead to considerable complications. While the attractive force between two uncharged metal plates has been calculated by Casimir as far back as 1948 [1], this effect for perfectly conducting spherical shell in vacuum was computed by Boyer only in 1968 [2] (see also the latter calculations [3, 4, 5, 6]). If an infinitely thin spherical shell separates media with arbitrary dielectric $(\varepsilon_1, \varepsilon_2)$ and magnetic (μ_1, μ_2) characteristics, this problem is not solved till now. The main drawback here is the lack of a consistent method for removing the divergences. Besides an attempt to revive the quasiclassical model of an extended electron proposed by Casimir [7], interest in this problem was also initiated by investigations of the bag models in hadron physics [8, 9, 10] and recently by search for the mechanism of sonoluminescence [11].

In the case of nonmagnetic media $(\mu_1 = \mu_2 = 1)$ with permittivities $(\varepsilon_1, \varepsilon_2)$ slightly different $\xi^2 \ll 1, \quad \xi = (\varepsilon_2 - \varepsilon_1)/(\varepsilon_1 + \varepsilon_2)$ the Casimir energy for this configuration has been found in [12]. Unlike the perfectly conducting spherical shell [3, 6] the Casimir energy of a dilute dielectric ball proved to be negative. By making use of an estimation of this energy [13, 14] a conclusion was drawn that the Casimir effect for air bubbles in liquid cannot explain the sonoluminescence as it was suggested by Schwinger [11].

In this paper we calculate the Casimir energy of a solid ball by making use of the direct summation of eigenfrequencies of vacuum electromagnetic field employing the contour integration [15, 16]. A definite advantage of this method, compared with the Green's function technique employed in [12, 13, 14, 17], is its simplicity and visualization. First we consider a dielectric ball placed in an infinite medium when the condition $\varepsilon_1\mu_1 = \varepsilon_2\mu_2$ holds. In this way we attain some generalization and refinement of the results obtained in this problem earlier [17]. Further we address ourselves to the case of a dilute dielectric ball ($\mu_1 = \mu_2 = 1$ and $\xi^2 \ll 1$). The Casimir energy for this configuration proves to be positive, it being increased when the radius of the ball decreases. The latter eliminates completely the possibility of explaining, via the Casimir effect, the sonoluminescence for bubbles in a liquid.

The layout of the paper is as follows. In Sect. II we derive the Casimir energy of a solid ball in a infinite surrounding under condition $\varepsilon_1\mu_1 = \varepsilon_2\mu_2 = c^{-2}$, where c is an arbitrary constant not necessary equal to one (it is the light velocity in the medium), the mode-by-mode summation of eigenfrequencies being employed. Use of the uniform asymptotics of the Bessel function enables us to derive the first two terms of the expansion of the Casimir energy in hand with respect to ξ^2. In Sect. III the general formula derived for the Casimir energy is applied to a dielectric ball under the condition $\xi^2 \ll 1$. The implication of the obtained result to the Schwinger attempt to explain the sonoluminescence by the Casimir effect is also considered. In Conclusion (Sect. IV) the results of the paper are briefly discussed. Dispersive effects are ignored in our paper.

2 Casimir energy of a solid ball under the condition $\varepsilon_1\mu_1 = \varepsilon_2\mu_2$

Let us consider the Casimir energy of a solid ball of radius a, consisting of a material which is characterized by permittivity ε_1 and permeability μ_1. We assume that the ball is placed in an infinite medium with permittivity ε_2 and permeability μ_2. We also suppose

that the conductivity of the ball material and its surroundings is equal to zero.

