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PROPER TIME AXIS OF A CLOSED RELATIVISTIC SYSTEM

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Let us consider a closed relativistic system. In our consideration the space-time is a four dimensional pseudo-Euclidean space $E_{(1,3)}$ with the following metric:

$$\theta_{ab}\, dx^a\, dx^b = dt^2 - \frac{1}{c^2}\left(dx^2 + dy^2 + dz^2\right), \qquad (1)$$

$$a = 0, 1, 2, 3,$$

c — being the light velocity.

Let us denote by p^a the 4-momentum of a particle, where

$$p^a = p^0\, \frac{dx^a}{dt},$$

$$p^0 = m \Big/ \sqrt{1 - \frac{1}{c^2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right)}, \qquad (2)$$

m — being the mass of the particle. Consequently,

$$m^2 = \theta_{ab}\, p^a\, p^b. \qquad (3)$$

We denote by p_a the following components:

$$p_a = \theta_{ab}\, p^b. \qquad (4)$$

Let us denote by m^{ab} the angular 4-momentum of the particle. It equals

$$m^{ab} = x^a\, p^b - x^b\, p^a, \qquad (5)$$

where x^a are the coordinates of the particle. This momentum is associated with the origin of coordinates.

Now let us consider a Cauchy hyper-surface Σ in $E_{(1,3)}$ and a system of N particles on it, not yet their interacting. The Cauchy hyper-surface in $E_{(1,3)}$ is a space-like one. Let us define the 4-momentum of the system as

$$P^a = \sum_{k=1}^{N} p^a_{(k)} \tag{6}$$

and the angular 4-momentum of the system as

$$M^{ab} = \sum_{k=1}^{N} m^{ab}_{(k)}. \tag{7}$$

Here the index k numbers our particles,

$$m^{ab}_{(k)} = x^a_{(k)}\, p^b_{(k)} \;-\; x^b_{(k)}\, p^a_{(k)}. \tag{8}$$

The points $x^a_{(k)}$ and the momenta $p^a_{(k)}$ are considered on the hyper-surface Σ.

During the interaction the number of particles N may be changed, but not the vector P^a and the bivector M^{ab}: the last ones are integrals of the equations of motion, since our system is closed.

The angular momenta (7) and (8) are associated with the origin coordinates O. Let us replace the point O with a point $\breve{O}(\breve{t}, \breve{x}, \breve{y}, \breve{z})$. Correspondingly, we replace $x^a_{(k)}$ with $(x^a_{(k)} - \breve{x}^a)$, $m^{ab}_{(k)}$ with

$$\breve{m}^{ab}_{(k)} = m^{ab}_{(k)} \;-\; \breve{x}^a\, p^b_{(k)} \;+\; \breve{x}^b\, p^a_{(k)}, \tag{9}$$

and M^{ab} with

$$\breve{M}^{ab} = M^{ab} \;-\; \breve{x}^a\, P^b \;+\; \breve{x}^b\, P^a. \tag{10}$$

Consequently, if $\breve{x}^a = \lambda\, P^a$, then $\breve{M}^{ab} = M^{ab}$. Here λ is an arbitrary number.

Now let us find such points, so that

$$\breve{M}^{ab}\, P_b = 0. \tag{11}$$

Inserting here (10), we receive the following equation for
the proper time axis of our system:

$$(\breve{x}^a\, P^b \;-\; \breve{x}^b\, P^a)\, P_b \;=\; M^{ab}\, P_b. \tag{12}$$

One can see that if a point $\breve{x}^a_0$ is a solution of the equation (12), then the point $\breve{x}^a_0 + \lambda\, P^a$ is a solution of the same equation too.

In our case

$$P^0 > 0,\ (P, P) = P^a P_a = \theta_{ab}\, p^a\, p^b > 0; \tag{13}$$

the set of solutions of equation (12) is a time-like straight line

$$\breve{x}^a = \breve{x}^a_0 + \lambda\, P^a \tag{14}$$

in the space-time $E_{(1,3)}$. This straight line is the proper time axis of our system.

Let us write down equation (12) in the form:

$$\breve{x}^a\, (P, P) \;-\; P^a\, (P, \breve{x}) \;=\; M^{ab}\, P_b. \tag{15}$$

$$(P,\breve{x}) = P_b\,\breve{x}^b.$$

Setting $a=0$, we have

$$\breve{t}\,(P,P) - P^0\,(P,\breve{x}) = M^{0b}\,P_b. \tag{16}$$

From (15) and (16) we get

$$\breve{x}^a = \frac{P^a}{P^0}\,\breve{t} + \frac{P^0 M^{ab} P_b - P^a M^{0b} P_b}{P^0(P,P)}. \tag{17}$$

This is another form of writing down Eq (14).

If $P^1 = P^2 = P^3 = 0$, then

$$\breve{x}^\alpha = \frac{M^{\alpha 0}}{P^0}. \tag{18}$$

In this case the proper time axis is parallel to the coordinate time axis.

If $P^1 = P^2 = P^3 = 0$, and $M^{10} = M^{20} = M^{30} = 0$, then

$$\breve{x}^\alpha = 0. \tag{19}$$

In the last case the proper time axis coincides with the coordinate time axis.

Let us consider now the case when the system moves in the plane $z=0$. Then $p^3_{(k)} = 0$ and $m^{a3}_{(k)} = 0$. Consequently, in this case $P^3 = 0$, $M^{a3} = 0$ and $\breve{M}^{a3} = 0$. If we denote

$$\breve{M}_0 = \breve{M}^{12},\ \ \breve{M}_1 = \breve{M}^{20},\ \ \breve{M}_2 = \breve{M}^{01}, \tag{20}$$

then

$$\breve{M}^{0b} P_b = \breve{M}_2\,P_1 - \breve{M}_1\,P_2,$$

$$\breve{M}^{1b} P_b = \breve{M}_0\,P_2 - \breve{M}_2\,P_0,\ \ \breve{M}^{2b} P_b = \breve{M}_1\,P_0 - \breve{M}_0\,P_1. \tag{21}$$

From Eqs. (11) and (21) it follows

$$\frac{\breve{M}_0}{P_0} = \frac{\breve{M}_1}{P_1} = \frac{\breve{M}_2}{P_2}. \tag{22}$$

Inserting here (10), we receive the following equations for the proper time axis of our system in the case $z=0$:

$$\frac{M_0 - \breve{x}^1\,P^2 + \breve{x}^2\,P^1}{P_0} =$$

$$= \frac{M_1 - \breve{x}^2\,P^0 + \breve{x}^0\,P^2}{P_1} = \frac{M_2 - \breve{x}^0\,P^1 + \breve{x}^1\,P^0}{P_2}. \tag{23}$$

In engineering statics of an absolutely rigid body the problem about reducing of a system of forces to the dynamic screw is considered (see for instance [4]). The dynamic screw consists of a force and a couple, lying in a plane, perpendicular to this force.

In the case $z=0$, starting from equations (23), we come upon the problem about the dynamic screw, assuming that the light velocity c is equal to the imaginary unit i. In such a way the particle momenta play the role of forces, acting on the rigid body.

In the general case, setting up in Eq.(1) $c^2 = -1$ and starting from (12), we arrive at the problem about the dynamic screw in a four-dimensional Euclidean space.

References

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