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HAMILTONIAN REDUCTION OF *SU*(2) DIRAC — YANG-MILLS MECHANICS

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I. INTRODUCTION

The correct canonical formulation of the quantum theory of non-Abelian fields assumes a detailed knowledge of the corresponding classical generalized Hamiltonian dynamics [1]-[5]. Since the introduction of non-Abelian gauge fields by C.N. Yang and R.L. Mills [6] over fourty years ago essential progress in this direction has been made. Rigorous statements about the geometrical structure of the configuration and the phase space have been established. It turned out that due to the underlying non-Abelian gauge symmetry the "true phase space" of Yang-Mills theory, namely the quotient space of phase space by the action of gauge transformations, possesses a rich topological structure [7]. In the framework of traditional perturbation theory these geometrical peculiarities are not taken into account and as a result the description of large scale effects, including confinement, is beyond its scope. The most important lesson one has learned is that, in order to reach a complete description, it is necessary to first reformulate Yang-Mills theory in terms of gauge-invariant variables and only after this step apply any approximation method. With this aim several different representations for the physical degrees of freedom of non-Abelian theories [8]- [19] have been proposed. All these approaches lead to an unconstrained Hamiltonian system, which exhibits non-perturbative features and are in some sense alternatives to the conventional perturbative approach. Whereas perturbation theory is appropriate for the computation of short distance effects, the unconstrained formulation is adapted to the study of large scale phenomena if the gauge invariant expressions are evaluated in a derivative expansion. Since the work by Matinyan et al. [20], the corresponding zeroth order or long-wavelength approximation, the Yang-Mills mechanics of spatially homogeneous gauge fields, has been studied extensively from different points of view (see e.g. [21] - [24] and references therein). In the present note we shall continue the study of the model arising in this approximation, pursuing the aim to prepare the necessary background for studying the problem of construction of the reduced phase space of QCD. Due to the spatial homogeneity condition conventional Dirac-Yang-Mills theory reduces to a theory describing a finite dimensional system which is incomparably simpler than the exact field system. At the same time, however, it possesses all the main peculiarities of the full theory and can be used as a laboratory for testing the viability of ideas and techniques that could be applied in the general case.

Below we shall isolate the true dynamical degrees of freedom of $SU(2)$ Dirac-Yang-Mills theory in the long-wavelength approximation using the gaugeless approach [1] to the reduction in the number of degrees of freedom instead of the conventional gauge fixing method. [2] The cornerstones for this method applied to a system with first class constraints are the procedure of Abelianization of constraints (replacement of the original

[1] Presumably, S.Shanmugadhasan [25] was the first to employ the classical Lee-Cartan method of reduction (see e.g. [26] - [30]) in the framework of generalized Hamiltonian dynamics.

[2] We point out here that the idea of constructing the physical variables entirely in internal terms without using any additional gauge conditions is connected with the desire not to distort the global properties of the theory and to have all dynamical degrees of freedom under control.

non-Abelian constraints by an equivalent set of Abelian ones) and the canonical transformation to new variables where a subset of the new momenta is equal to the new Abelian constraints. The system of interacting gauge and spinor fields considered in this article represent a Hamiltonian system with mixed first and second class constraints. In this case the reduction procedure additionally includes the separation of first and second class constraints and putting them into the canonical form.

The paper is organized as follows. In Section II we briefly recall how to obtain the unconstrained Hamiltonian system from the initially gauge symmetric one in the framework of Dirac constraint theory in order to set the formalism. The Dirac and the Faddeev gauge fixing methods as well as the gaugeless method are described. In Section III the gaugeless method is exemplified by considering the Yang-Mills system in $0+1$- dimensions. In Section IV we perform the reduction of the Dirac-Yang-Mills system by explicitly separating the first and second class constraints, putting the second class constraints into the canonical form and Abelianizing the first class constraints. We construct the corresponding reduced Hamiltonian system by first eliminating the unphysical gauge degrees of freedom and then using the classical scheme of Hamiltonian reduction due to the existence of three first integrals of motion. Section V finally gives our conclusions and remarks.

