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DUAL SYMMETRY IN GAUGE THEORIES

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1 Introduction

The position of Gauge Theories in the modern physics is exceptionally important. The very name includes the adjective pointing out to the important quality in the theories – gauge symmetry. Such theories involve also other symmetries, not less important ones. Dual symmetry has taken its own place in the number as well. It does not seem to be honorable enough.

One cannot say that the dual symmetry was studied little – besides a lot of articles there are also a few books. Especially this topic was discussed very actively when the magnetic monopole problem claimed attention of physicists. Another related topic is instantons and monopoles in the non-Abelian theories.

Still, it should be once more emphasized that this field has not been studied enough. The Dual Symmetry (DUSYa) is the stepdaughter of Field Theory while Gauge Symmetry is the favorite one.

It will be reminded the dual symmetry like gauge one is a necessary attribute of gauge theories. And in this fact an idea of Unification, passing through the modern physics, is displayed. However, what could be said of duality in Gravitation? Nearly nothing! But this is one of the important gauge theories. In this paper it is shown that there exists two-parameter dual group in Gravitation.

A lot of attempts to solve the magnetic monopole problem in electrodynamics by means of duality have been mentioned to be undertaken. There are many references to this subject in [1] and [2]. More modern approaches can be seen, e.g., in [3, 4].

Dual invariance of the pure electrodynamics equations was found rather long ago. Having at first appeared as a discrete symmetry, this invariance was then described as continuous one (dual turning) by G. Rainich [5], C. Misner and J. Wheeler [6].

The interesting and strange fact is that the pure Maxwell electrodynamics equations are invariant under the dual turning, and the Lagrangian does not possess such a symmetry.

And what about introducing the dual symmetry in theory from the very beginning in a standard manner, having chosen the dual invariant as a Lagrangian? The obvious objection to this idea is that the motion equations are badly nonlinear. Nevertheless, it will be seen further that the inserted dual symmetry models are of interest. For example, there may be an opportunity to interpret the nonlinear dual theory as a linear Maxwell theory with source. In this case the original theory nonlinearity is hidden into the source. Non-Abelian fields can be considered according to a similar scheme.

On the other hand, the dual-covariant electrodynamics equations may well be interpreted as linear equations of the electrodynamics with magnetic charge. Transition from the picture with electrical charge to the picture with magnetic one is provided by means of local dual transformation. Local dual angle arises in the theory in a natural way. A similar angle (phase, complexion) was discussed by C. Misner and J. Wheeler [6].

Establishing the dual group in Gravitation enables us to speak about gravitational instantons as solutions to the duality equations like it happens in the Yang–Mills theory. Moreover, the instanton sector is here far richer than in the Yang–Mills case since there are two irreducible representations of the dual group. In addition this fact gives new opportunities to obtain new gravitational equations, to construct the Lagrangians.

2 Dual Symmetry in Electrodynamics

2.1 The Dual Transformations

In the modern form the dual transformations were introduced by Rainich [5], and Misner and Wheeler [6]:

$$F'_{\mu\nu} = F_{\mu\nu}\cos\theta + *F_{\mu\nu}\sin\theta \equiv e^{*\theta}F_{\mu\nu},$$

$$*F'_{\mu\nu} = -F_{\mu\nu}\sin\theta + *F_{\mu\nu}\cos\theta \equiv e^{*\theta} * F_{\mu\nu}, \qquad (1)$$

where $*F_{\mu\nu} = 1/2\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$.

In view of formal similarity of the dual conjugation operator and imaginary unit and because of the fact that the transition operator from $\mathbf{F}$ to $\mathbf{F}'$ looks like turning in some plane as well, it becomes evident that transformation (1) is referred to as the dual turning.

The pure Maxwell electrodynamics equations with the conventional quadratic Lagrangian

$$\mathcal{L} = gF^{\mu\nu}F_{\mu\nu} \equiv g\mathbf{F}^2$$

prove to be invariant under the transformations (1) while the Lagrangian itself does not possess the invariance! That is, there is no symmetry but this points out that the symmetry might be realized nonlinearly (compare with spontaneous symmetry breaking). Well-known as though,this rather unusual fact did not claim due attention of the theorists.

Meanwhile, the authors of the book [1], playing on invariance of equations but ignoring the Lagrangian noninvariance, try to introduce the "conservative" dual current what seems to us to be not quite correct.

It would be quite in spirit of the modern approaches to consider the Lagrangian with the explicit dual symmetry. It is known there is the only dual invariant which is also Lorentz one. It has not been considered (as far as I know) as a Lagrangian what is quite clear – it would lead to the badly nonlinear motion equation. Nevertheless it is of interest to consider the model of pure electrodynamics with dual-invariant Lagrangian, first, in virtue of the unusual and very nice properties of the model and, second, since this essentially nonlinear theory may well relate in some way both to the Maxwell electrodynamics and to the magnetic charge problem.