In our consideration the main part will be played by equations determining the eigenfrequencies ω of the electromagnetic oscillations for this configuration [18]. It is convenient to rewrite these equations in terms of the Riccati-Bessel functions

$$\tilde{s}_l(x) = xj_l(x), \quad \tilde{e}_l(x) = xh_l^{(1)}(x), \tag{2.1}$$

where $j_l(x) = \sqrt{\pi/2x}J_{l+1/2}(x)$ is the spherical Bessel function and $h_l^{(1)}(x) = \sqrt{\pi/2x}H_{l+1/2}^{(1)}(x)$ is the spherical Hankel function of the first kind. For the TE-modes the frequency equation reads

$$\Delta_l^{TE}(a\omega) \equiv \sqrt{\varepsilon_1\mu_2}\,\tilde{s}_l'(k_1a)\tilde{e}_l(k_2a) - \sqrt{\varepsilon_2\mu_1}\,\tilde{s}_l(k_1a)\tilde{e}_l'(k_2a) = 0, \tag{2.2}$$

where $k_i = \sqrt{\varepsilon_i\mu_i}\omega, \quad i = 1, 2$ are the wave numbers inside and outside the ball, respectively; prime stands for the differentiation with respect to the argument (k_1a or k_2a) of the corresponding Riccati-Bessel function. The frequencies of the TM-modes are determined by

$$\Delta_l^{TM}(a\omega) \equiv \sqrt{\varepsilon_2\mu_1}\,\tilde{s}_l'(k_1a)\tilde{e}_l(k_2a) - \sqrt{\varepsilon_1\mu_2}\,\tilde{s}_l(k_1a)\tilde{e}_l'(k_2a) = 0. \tag{2.3}$$

The orbital quantum number l in (2.2) and (2.3) assumes the values $1, 2, \ldots$. Under mutual change $\varepsilon_i \leftrightarrow \mu_i, \quad i = 1, 2$ frequency equations (2.2) and (2.3) transform into each other.

It is worth noting that the frequencies of the electromagnetic oscillations determined by Eq. (2.2) and (2.3) are the same inside and outside the ball. The physical reason for this is that photons do not perform work when passing through the boundary at $r = a$. This is in contrast to the case of perfectly conducting spherical shell in vacuum [6], where eigenfrequencies inside the shell and outside it are determined by different equations [18].

As usual we define the Casimir energy by the formula

$$E = \frac{1}{2}\sum_s(\omega_s - \bar{\omega}_s), \tag{2.4}$$

where ω_s are the roots of Eqs. (2.2) and (2.3) and $\bar{\omega}_s$ are the same roots under condition $a \to \infty$. Here s is a collective index that stands for a complete set of indices for the roots of Eqs. (2.2) and (2.3). Denoting the roots of Eqs. (2.2) and (2.3) by $\omega_{nl}^{(1)}$ and $\omega_{nl}^{(2)}$, respectively, we can cast Eq. (2.4) in the explicit form

$$E = \frac{1}{2}\sum_{\alpha=1}^{2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\sum_{n=1}^{\infty}\left(\omega_{nl}^{(\alpha)} - \bar{\omega}_{nl}^{(\alpha)}\right) = \sum_{l=1}^{\infty} E_l, \qquad (2.5)$$

where the notation

$$E_l = (l+1/2)\sum_{\alpha=1}^{2}\sum_{n=1}^{\infty}\left(\omega_{nl}^{(\alpha)} - \bar{\omega}_{nl}^{(\alpha)}\right) \qquad (2.6)$$

is introduced. Here we have taken into account that the eigenfrequencies $\omega_{nl}^{(\alpha)}$ do not depend on the azimuthal quantum number m. For partial energies E_l we use representation in terms of the contour integral provided by the Cauchy theorem [19]

$$E_l = \frac{l+1/2}{2\pi i}\oint_C dz\, z\,\frac{d}{dz}\ln\frac{\Delta_l^{TE}(az)\Delta_l^{TM}(az)}{\Delta_l^{TE}(\infty)\Delta_l^{TM}(\infty)}, \qquad (2.7)$$