II. REDUCTION OF CONSTRAINED SYSTEMS WITH FIRST CLASS CONSTRAINTS

The procedure of reduction of phase space of a singular system is a generalization of the method of reduction of a system of differential equations possessing a Lie group symmetry. The well-known results for this type of reduction in the number of the degrees of freedom are embodied in the famous J.Liouville theorem on first integrals in involution. Interest in these has revived in connection with the study of Hamiltonian systems with a local (gauge) symmetry. Since the work of P. Bergmann and P.A.M. Dirac at the beginning of the fifties it has become clear that the role of integrals of motion in a Hamiltonian system with gauge symmetry is played by the first class constraints. Although the reduction in the number of degrees of freedom due to first class constraints has many features in common with the classical case, there are very important differences. In order to explain these peculiarities of the reduction procedure and to make the paper self-contained we first have to summarize some definitions and to put facts from the Dirac theory of generalized Hamiltonian dynamics into the appropriate context. In view of the main purpose of our paper, namely to study the finite dimensional system of homogenous Yang-Mills fields, we shall discuss the above ideas for a mechanical system, i.e. a system with a finite number of degrees of freedom.

A. The definition of reduced phase space

Let us consider a system with the $2n$ - dimensional Euclidean phase space Γ spanned by the canonical coordinates q_i and their conjugate momenta p_i and endowed with the canonical simplectic structure $\{q_i, p^j\} = \delta_i^j$. Suppose that the dynamics is constrained to a

certain $(2n-m)$ — dimensional subrnanifold Γ_c determined by m functionally independent constraints

$$\varphi_\alpha(p,q) = 0\,, \tag{2.1}$$

which we assume to be first class

$$\{\varphi_\alpha(p,q), \varphi_\beta(p,q)\} = f_{\alpha\beta\gamma}(p,q)\varphi_\gamma(p,q) \tag{2.2}$$

and complete in the sense that

$$\{\varphi_\alpha(p,q), H_C(p,q)\} = g_{\alpha\gamma}\varphi_\gamma(p,q)\,, \tag{2.3}$$

where $H_C(p,q)$ is the canonical Hamiltonian. Due to the presence of these constraints the Hamiltonian system admits generalized dynamics described by the extendend Poincare-Cartan form

$$\Theta := \sum_{i=1}^{n} p_i dq_i - H_E(p,q)dt \tag{2.4}$$

with the extended Hamiltonian $H_E(p,q)$ that differs from the canonical $H_C(p,q)$ by a linear combination of constraints with arbitrary multipliers $u_\alpha(t)$

$$H_E(p,q) := H_C(p,q) + u_\alpha(t)\varphi_\alpha(p,q)\,. \tag{2.5}$$

From the condition of completeness (2.3) with H_C replaced by H_E it follows that for first class constraints the functions $u_\alpha(t)$ can not be fixed in internal terms of the theory. This implies that the system possesses a local symmetry and that the coordinates split up into two sets, one set whose dynamics is governed in an arbitrary way and another set with an uniquely determined behaviour. Recalling the Dirac definition [31] of a *physical variable* as a dynamical variable F with the property

$$\{F(p,q), \varphi_\alpha(p,q)\} = d_{\alpha\gamma}(p,q)\varphi_\gamma(p,q)\,, \tag{2.6}$$

one can conclude that the first set of coordinates does not affect the physical quantities which are defined on some subspace of the constraint surface Γ_c. Indeed, if one considers (2.6) as a set of m first order linear differential equations for F, then due to the integrability condition (2.2) this function can be completely determined by its values in the $2(n-m)$ submanifold of its initial conditions [32], [2]. This subspace of constraint shell represents the *reduced phase space* Γ^*. This definition of reduced phase space is implicit. The main problem is to find the set of $2(n-m)$ "physical coordinates" Q_i^*, P_i^* that span this reduced phase space and pick out the other additional m pairs which have no physical significance and represent the pure gauge degrees of freedom. Several approaches to its solution are known. Below we shall briefly discuss the corresponding methods of practical construction of the physical and the gauge degrees of freedom with and without gauge fixing.