One can obtain the only dual-invariant expression as follows. Let us consider a complex quadratic form

$$\mathbf{F}^2 + i * \mathbf{F}\mathbf{F}.$$

This form is usually regarded in complexifying $\mathbf{F}$-space. It is invariant under the complex orthogonal transformations $O(3,C)$ [7]. With respect to the dual group (1) the form transforms as follows:

$$\mathbf{F}^2 + i * \mathbf{FF} = (\mathbf{F}'^2 + i * \mathbf{F}'\mathbf{F}')e^{2i\theta}.$$

First, this formula establishes isomorphism between the dual rotations and the ordinary phase transformations. Second, it is getting clear that the transformations (1) are not a subgroup of $O(3, C)$. That is, the complex orthogonal transformations should not mix up with the dual ones. Now it is quite clear that

$$(\mathbf{F}^2)^2 + (*\mathbf{FF})^2 = inv.$$

It is the only expression to be invariant under either the Lorentz or the dual transformations. Occasionally, it should be noted that for any component $F_{\mu\nu} = f$

$$f + i * f = (f' + i * f')e^{i\theta} \quad \longrightarrow f^2 + *f^2 = inv.$$

Let the Lagrangian of the pure dual electrodynamics be chosen as follows:

$$\mathcal{L} = \sqrt{(\mathbf{F}^2)^2 + (*\mathbf{FF})^2}.$$

Variational principle brings about the equations

$$\left(F^{\mu\nu} \frac{\mathbf{F}^2}{\sqrt{(\mathbf{F}^2)^2 + (*\mathbf{FF})^2}} + *F^{\mu\nu} \frac{*\mathbf{FF}}{\sqrt{(\mathbf{F}^2)^2 + (*\mathbf{FF})^2}} \right)_{,\nu} = 0.$$

It is seen that the local angle φ determined by electromagnetic strength at the point of space-time appears:

$$\cos\varphi = \frac{\mathbf{F}^2}{\sqrt{(\mathbf{F}^2)^2 + (*\mathbf{FF})^2}}, \quad \sin\varphi = \frac{*\mathbf{FF}}{\sqrt{(\mathbf{F}^2)^2 + (*\mathbf{FF})^2}}.$$

And now the motion equations can be written in a shorter form

$$(F^{\mu\nu}\cos\varphi + *F^{\mu\nu}\sin\varphi)_{,\nu} = 0.$$

Denoting

$$F^{\mu\nu}\cos\varphi + *F^{\mu\nu}\sin\varphi = e^{*\varphi(x)}F^{\mu\nu},$$

one can rewrite these equations in a more elegant way:

$$(e^{*\varphi(x)}F^{\mu\nu})_{,\nu} = 0. \tag{2}$$

Adding the identity

$$*F^{\mu\nu}{}_{,\nu} \equiv 0, \tag{3}$$

we obtain the complete system (set) of dual electrodynamics equations (2),(3) which is essentially non-linear.

2.2 Transformation Properties of the Model under the Dual Group

Further it is written down how some values transform.

$$F_{\mu\nu} = e^{-*\theta}F'_{\mu\nu}, \quad *F_{\mu\nu} = e^{-*\theta} * F'_{\mu\nu}, \tag{4}$$

$$\cos\varphi = \cos(\varphi' + 2\theta), \quad \sin\varphi = \sin(\varphi' + 2\theta),$$

$$e^{*\varphi}F_{\mu\nu} = e^{*\theta}(e^{*\varphi'}F'\mu\nu). \tag{5}$$

All these formulae could be obtained directly. We can see that the left-hand sides of the system (2),(3) transform in various ways. The left-hand side of (3) transforms like (4), i.e., *co-gradiently* with respect to the field $F_{\mu\nu}$. As seen from (5), the left-hand side of (2) transforms by means of invert transformation, i. e. *contra-gradiently* with respect to $F_{\mu\nu}$.

2.3 Dual Electrodynamics as a Maxwell System with Source

One can rewrite the set (2),(3) otherwise, just resolving the first of the equations with respect to $F^{\mu\nu}{}_{,\nu}$:

$$F^{\mu\nu}{}_{,\nu} = \varphi_{,\nu}(F^{\mu\nu}\tan\varphi - *F^{\mu\nu}) \equiv j^\mu, \tag{6}$$

$$*F^{\mu\nu}{}_{,\nu} \equiv 0.$$

One can consider such representation of (2),(3) as a way to break down the symmetry since both sides of (6) transform in different ways . Thereby, the equation (6) takes the form of the Maxwell equation with source. Let us suppose that the nonlinear set (6) has a solution of the form

$$F_{0\mu} = \mathbf{E} \sim \frac{1}{r}, \qquad j^0 = \mathcal{O}\left(\frac{1}{r}\right), \quad r \to \infty.$$

Then perhaps the notation of the right-hand side of (6) as a current could be justified. It will be noted that genuine central-symmetry field makes the right-hand side of (6) to be equal zero. That is all right. But it would be better to find an *asymptotically* central-symmetric solution.

Going on to speculate on this matter, it could be noted that quantization of charge which is effective in such a theory would arise as a result of quantizing the nonlinear field theory.

2.4 Instantons in Electrodynamics

Instantons in electrodynamics are known to be absent because of the gauge group topology triviality. Although the term "instanton" should be made more precise.