where the contour C surrounds, counterclockwise, the roots of the frequency equations in the right half-plane. Location of the roots of Eqs. (2.2) and (2.3) enables one to deform the contour C into a segment of the imaginary axis $(-i\Lambda, i\Lambda)$ and a semicircle of radius Λ in right half-plane. At a given value of Λ a finite number of the roots of frequency equations is taken into account. Thus Λ plays the role of a regularization parameter for the initial sum in Eq. (2.6) which should be subsequently removed to infinity. In this limit the contribution of the semicircle of radius Λ into integral (2.7) vanishes. From physical considerations it is clear that multiplier z in (2.7) is understood to be the $\lim_{\mu\to 0}\sqrt{z^2+\mu^2}$, where μ is the photon mass. Therefore in the integral along the segment $(-i\Lambda, i\Lambda)$ we can integrate once by

parts, the nonintegral terms being canceled. As a result Eq. (2.7) acquires the form

$$E_l = \frac{l+1/2}{\pi a}\int_0^\infty dy \ln\frac{\Delta_l^{TE}(iy)\Delta_l^{TM}(iy)}{\Delta^{TE}(i\infty)\Delta_l^{TM}(i\infty)}. \qquad (2.8)$$

Now we need the modified Riccati-Bessel functions

$$s_l(x) = \sqrt{\frac{\pi x}{2}}I_\nu(x), \qquad e_l(x) = \sqrt{\frac{2x}{\pi}}K_\nu(x), \quad \nu = l+1/2, \qquad (2.9)$$

where $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions [20]. With allowance for the asymptotics of $s_l(x)$ and $e_l(x)$ at $x\to\infty$ and fixed l

$$s_l(x) \simeq \frac{1}{2}e^x, \qquad (2.10)$$

$$e_l(x) \simeq e^{-x} \qquad (2.11)$$

equation (2.8) can be rewritten as

$$E_l = \frac{l+1/2}{\pi a}\int_0^\infty dy \ln\left\{\frac{4e^{-2(q_1-q_2)}}{(\sqrt{\varepsilon_1\mu_2}+\sqrt{\varepsilon_2\mu_1})^2}\right.$$
$$\times\left[\sqrt{\varepsilon_1\varepsilon_2\mu_1\mu_2}\left((s_l'(q_1)e_l(q_2))^2 + (s_l(q_1)e_l'(q_2))^2\right)\right. \qquad (2.12)$$
$$\left.\left. -(\varepsilon_1\mu_2+\varepsilon_2\mu_1)s_l(q_1)s_l'(q_2)e_l(q_2)e_l'(q_2)\right]\right\},$$

where $q_i = \sqrt{\varepsilon_i\mu_i}y$, $i = 1, 2$. We shall use this general equation in the next Section but here we address ourselves to the special case when the condition

$$\varepsilon_1\mu_1 = \varepsilon_2\mu_2 = c^{-2} \qquad (2.13)$$

is fulfilled. Here c is an arbitrary positive constant (the light velocity in medium). Physical implications of this condition at $c = 1$ can be found in [21]. Now Eq. (2.12) is simplified considerably

$$E_l = \frac{c(l+1/2)}{\pi a}\int_0^\infty dy \ln\Bigg\{\frac{4}{\varepsilon+\varepsilon^{-1}+2}\Big[(s_l'(y)e_l(y))^2+(s_l(y)e_l'(y))^2 -(\varepsilon+\varepsilon^{-1})s_l(y)s_l'(y)e_l(y)e_l'(y)\Big]\Bigg\}, \quad (2.14)$$

where $\varepsilon = \varepsilon_1/\varepsilon_2$. The argument of the logarithm in (2.14) can be transformed, if the following two equalities for the functions $s_l(y)$ and $e_l(y)$

$$s_l'(y)e_l(y) - s_l(y)e_l'(y) = -1, \quad (2.15)$$

$$s_l'(y)e_l(y) + s_l(y)e_l'(y) = (s_l(y)e_l(y))'. \quad (2.16)$$

are taken into account. It gives

$$E_l = \frac{c(l+1/2)}{\pi a}\int_0^\infty dy \ln\left\{1-\xi^2\left[(s_l(y)e_l(y))'\right]^2\right\} \quad (2.17)$$