B. Reduced phase space with the Dirac gauge fixing method

General principles for imposing gauge fixing constraints onto the canonical variables in the Hamiltonian approach were proposed by Dirac in connection with the canonical formulation of gravity [33]. According to the Dirac gauge fixing prescription, one starts with the introduction of as many new "gauge" constraints

$$\chi_\alpha(p,q) = 0 \tag{2.7}$$

as there are first class constraints (2.1), with the requirement

$$\det \|\{\chi_\alpha(p,q), \varphi_\beta(p,q)\}\| \neq 0 . \tag{2.8}$$

This allows one to find the unknown Lagrange multipliers $u_\alpha(t)$ from the requirement of conservation of the gauge conditions (2.7) in time [3]

$$\dot{\chi}_\alpha = \{\chi_\alpha, H_C\} + \sum_\beta \{\chi_\alpha, \varphi_\beta\} u_\beta = 0 \tag{2.9}$$

and thus to determine the dynamics of system in a unique manner. A striking result of Dirac consists in the observation that such kind of fixation of Lagrange multipliers $u(t)$ is equivalent to the following way of proceeding. One can drop both the constraints (2.1) and the gauge fixing conditions (2.7) and at the same time achieve the reduction to the unconstrained theory by using the Dirac brackets

$$\{F,G\}_D := \{F,G\} - \{F,\xi_s\} C^{-1}_{ss'} \{\xi_{s'}, G\} , \tag{2.10}$$

instead of the Poisson brackets. Here ξ denotes the set of all constraints (2.1) and (2.7) and C^{-1} is the inverse of the Poisson matrix $C_{\alpha\beta} := \{\xi_\alpha, \xi_\beta\}$. In this method all coordinates of the phase space are treated on an equal footing and all information on both initial and gauge constraints is absorbed into the Dirac brackets, which describe the effective reduction in the number of degrees of freedom from n to $n-m$

$$\sum_{i=1}^n \{q_i, p_i,\}_{P.B.} = n , \quad \sum_{i=1}^n \{q_i, p_i,\}_{D.B.} = n - m .$$

The inclusion of gauge constraints in addition to the initial constraints allows one to takethe constraint nature of the canonical variables into account by changing the initial canonical symplectic structure to a new one defined by the Dirac brackets. The new canonical structure, being dependent on the choice of gauge fixing-conditions, is very complicated in general and it is not clear how to deal with it, in particular, when we are quantizing the theory. However, there is a special case when the Dirac bracket coincides

[3] Everywhere in the article the dot over the letter denotes the derivative with respect to the time variable

with the canonical one and looks like the Poisson bracket for an unconstrained system defined on Γ^*

$$\{F,G\}_D\Big|_{\varphi=0,\ \chi=0} = \sum_{i=1}^{n-m} \left\{ \frac{\partial \overline{F}}{\partial Q_i^*} \frac{\partial \overline{G}}{P_i^*} - \frac{\partial \overline{F}}{\partial P_i^*} \frac{\partial \overline{G}}{Q_i^*} \right\} . \tag{2.11}$$

This representation of the Dirac bracket means that in terms of the conjugate coordinates Q_i^*, P_i^* $(i = 1, \dots, n-m)$ the reduced phase space is parametrized so that constraints vanish identically and any function $F(p,q)$ given on the reduced phase space becomes [3]

$$F(p,q)\Big|_{\varphi=0,\ \chi=0} = \overline{F}(P^*, Q^*) .$$

Thus in the Dirac gauge-fixing method the problem of definition of the "true dynamical degrees" of freedom reduces to the problem of a "lucky" choice of the gauge condition.

C. Reduced phase space with the Faddeev gauge fixing method

An alternative to the Dirac gauge-fixing procedure has been proposed in the well-known paper by L.D. Faddeev [32], devoted to the method of path integral quantization of a constrained system. In contrast to the Dirac method, the main idea of the Faddeev method is to introduce an explicit parametrization of the reduced phase space. As in the Dirac method, one introduces gauge fixing constraints $\chi_\alpha(p,q) = 0$, but now with the additional " Abelian" property

$$\{\chi_\alpha(p,q), \chi_\beta(p,q)\} = 0 , \tag{2.12}$$

and the requirement (2.8) is fulfilled. Now, in accordance with the Abelian character of gauge conditions (2.12), there exists a canonical transformation to new coordinates

$$\begin{aligned} q_i &\mapsto Q_i := Q_i(q,p) \\ p_i &\mapsto P_i := P_i(q,p) \end{aligned} \tag{2.13}$$

such that m of the new P's coincide with the constraints χ_α

$$P_\alpha = \chi_\alpha(q,p) . \tag{2.14}$$

The condition (2.8) allows one to resolve the constraints (2.1) for the coordinates Q_α in terms of the $(n-m)$ cannonical pairs (Q_i^*, P_i^*), which span the $2(n-m)$ - dimensional surface Σ determined by the equations

$$\begin{aligned} P_\alpha &= 0 , \\ Q_\alpha &= Q_\alpha(Q^*, P^*) . \end{aligned} \tag{2.15}$$

After this construction has been carried out, the problem is to prove that the surface Σ coincides with the true reduced phase space Γ^*, independent of the choice of the gauge fixing conditions. In other words, it is necessary to find a criterion for gauge conditions to be admissible. A radical method to solve this problem is not to use any gauge conditions at all. The following subsection will give a brief description of such an alternative gaugeless scheme to construct the reduced phase space without using gauge fixing functions.