Let us notice the motion equations are satisfied if the condition

$$F_{\mu\nu} \cos\varphi + *F_{\mu\nu} \sin\varphi = 0 \tag{7}$$

is fulfilled. Let it be called the generalized instanton equation. Why instanton it will be seen later when discussing an analogous equation in the non-Abelian theory. It is convenient to denote $a = \mathbf{F}^2$, $b = *\mathbf{FF}$. Then, projecting (7) onto $\mathbf{F}$, one obtains $a^2 + b^2 = 0$, or, on account of the reality of the fields $*\mathbf{F}$ and $\mathbf{F}$, $a = 0$, $b = 0$. Thus, the electrodynamical instantons are, e.g., plane waves.

2.5 The Local Dual Transformations

Appearance of the angle $\varphi(x)$ in a natural manner prompts to introduce the space-time point-dependent dual transformations. So

$$\tilde{F}^{\mu\nu} = e^{*\varphi} F^{\mu\nu}. \tag{8}$$

To go further, it is necessary to know what is $*\tilde{\mathbf{F}}$? Its definition is introduced as follows. For any function f to be given on $\mathbf{F}$-space the dual conjugation procedure is

$$* f(\mathbf{F}) = f(*\mathbf{F}). \tag{9}$$

Then, for example,

$$* \tilde{F}_{\mu\nu}(\mathbf{F}) = \tilde{F}_{\mu\nu}(*\mathbf{F}). \tag{10}$$

Noticing that

$$\cos\varphi(*\mathbf{F}) = -\cos\varphi, \qquad \sin\varphi(*\mathbf{F}) = -\sin\varphi,$$

we find by means of (10)

$$* \tilde{F}_{\mu\nu} = F_{\mu\nu} \sin\varphi - *F_{\mu\nu} \cos\varphi = -e^{*\varphi} * F_{\mu\nu}. \tag{11}$$

Now one can easily prove the properties

$$(\tilde{\mathbf{F}}^2)^2 + (*\tilde{\mathbf{F}}\tilde{\mathbf{F}})^2 = (\mathbf{F}^2)^2 + (*\mathbf{FF})^2,$$

$$\cos\varphi = \cos\tilde{\varphi}, \qquad \sin\varphi = \sin\tilde{\varphi}. \tag{12}$$

For instance,

$$\begin{aligned}\cos\tilde{\varphi} &= \frac{\tilde{\mathbf{F}}^2}{\sqrt{(\tilde{\mathbf{F}}^2)^2 + (*\tilde{\mathbf{F}}\tilde{\mathbf{F}})^2}} \\ &= \frac{\mathbf{F}^2(\cos^2\varphi - \sin^2\varphi) + 2(*\mathbf{FF})\sin\varphi\cos\varphi}{\sqrt{(\mathbf{F}^2)^2 + (*\mathbf{FF})^2}} \\ &= \cos\varphi\cos 2\varphi + \sin\varphi \sin 2\varphi = \cos\varphi.\end{aligned}$$

Taking into account (9), (10) and (12), one can invert the formulae (8),(11):

$$F_{\mu\nu} = e^{*\varphi}\tilde{F}_{\mu\nu} = e^{*\tilde{\varphi}}\tilde{F}_{\mu\nu}, \qquad *F_{\mu\nu} = -e^{*\varphi} * \tilde{F}_{\mu\nu} = -e^{*\tilde{\varphi}} * \tilde{F}_{\mu\nu}. \quad (13)$$

Now the system (2),(3) can be written in terms of the tensor $\tilde{F}_{\mu\nu}$

$$\tilde{F}^{\mu\nu}{}_{,\nu} = 0, \qquad \left(e^{*\tilde{\varphi}} * \tilde{F}^{\mu\nu}\right)_{,\nu} = 0,$$

and admits, like the set (6), to be written in the form

$$\tilde{F}^{\mu\nu}{}_{,\nu} = 0, \qquad *\tilde{F}^{\mu\nu}{}_{,\nu} = \tilde{\jmath}^{\mu},$$

where $\jmath^{\mu}$ is formally a magnetic charge current.

In conclusion write down the relation

$$e^{*a\varphi}e^{*\phi}F_{\mu\nu} = e^{*(1-a)\varphi}F_{\mu\nu},$$

where a is any number. The property to be expressed by the formula is due to the definition (10) and makes it difficult to introduce "partial" local transformation, by means of which one could have both electrical sources and magnetic ones.

The transformations (8),(11) and (13) are rather similar to discrete ones by its properties. Do they form a group? What is deeper sense of the field $\tilde{\mathbf{F}}$ and its prototype $\mathbf{F}$?