where $\xi = (1-\varepsilon)/(1+\varepsilon)$. Expression (2.17) agrees with the results obtained in [22, 13], if one performs a partial integration of the expression for E given in these references and puts the cutoff parameter δ equal to zero. However, Eq. (2.17) differs from the energy corresponding the Casimir force (Eq. (2.42)) derived in Ref. [17, (1982)]. There seems to have occurred a calculational error in that reference. If $\varepsilon = 0$ or ∞ and $c = 1$ then, as one could expect, Eq. (2.17) turns into the analogous expression for the perfectly conducting spherical shell in vacuum [3, 6]. In the further consideration we shall follow Ref. [6]. Integral in (2.17) converges as at large y and fixed l we have [20]

$$(s_l(y)e_l(y))' \equiv (yI_\nu(y)K_\nu(y))' \simeq \frac{4\nu^2-1}{8y^3}. \quad (2.18)$$

On the other hand, for $y \to 0$

$$(s_l(y)e_l(y))' \to \frac{1}{2l+1}. \quad (2.19)$$

The behavior of E_l at large l is deduced by applying a uniform asymptotic expansion of the Bessel functions [20]. This gives

$$\frac{a}{c}E_l\Big|_{l\to\infty} \simeq -\frac{3}{64}\xi^2 + \frac{9}{16384\nu^2}\xi^2(6-7\xi^2) + \mathcal{O}(\nu^{-4}). \quad (2.20)$$

We find the sum over l in (2.5) by making use of the Hurwitz zeta-function technique [23]

$$\begin{aligned} E &= \sum_{l=1}^\infty E_l = \sum_{l=1}^\infty\left(E_l + \frac{3c}{64a}\xi^2 - \frac{3c}{64a}\xi^2\right) \\ &= \sum_{l=1}^\infty \bar{E}_l - \frac{3c}{64a}\xi^2\sum_{l=1}^\infty (l+1/2)^0 \\ &= \sum_{l=1}^\infty \bar{E}_l - \frac{3c}{64a}\xi^2[\zeta(0,1/2)-1], \end{aligned} \quad (2.21)$$

where $\bar{E}_l$ is the renormalized partial Casimir energy

$$\bar{E}_l = E_l + \frac{3c}{64a}\xi^2, \quad (2.22)$$

with $\zeta(z,q)$ being the Hurwitz zeta function [19]

$$\zeta(z,q) = \sum_{n=0}^\infty \frac{1}{(q+n)^z}. \quad (2.23)$$

For $q = 1/2$ the relation [24]

$$\zeta(z,1/2) = (2^z-1)\zeta(z) \quad (2.24)$$

holds. Whence it follows in particular that $\zeta(0,1/2) = 0$. In view of this the Casimir energy (2.21) acquires the form

$$E = \sum_{l=1}^\infty \bar{E}_l + \frac{3c}{64a}\xi^2. \quad (2.25)$$

The sum $\sum_{l=1}^{\infty} \bar{E}_l$ is finite because we have for large l from Eq. (2.20)

$$\frac{a}{c}\bar{E}_l \simeq \frac{9}{2^{14}\nu^2}\xi^2(6-7\xi^2). \tag{2.26}$$

With allowance for this we obtain the estimation for the sum $\sum_{l=1}^{\infty} \bar{E}_l$

$$\begin{aligned} \frac{a}{c}\sum_{l=1}^{\infty}\bar{E}_l &\simeq \frac{9}{2^{14}}\xi^2(6-7\xi^2)\sum_{l=1}^{\infty}\frac{1}{(l+1/2)^2} \\ &= \frac{9}{2^{14}}\xi^2(6-7\xi^2)\left(\frac{\pi^2}{2}-4\right) \\ &= 5.135\cdot 10^{-4}\xi^2(6-7\xi^2). \end{aligned} \tag{2.27}$$

Thus the basic contribution into Eq. (2.25) is due to the second term. Therefore with a fairly good accuracy (a few percents) one can put

$$E \simeq \frac{3c}{64a}\xi^2. \tag{2.28}$$

Taking into account Eq. (2.27) we obtain, in place of (2.28), a more precise formula

$$E \simeq \frac{3c}{64a}\xi^2(1.066-0.077\xi^2). \tag{2.29}$$

At all the values of the parameter ξ, $0 \le \xi^2 \le 1$ the Casimir energy of a solid ball is positive. For $c=1$ Eq. (2.28) has been derived in [17] by making use of the Green's function technique. Formula (2.29) even at $c=1$ differs from Eq. (2.60) in Ref. [17, (1982)]. More precise result for the Casimir energy E can be obtained by direct integration in Eq. (2.17) for the first values of l and only after that using the asymptotics (2.26). Certainly, it can be done only for a given numerical value of the parameter ξ.

Concluding it is worth to note once more that when calculating the Casimir energy of a solid ball under condition (2.13) the divergences have been removed in the same way as in the case of perfectly conducting spherical shell [3, 6].

3 Casimir energy of a dilute dielectric ball

Now we address ourselves to the consideration of the Casimir energy of a dielectric ball, when its permittivity and permittivity of surrounding differ slightly

$$\varepsilon_1+\varepsilon_2 = 2\varepsilon, \quad \varepsilon_2-\varepsilon_1 = 2\Delta\varepsilon, \quad |\Delta\varepsilon|/\varepsilon \ll 1. \tag{3.1}$$

The permeabilities μ_1 and μ_2 are assumed to be equal to 1. When this condition is satisfied, the general formula for the Casimir energy (2.12) can be simplified putting there $q_1 = q_2$. Making use of Eqs. (2.15) and (2.16) again we arrive at Eq. (2.12) with

$$\xi^2 = \left(\frac{\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}}\right)^2 \simeq \frac{1}{4}\left(\frac{\Delta\varepsilon}{\varepsilon}\right)^2 \text{ and } c = \frac{1}{\sqrt{\varepsilon}}. \tag{3.2}$$

According to our assumption (3.1) $\xi^2 \ll 1$, therefore we can expand the logarithm in Eq. (2.17)

$$\begin{aligned} \ln\{1-\xi^2[(s_l(y)e_l(y))']^2\} &\simeq -\xi^2[(s_l(y)e_l(y))']^2\big|_{\nu\to\infty} \\ &\simeq -\frac{\xi^2}{4\nu^2}\frac{1}{(1+z^2)^3}, \end{aligned} \tag{3.3}$$

where z is defined by $y = \nu z$ and $\nu = l + 1/2$ (cf. Eq. (2.54) in Ref. [17, (1982)]). This leads to

$$\begin{aligned} E_{ball} &\simeq \frac{1}{2\pi a}\sum_{l=1}^{\infty}\int_0^{\infty}\nu\, dz\, 2\nu\left(-\frac{\xi^2}{4\nu^2}\right)\frac{1}{(1+z^2)^3} \\ &= -\frac{\xi^2}{4\pi a}\sum_{l=1}^{\infty}\nu^0\int_0^{\infty}\frac{dz}{(1+z^2)^3} = -\frac{\xi^2}{4\pi a}\frac{3\pi}{16}\sum_{l=1}^{\infty}\nu^0. \end{aligned} \tag{3.4}$$

With $\sum_{l=1}^{\infty}\nu^0 = -1$, as before, we obtain, in dimensional units, the Casimir energy of a dilute dielectric ball [25]

$$E_{ball} \simeq \frac{3\hbar c}{64a}\xi^2 \simeq \hbar c\frac{3}{4}\frac{(\varepsilon_2-\varepsilon_1)^2}{256\varepsilon^{3/2}a}. \tag{3.5}$$