3 Duality in Non-Abelian Theory

3.1 The Lagrangian and Motion Equations

The dual-invariant Lagrangian for non-Abelian fields can be chosen in the form

$$\mathcal{L} = \sqrt{(\mathrm{Tr}\mathbf{F}^2)^2 + (\mathrm{Tr}(*\mathbf{FF}))^2},$$

where the fields $F_{\mu\nu}$ and the potentials A_μ take values in Lie algebra of some group. The dual transformations

$$F'_{\mu\nu} = F_{\mu\nu}\cos\theta + *F_{\mu\nu}\sin\theta = e^{*\theta}F_{\mu\nu}, \qquad F_{\mu\nu} = e^{-*\theta}F'_{\mu\nu},$$

$$*F'_{\mu\nu} = *F_{\mu\nu}\cos\theta - F_{\mu\nu}\sin\theta = e^{*\theta} * F_{\mu\nu}, \qquad *F_{\mu\nu} = e^{-*\theta} * F'_{\mu\nu}$$

are the same as in the electrodynamics. Again for convenience notations are introduced:

$$a = \mathrm{Tr}\mathbf{F}^2, \quad b = \mathrm{Tr}(*\mathbf{FF}), \quad \cos\varphi = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\varphi = \frac{b}{\sqrt{a^2+b^2}},$$

$$\tilde{F}^{\mu\nu} = F^{\mu\nu}\cos\varphi + *F^{\mu\nu}\sin\varphi = e^{*\varphi}F^{\mu\nu}.$$

The motion equations are

$$\tilde{F}^{\mu\nu}{}_{,\nu} + i[\tilde{F}^{\mu\nu}, A_\nu] = 0.$$

These equations are far more nonlinear than the Yang–Mills ones. To know transformation properties of these equations one need to know how the vector-potential A_μ transforms. This is the old problem [1]. Perhaps it would be useful to apply gauge transformations.

3.2 The Instantons

The motion equations will be satisfied , the condition

$$\tilde{F}^{\mu\nu} = F^{\mu\nu}\cos\varphi + *F^{\mu\nu}\sin\varphi = 0 \qquad (14)$$

having been fulfilled. Any solution to this equation will be referred to as a *generalized instanton*. It is not difficult to see that the Belavin-Polyakov-Schwartz-Tyupkin (BPST) instanton [8] obeys the equation. Really, the BPST instanton is the solution to the (anti-)self-duality equation (the space-time is pseudo-Euclidean)

$$*F_{\mu\nu} = \pm iF_{\mu\nu}.$$

Projecting this and also the equation (14) onto $F_{\mu\nu}$ with tracing results in

$$b = \pm ia, \qquad \sqrt{a^2+b^2} = 0.$$

Since in the non-Abelian case the fields are complex in the general case $a, b \neq 0$. Taking into account the duality equation, the equality (14) can be written in the form

$$b = \pm ia, \quad \frac{\pm i(b \mp ia)}{\sqrt{(b + ia)(b - ia)}} F_{\mu\nu} = 0.$$

It is fulfilled for the (anti-)self-dual fields because the degree of zero over the fraction bar is higher than below bar.

One can show that conventional Yang–Mills equations will be satisfied if together with (14) takes place the condition

$$*F^{\mu\nu}(\tan\varphi)_{,\nu} = 0$$

which is valid for the (anti-)self-dual fields.

Non-self-dual solutions were searched for in [9] by means of generalization of the duality equations. It is the equation (14) that may well be regarded as such a generalization. To find new solutions to the equation (14) is an important task.

4 Dual Symmetry in Gravitation

4.1 Some Notations

First one needs to introduce some notations and recall some facts writing them in the form convenient to use further.

Curvature tensor $R_{\mu\nu\rho\sigma}$ in gravitation is the analog of the electromagnetic strength. There are two channels (two pair of indices) not to be quite independent for which the dual conjugation operation can be introduced. So

$$*R_{ijkl} = \frac{1}{2} E_{ijmn} R^{mn}{}_{kl}, \quad R*_{ijkl} = \frac{1}{2} R_{ij}{}^{mn} E_{mnkl}, \quad E_{ijkl} = \frac{1}{\sqrt{-g}} \varepsilon_{ijkl}.$$

The properties can be easily verified:

$$**\mathbf{R} = \mathbf{R}** = -\mathbf{R}.$$

It is convenient to rewrite a number of known properties of the curvature tensor in terms of the right-handed and/or the left-handed dual-conjugated tensor.

The circular transposition identity is rewritten in the form

$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0 \quad \longrightarrow \quad *R^{\mu}_{\nu} = 0 \text{ and/or } R*^{\mu}_{\nu} = 0.$$

The Bianchi identity

$$R_{\mu\nu\rho\sigma;\delta} + R_{\mu\nu\delta\rho;\sigma} + R_{\mu\nu\sigma\delta;\rho} = 0$$

can be written as follows

$$*R_{\mu}{}^{\nu}{}_{\rho\sigma;\nu} = 0 \text{ and/or } R*_{\mu\nu\rho}{}^{\sigma}{}_{;\sigma} = 0.$$

However

$$R*_{\mu}{}^{\nu}{}_{\rho\sigma;\nu} \neq 0 \text{ and/or } *R_{\mu\nu\rho}{}^{\sigma}{}_{;\sigma} \neq 0.$$

Often one considers the so-called twice dual curvature tensor (TDCT) [10]

$$*R*_{\mu\nu\rho\sigma} = \frac{1}{4} E_{\mu\nu\alpha\beta} R^{\alpha\beta\gamma\delta} E_{\gamma\delta\rho\sigma}.$$