As one could expect, this is precisely Eq. (2.28) with a new definition of ξ^2 given in (3.2). We remind that Eq. (2.28) has been derived by taking at first the limit $\nu \to \infty$. It is worth giving here some numerics. Take $|\xi| = 0.1$, $a = 4 \cdot 10^{-4}$ cm. Then $E_{ball} \sim 2 \cdot 10^{-5}$ eV. This is immensely smaller than the amount of energy ($\sim$ 10 MeV) emitted in a sonoluminescent flash. Moreover the Casimir energy (3.5) is always positive and increases when the radius of the ball, a, decreases. The latter, obviously, completely eliminates possibility of using the Casimir effect for explanation of the sonoluminescence. As known [26], emission of light takes place at the end of collapsing the bubbles in liquid. In this respect, more precise formula (2.29) with allowance for (3.2) does not give anything new. It should be emphasized, however, that all our arguments are concerned with the static Casimir effect only.

Comparing our result for the Casimir energy of a dilute dielectric ball (3.5) with other calculations of this energy we see that it is close to Eqs. (3.17) and (3.26) in Ref. [13] differing only by the factor $9\pi/46 \simeq 0.6$. This is important for justification of our consideration because Eqs. (3.17) and (3.26) in [14] have been derived in the framework of absolutely different but physically clear approach by a direct summation of the van der Waals forces. Our result (3.5) differs by the factor $-3/4$ from Eq. (7.5) in [13] and by the dependence on $\Delta\varepsilon$ from the calculation in [27].

4 Conclusion

Our method for calculating the Casimir energy E by means of the contour integral (2.7) proves to be very convenient and effective. As known, there are in principle at least two different methods for calculating E: one can follow a local approach, implying use of the Green's function to find the energy density (or the surface force density). Or, one can sum the eigenfrequencies directly. Equation (2.7) thus means that we have adopted the latter method here. The Cauchy integral formula turns out to be most useful in other context also, such as in the calculation of the Casimir energy for a piece wise uniform relativistic string [28]. A survey on this subject can be found in [29]. The great advantage of the method is that the multiplicity of zeros in the dispersion function is automatically taken care of, i.e., one does not have to plug in the degeneracy in the formalism by hand.

A remarkable feature of our approach is also the ultimate formula for the Casimir energy having the form of the spectral representation, i.e., of an integral with respect to frequency between the limits $(0, \infty)$ of a smooth function, spectral density. Evidently, for physical applications one needs to know the frequency range which gives the main contribution into the spectral density. An example of this representation for the partial energies E_l is Eq. (2.17), where the substitution $y = \omega a$ should be done. As shown above, the partial energies E_l decrease rapidly as l increases. Therefore the most interesting is a few first values of l. In this case, as one could expect, the spectral density is different from zero when $\omega a \simeq 1$. Keeping in mind the search for the origin of the sonoluminescence we put [13, 26] $a = 4 \cdot 10^{-4}$ cm. Then the wave length of the photon in question turns out to be $25.0 \cdot 10^{-4}$ cm, i.e., this radiation belongs to infrared region, while in experiments on sonoluminescence the blue light is observed [26]. This fact also testifies against the possibility of explaining the sonoluminescence by the Casimir effect.

It is worth noting that the spectral distribution of the Casimir energy is practically not discussed in literature while the space density of this energy has been investigated in detail (see, for example [17, (1983)]. From the physical point of view the space density and spectral density of energy in this problem should be treated on the same footing. One can remind here the treatment of the Casimir effect as a manifestation of the fluctuations of the vacuum fields [30], these fluctuations being occurred in space and time simultaneously.

It should be emphasized that in this paper we have neglected the dispersion effects when calculating the Casimir energy. Importance of this point has been demonstrated in [27]. As for the elucidation of the sonoluminescence origin, we have to stress once more that in our consideration we have contented ourselves with the static Casimir effect only.

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References

[1] H. B. K. Casimir, Proc. Kon. Ned. Akad. Wet. **51**, 793 (1948).

[2] T. H. Boyer, Phys. Rev. **174**, 1764 (1968).

[3] K. A Milton, L. L. DeRaad Jr., and J. Schwinger, Ann. Phys. (N.Y.) **115**, 338 (1978).

[4] R. Balian and B. Duplantier, Ann. Phys. (N.Y.) **112**, 165 (1978).

[5] S. Leseduarte and A. Romeo, Ann. Phys. (N.Y.) **250**, 448 (1996)

[6] V. V. Nesterenko and I. G. Pirozhenko, *Simple method for calculating the Casimir energy for sphere*, Preprint JINR E2-97-240, Dubna, 1997; hep-th/9707253, to be published in Phys. Rev. D.

[7] H. B. K. Casimir, Physica, **19**, 846 (1956).

[8] K. A. Milton, Phys. Rev. D **22**, 1441, 1444 (1980); Ann. Phys. (N.Y.) **127**, 49 (1980); **150**, 432 (1983).

[9] M. Bordag, E. Elizalde, K. Kirsten, and S. Leseduarte, *Casimir energies for massive fields in the bag*, Preprint Barcelona University UB-ECM-PF 96/14, hep-th/9608071.

[10] E. Elizalde, M. Bordag, and K. Kirsten, *Casimir energy in the MIT bag model*, hep-th/9707083.

[11] J. Schwinger, Proc. Nat. Acad. Sci. U.S.A. **90**, 958, 2105, 4505, 7285 (1993); **91**, 6473 (1994).

[12] K. A. Milton, Ann. Phys. (N.Y.) **127**, 49 (1980).

[13] K. A. Milton and Y. J. Ng, Phys. Rev. E **55**, 4207 (1997).

[14] K. A. Milton and Y. J. Ng, *Observability of the bulk Casimir effect: Can the dynamical Casimir effect be relevant to sonoluminescence?*, Preprint OKHEP-97-04, hep-th/9707122.

[15] G. Lambiase and V. V. Nesterenko, Phys. Rev. D **54**, 6387 (1996).

[16] V. V. Nesterenko and I. G. Pirozhenko, *Justification of the zeta function renormalization in rigid string model*, Preprint JINR E2-97-75, Dubna, 1997; hep-th/9703097, to be published in J. Math. Phys. (N.Y.).

[17] I. Brevik and H. Kolbenstvedt, Ann. Phys. (N.Y.) **143**, 179 (1982); **149**, 237 (1983).

[18] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).

[19] E. T. Wittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1969).

[20] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, D. C., 1964; reprinted by Dover, New York, 1972).

[21] T. D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood, New York, 1981).

[22] K. A. Milton, in *Proceedings of the Third Workshop on Quantum Field Theory Under the Influence of External Conditions, Leipzig, 1995*, edited by M. Bordag, (Teubner, Stuttgart, 1996), p. 13.

[23] A comprehensive introduction into this method is provided by: E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerlini, *Zeta regularization techniques with applications* (World Scientific, Singapore, 1994).

[24] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1980).

[25] For definiteness we refer to the Casimir energy of a dielectric ball although it stands no reason that our consideration is also applicable to a spherical cavity in infinite slab of dielectric.

[26] L. A. Crum, Physics Today, **47**, № 9, 22 (1994); B. P. Barber, R. A. Hiller, R. Löfstedt, S. J. Putterman, and K. Weniger, Phys. Rep. **281**, 65 (1997).

[27] I. Brevik, H. Skurdal, and R. Sollie, J. Phys. A **27**, 6853 (1994).

[28] I. Brevik and E. Elizalde, Phys. Rev. D **49**, 5319 (1994).

[29] M. H. Berntsen, I. Brevik, and S. D. Odintsov, Ann. Phys. (N.Y.) **257**, 84 (1997).

[30] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1960).

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