Using the expression for the antisymmetric ε-symbols product in terms of the Kronecker δ-symbols [11], it is not difficult to express TDCT in terms of the curvature tensor:

$$*R*^{\mu\nu}{}_{\rho\sigma} = -R^{\mu\nu}{}_{\rho\sigma} + \delta^{\mu}_{\sigma} R^{\nu}_{\rho} + \delta^{\mu}_{\rho} R^{\nu}_{\sigma} - \delta^{\mu}_{\sigma} R^{\nu}_{\rho} - \delta^{\nu}_{\rho} R^{\mu}_{\sigma} + \frac{1}{2} R(\delta^{\nu}_{\rho}\delta^{\mu}_{\sigma} - \delta^{\mu}_{\rho}\delta^{\nu}_{\sigma}). \tag{15}$$

4.2 Dual Group in Gravitation

By means of TDCT the curvature tensor in a natural and invariant manner is expanded into a sum of two irreducible parts

$$R_{\mu\nu\rho\sigma} = \mathcal{R}_{\mu\nu\rho\sigma} + \mathcal{S}_{\mu\nu\rho\sigma},$$

where

$$\mathcal{R}_{\mu\nu\rho\sigma} = \frac{1}{2}(R_{\mu\nu\rho\sigma} - *R*_{\mu\nu\rho\sigma}), \quad \mathcal{S}_{\mu\nu\rho\sigma} = \frac{1}{2}(R_{\mu\nu\rho\sigma} + *R*_{\mu\nu\rho\sigma}).$$

The properties may easily be proved to take place (sometimes no indices notations are used):

$$*\mathcal{R} = \mathcal{R}*, \quad *\mathcal{R}* = **\mathcal{R} = \mathcal{R}** = -\mathcal{R},$$
$$\mathcal{S} = -\mathcal{S}*, \quad *\mathcal{S}* = \mathcal{S} = -**\mathcal{S} = -\mathcal{S}**.$$

For instance,

$$*\mathcal{R} = *\frac{1}{2}(\mathbf{R} - *\mathbf{R}*) = \frac{1}{2}(*\mathbf{R} + \mathbf{R}*) = \mathcal{R}*,$$

$$*\mathcal{S} = *\frac{1}{2}(\mathbf{R} + *\mathbf{R}*) = \frac{1}{2}(*\mathbf{R} - \mathbf{R}*) = -\mathcal{S}*$$

etc. Since the tensors $\mathcal{R}$ and $\mathcal{S}$ transform simply in dual conjugating, the notations can be somewhat improved. We take into account that among the left-handed and right-handed dual-conjugated tensors (e.g., $*\mathcal{R}, \mathcal{R}*$) the independent tensor is alone

$$*\mathcal{R} = \mathcal{R}* = \overset{*}{\mathcal{R}}, \qquad *\mathcal{S} = -\mathcal{S}* = \overset{*}{\mathcal{S}}.$$

Using the explicit expression for the TDCT, (15) one can write down the explicit expressions for $\mathcal{R}$ and $\mathcal{S}$

$$\begin{aligned}\mathcal{R}_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} + \frac{1}{2}(g_{\mu\sigma}R_{\nu\rho} + g_{\nu\rho}R_{\mu\sigma} - g_{\mu\rho}R_{\nu\sigma} - g_{\nu\sigma}R_{\mu\rho}) \\ &\quad + \frac{R}{4}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \\ &= C_{\mu\nu\rho\sigma} + \frac{R}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \qquad (16) \\ \mathcal{S}_{\mu\nu\rho\sigma} &= \frac{1}{2}(g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma}) \\ &\quad + \frac{R}{4}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}),\end{aligned}$$

where $C_{\mu\nu\rho\sigma}$ is conformal Weil's tensor. It is also easy to obtain

$$\mathcal{R}_{\mu\nu}{}^{\mu}{}_{\sigma} = \frac{R}{4}g_{\nu\sigma}, \qquad \mathcal{R}_{\mu\nu}{}^{\mu\nu} = R, \qquad (17)$$

$$\mathcal{S}_{\mu\nu}{}^{\mu}{}_{\sigma} = R_{\nu\sigma} - \frac{R}{4}g_{\nu\sigma}, \qquad \mathcal{S}_{\mu\nu}{}^{\mu\nu} = 0.$$

Now one gets ready to introduce a dual group in gravitation. Let us consider the transformation

$$\mathbf{R} \to \mathbf{R}': \quad \mathbf{R}' = e^{*\alpha}\mathbf{R}e^{*\beta}, \qquad (18)$$

where α and β are independent dual angles. More precisely, the somewhat symbolic notation (18) means

$$\mathbf{R}' = \mathbf{R}\cos\alpha\cos\beta + *\mathbf{R}\sin\alpha\cos\beta + \mathbf{R}*\cos\alpha\sin\beta + *\mathbf{R}*\sin\alpha\sin\beta. \qquad (19)$$

Quite clear, the transformations of kind (19) form an Abelian two-parameter group.

Next, one finds that the tensors $\mathcal{R}, \mathcal{S}$ transform *simply* when acted by the group (19)

$$\mathcal{R}' = e^{*(\alpha+\beta)}\mathcal{R} = \mathcal{R}e^{*(\alpha+\beta)}, \quad \mathcal{S}' = e^{*(\alpha-\beta)}\mathcal{S} = \mathcal{S}e^{-*(\alpha-\beta)} \qquad (20)$$

and actually realize an irreducible representation of the dual group in gravitation.

By analogy with the dual electrodynamics, as a consequence of (20) we immediately obtain two dual gravitational invariants:

$$I_1 = (\mathcal{R}^2)^2 + (\overset{*}{\mathcal{R}}\mathcal{R})^2, \quad I_2 = (\mathcal{S}^2)^2 + (\overset{*}{\mathcal{S}}\mathcal{S})^2.$$

Here $\mathcal{R}^2 = \mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}$ etc. Perhaps, it would be far more convenient to use the linear combinations of these invariants which are expressed in terms of this curvature tensor:

$$\begin{aligned}J_1 &= (\mathbf{R}^2)^2 + (*\mathbf{RR})^2 + (\mathbf{RR}*)^2 + (*\mathbf{RR}*)^2, \\ J_2 &= \mathbf{R}^2(*\mathbf{RR}*) - (*\mathbf{RR})(\mathbf{RR}*),\end{aligned}$$

where in reality

$$*\mathbf{RR} = \mathbf{RR}*, \quad *\mathbf{RR}* = \mathbf{R}(*\mathbf{R}*).$$

4.3 The Gravitational Instantons

4.3.1 Discussion

The current state of the gravitational instantons question seems to be somewhat intricate. Penrose's [12] instantons (nonlinear gravitons) are the (anti)self-dual complex solutions to Einstein's equations. Hawking introduces the instantons as Euclidean solutions to Einstein's equations with finite action [13]. This matter is reviewed e.g. in [14]. In either event gravitational instantons are related to solutions to Einstein's equations in the *Euclidean* space

$$R_{\mu\nu} = \lambda g_{\mu\nu}$$

and should obey the (anti-)self-duality equations to be understood as follows

$$*C_{\mu\nu\rho\sigma} = \pm C_{\mu\nu\rho\sigma},$$

where $C_{\mu\nu\rho\sigma}$ is Weil's tensor. [1] As seen, for example, from non-Abelian theory, the instantons, to a certain degree, do not depend on dynamics. They rather display deeper kinematic-topological properties. For Einstein's gravitation it is not the point.

It is not worthwhile to relate instantons to any dynamical equations, to Einstein's ones in particular. These equations, as distinct from Yang–Mills ones, place too hard restrictions on the curvature tensor from the point of view of the (anti-)self-duality properties.

Let us call as gravitational instantons the solutions to the duality equations in *pseudo-Euclidean* space for the tensors $\mathcal{R}$ and S

$$\overset{*}{\mathcal{R}} = \pm i\mathcal{R}, \qquad \overset{*}{\mathcal{S}} = \pm i\mathcal{S}. \tag{21}$$

These equations are quite equivalent to the duality conditions in the Yang–Mills theory. Of course, the **R**-space is real and the equations (21) are reduced to

$$\mathcal{R}_{\mu\nu\rho\sigma} = 0, \qquad \mathcal{S}_{\mu\nu\rho\sigma} = 0. \tag{22}$$

[1] This point has been cleared up to me by A. Popov.

So, the real gravitational instantons are determined by the equations (22). To avoid misunderstanding it should be emphasized that the equations (22) must not be regarded as a system (set).

Written in another form, such equations are given in the book [15] and they were obtained otherwise. Solutions to these equations (which are not obtained and are not presented in the book) are referred to as twice (anti-)self-dual ones, which are similar to usual instantons by their properties.

It follows from our approach that they are usual gravitational instantons.

Next a few central-symmetry solutions are shown.

4.3.2 The 4-central-symmetric solutions

The Metric Choice. Let us look for the solutions to (22) as a metric of the form

$$ds^2 = e^{\nu(\rho)}d\rho^2 - \rho^2[d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2)],$$

where

$$ds_0^2 = d\rho^2 - \rho^2[d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2)]$$

is a 4-spherical flat metric. Calculating the curvature tensor gives four nonzero (diagonal) components of the tensor $R_{\mu\nu}$ and six nonzero (diagonal) components of the tensors $R_{\mu\nu\rho\sigma}$.

$\mathcal{R}_{\mu\nu\rho\sigma} = \mathbf{0}$. Six components of the equation that do not turn into identities reduce to the only one of the *first* order

$$\nu'(\rho) = \frac{2}{\rho}(1 - e^{\nu})$$

The equation is easily solved:

$$e^{\nu(\rho)} = \frac{\rho^2}{\rho^2 - C}.$$

The metric is getting flat if $\rho \to \infty$ or $C = 0$.

$\mathcal{S}_{\mu\nu\rho\sigma} = \mathbf{0}$. Six components of the equation not to be reduced to identities reduce to a differential *first* order equation alone

$$\nu'(\rho) = \frac{2}{\rho}(e^{\nu} - 1),$$

which has the solution

$$e^{\nu} = \frac{1}{1 - C\rho^2}.$$

If $\rho \to 0$ or $C = 0$ the metric gets flat.

$\mathbf{R}_{\mu\nu} = \mathbf{0}$. Einstein's equations in empty space have a trivial solution only: $e^{\nu} = 0$. This merely emphasizes the fact that the gravitational instantons are poorly compatible with Einstein's equations. The solutions described in this section are quite analogous to the spherical-symmetric BPST's instanton [8].

4.3.3 The Static Central-Symmetric Solutions

The Metric Choice. We search for a solution as follows (the metric is just as in Landau and Lifshits [11]):

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

As a result of the curvature tensor calculation it turned out to consist of four nonzero (diagonal) components of the tensor $R_{\mu\nu}$ and six nonzero (diagonal) ones of $R_{\mu\nu\rho\sigma}$.

$\mathcal{R}_{\mu\nu\rho\sigma} = \mathbf{0}$. Six equations not to be reduced to identities reduce to the only differential equation of the *second* order

$$\lambda = \nu, \qquad \nu''(r) = \frac{2}{r^2}(e^{\nu} - 1).$$

This is a rather nontrivial equation. It can be exactly solved [16]. The solution is represented in two forms:

$$e^{\nu} = \frac{C_1^2}{2} r^2 \sin^{-2}\left[\frac{C_1}{\sqrt{2}}(r - C_2)\right]$$

or

$$e^{\nu} = \frac{C_1^2}{2} r^2 \sinh^{-2}\left[\frac{C_1}{\sqrt{2}}(r - C_2)\right].$$

With the constant C equal to zero, the solution becomes more simple:

$$e^{\nu} = \frac{r^2}{(r - C)^2}.$$

This solution is asymptotically flat at $r \to \infty$ and for a small r:

$$e^{\nu} \sim r^2, \qquad r \to 0.$$

$\mathcal{S}_{\mu\nu\rho\sigma} = \mathbf{0}$. Six equations not to be identities reduce to the only equation of the *second* order

$$\lambda = -\nu, \qquad \nu''(r) + \nu'^2(r) = \frac{2}{r^2}(1 - e^{-\nu}),$$

which is solved simply:

$$e^{\nu} = 1 + C_1 r^2 + \frac{C_2}{r}.$$

It is seen that this solution contains Schwartzschild's solution (if $C_1 = 0$). For large r the first metric coefficient goes to infinity, what points out that the metric may be closed.

4.3.4 Generalized Instantons

By analogy with the dual electrodynamics and non-Abelian theory, the generalized instantons are defined as follows (it is not a system of equations):

$$\mathcal{R}_{\mu\nu\rho\sigma} \cos\varphi + \overset{*}{\mathcal{R}}_{\mu\nu\rho\sigma} \sin\varphi = 0, \tag{23}$$

$$\mathcal{S}_{\mu\nu\rho\sigma} \cos\psi + \overset{*}{\mathcal{S}}_{\mu\nu\rho\sigma} \sin\psi = 0, \tag{24}$$

where

$$\cos\varphi = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\varphi = \frac{b}{\sqrt{a^2+b^2}},$$

$$\cos\psi = \frac{c}{\sqrt{c^2+d^2}}, \quad \sin\psi = \frac{d}{\sqrt{c^2+d^2}},$$

$$a = \mathcal{R}^2, \quad b = \mathcal{R}\overset{*}{\mathcal{R}}, \quad c = \mathcal{S}^2, \quad d = \mathcal{S}\overset{*}{\mathcal{S}}.$$

It follows from (23),(24)

$$a^2 + b^2 = 0, \qquad c^2 + d^2 = 0$$

or $a = 0, b = 0, c = 0, d = 0$. The instantons $\mathcal{R} = 0$ and $\mathcal{S} = 0$ are easily seen to obey the equations (23),(24). Perhaps, there exists some way to complexify **R**-space, so that the instanton notion in gravitation would be more comprehensive as it should be in non-Abelian theory.

4.4 The Gravitation Equations

If there are tensors that possess the basic symmetries of the curvature tensor, metric and the energy-momentum tensor of matter, new gravitational equations could be constructed. Let us begin from the trivial but visual example. It is possible to construct tensor by means of the metric and the energy-momentum tensor which has the curvature tensor symmetries. The following equation is postulated:

$$R_{\mu\nu\rho\sigma} = const(g_{\mu\rho}T_{\nu\sigma} + g_{\nu\sigma}T_{\mu\rho} - g_{\mu\sigma}T_{\nu\rho} - g_{\nu\rho}T_{\mu\sigma}). \qquad (25)$$

Is this equation good or bad? It is bad as it follows from below. Let the energy-momentum tensor be concentrated at finite range of the space. Out of the range the equation is given by

$$R_{\mu\nu\rho\sigma} = 0.$$

Thus, the equation (25) predicts the absence of gravity wherever the matter is absent.

Let us try once more to find the gravitational equation using the twice anti-self-dual part of the curvature tensor

$$\mathcal{R}_{\mu\nu\rho\sigma} = A(g_{\mu\rho}T_{\nu\sigma} + g_{\nu\sigma}T_{\mu\rho} - g_{\mu\sigma}T_{\nu\rho} - g_{\nu\rho}T_{\mu\sigma}). \qquad (26)$$

One can show that in the long run the equation reduces to the conformal flat Nordström's theory [17]. Really, reducing it with respect to the indices μ and ρ and taking into account (17), we obtain

$$\frac{1}{4}Rg_{\nu\sigma} = A(2T_{\nu\sigma} + g_{\nu\sigma}T). \qquad (27)$$

One more reduction gives

$$R = 6AT. \qquad (28)$$

Expressing $T_{\nu\sigma}$ from (27) and substituting it to (26), also taking into account (28),(16), we find eventually

$$\mathcal{R}_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \frac{R}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) = \frac{R}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}).$$

Thus, $C_{\mu\nu\rho\sigma} = 0$. Together with (28) this equation is a formulation of Nordström's conformal flat theory [10] which, for example, predicts no deviation of light in the gravity field.

Being consequent, we have to try constructing an equation by means of the tensor $\mathcal{S}$:

$$*\mathcal{S}_{\mu\nu\rho\sigma} = B(g_{\mu\rho}T_{\nu\sigma} + g_{\nu\sigma}T_{\mu\rho} - g_{\mu\sigma}T_{\nu\rho} - g_{\nu\rho}T_{\mu\sigma}).$$

One can show that this equation reduces to

$$R_{\mu\nu} - \frac{R}{4}g_{\mu\nu} = 2BT_{\mu\nu}, \qquad T = 0.$$

It would be of interest to consider the system: electromagnetic field – gravitation starting from these equations rather than the Maxwell-Einstein ones.

4.5 Duality and Variational Principle in Gravitation

We concerned the dynamical aspects just in the previous subsection trying to construct gravitational equations by means of the dual-symmetric parts of the curvature tensor only. It is of interest to establish a variational principle which is compatible with duality, say, in the way to be similar to the dual electrodynamics variational principle. As has been noted, Einstein's equation, and consequently, Hilbert's variational principle are poor compatible with duality.

However, direct attempt to create a gravitational theory with the dual symmetry to be involved faces troubles.

Let us treat maintaining the analogy to electrodynamics. It is known that the quadratic Lagrangian electrodynamics equations are dual invariant. In the gravitational case that all would have been analogous if the equations

$$R_{\mu\nu\rho}{}^{\sigma}{}_{;\sigma} = 0 \qquad , R*_{\mu\nu\rho}{}^{\sigma}{}_{;\sigma} \equiv 0 \tag{29}$$

had taken place. But for the quadratic in **R** Lagrangian one obtains the equation

$$R_{\mu(\nu\rho)}{}^{\sigma}{}_{;\sigma} = 0 \tag{30}$$

rather than the first one of (29). Christoffel's symbols are suggested to be related to metric in the usual way but connections are varied rather than the metric. However, the left-hand side of (30) is identically zero on account of the circular transposition identity. Thus we have no variational principle leading to the equations (29) and at this point the analogy to electrodynamics already vanishes.

Perhaps, one should consider *nonsymmetric* connections introducing in this way torsion. In any case this would enable one to avoid the equation (30). None forbid however to compound the dual-symmetric objects by means of the tensors $\mathcal{R}$ and $\mathcal{S}$. In effect this has been done in the subsection about the gravitational instantons. It is possible also to use directly two gravitational dual invariants. We have to repeat, however, that besides the problem to choose the

Lagrangian, there is another difficulty. If the basic dynamical variables are symmetric connections related to the metric in the usual fashion then the dual invariant Lagrangian theory will be ugly in view of (30).

5 Conclusion

In the paper an attempt to consider the consequences of the theory with the incorporated dual symmetry has been made. The dual symmetry in gravitation has been investigated as well. Such an approach to gravitation has not appeared before. The dual electrodynamics is the elegant nonlinear model which might be related to the linear Maxwell electrodynamics. The local dual angle arises in a natural fashion in this theory. This angle enables introducing the local dual transformations. By means of these transformations the theory may be reformulated in terms of the magnetic charge. The equations have been considered the solutions to which were referred to as the generalized instantons (for the BPST-instanton obeys these equations). Nontrivial gravitational instantons have been found. Irreducible representations of the dual group in gravitation give new opportunities to establish new gravitational equations. The problem related to variational principle notion for the dual-symmetric gravitation has been discussed.

The tasks of interest should be noted.

- Searching for solutions to the dual electrodynamics and non-Abelian theory (the instantons as well). The problem to find the *asymptotically* central-symmetric solutions is of great importance.
- The transformation property in the non-Abelian theory (in particular for vector-potential) under the dual group.
- Establishing the dual-symmetric variational principle in gravitation. Perhaps, the torsion should be included in the theory to obtain self-consistent theory.

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