D. The gaugeless method

If the theory contains only Abelian constraints one can find a parametrization of reduced phase space as follows. According to a well-known theorem (see e.g. [34]), it is always possible to find a canonical transformation to a new set of canonical coordinates

$$\begin{aligned} q_i &\mapsto Q_i := Q_i(q,p)\,, \\ p_i &\mapsto P_i := P_i(q,p)\,, \end{aligned} \tag{2.16}$$

such that m of the new momenta, $(\overline{P}_1, \ldots, \overline{P}_m)$, become equal to the Abelian constraints φ_α

$$\overline{P}_\alpha = \varphi_\alpha(q,p)\,. \tag{2.17}$$

In terms of the new coordinates $(\overline{Q}, \overline{P})$, and (Q^*, P^*) the canonical equations read

$$\begin{aligned} \dot{Q}^* &= \{Q^*, H_{phys}\}\,, \qquad & \dot{\overline{Q}} &= u(t)\,, \\ \dot{P}^* &= \{P^*, H_{phys}\}\,, \qquad & \dot{\overline{P}} &= 0\,, \end{aligned} \tag{2.18}$$

with the physical Hamiltonian

$$H_{phys}(P^*, Q^*) \equiv H_C(P,Q)\Big|_{\overline{P}_\alpha=0}. \tag{2.19}$$

H_{phys} depends only on the $(n-m)$ pairs of new gauge invariant canonical coordinates (Q^*, P^*) and the form of the canonical system (2.18) expresses the explicit separation of the phase space into physical and unphysical sectors

$$2n\left\{\begin{pmatrix} q_1 \\ p_1 \\ \vdots \\ q_n \\ p_n \end{pmatrix}\right. \mapsto \begin{matrix} 2(n-m)\left\{\begin{pmatrix} Q^* \\ P^* \end{pmatrix}\right. & \textit{physical variables} \\ \\ 2m\left\{\begin{pmatrix} \overline{Q} \\ \overline{P} \end{pmatrix}\right. & \textit{unphysical variables} \end{matrix} \tag{2.20}$$

The arbitrary functions $u(t)$ enter into that part of the system of equations, which containes only the ignorable coordinates $\overline{Q}_\alpha$ and momenta $\overline{P}_\alpha$. A straightforward generalization of this method to the non-Abelian case is not possible, since the identification of momenta with constraints is forbidden due to the non-Abelian character of the constraints. However, there exists the possibility of a replacement of the constraints φ_α by an equivalent set of new constraints Φ_α

$$\Phi_\alpha = D_{\alpha\beta}\varphi_\beta\,, \qquad \det\|D\|\Big|_{\varphi=0} \neq 0\,, \tag{2.21}$$

describing the same surface Γ_c but forming an Abelian algebra. There are different proofs of this statement, based on the resolution of constraints [3] – [5], exploiting gauge-fixing conditions [35], or using the direct method of constructing the Abelianization matrix as the solution of a certain system of linear first order differential equations [36] [4]. For non-Abelian systems therefore, the construction of the Abelianization matrix and the implementation of the above mentioned transformation (2.16) to the new set of Abelian constraint functions Φ_α completes the reduction of the phase space without using gauge fixing functions, solely in internal terms of the theory.

Before applying the gaugeless method to the construction of the reduced phase space of homogeneous Yang-Mills fields in 3+1-dimensional space it seems worth setting forth our approach to the same problem in 0+1-dimensional space.

III. $SU(2)$ YANG-MILLS FIELDS IN 0+1 DIMENSIONS

In order to explain our main idea how to construct the physical variables we shall start with the non-Abelian Christ & Lee model [12], [39]. The Lagrangian of this model is

$$L := \frac{1}{2}\,(D_t x)_i (D_t x)_i - \frac{1}{2}\,V(x^2)\,, \tag{3.1}$$

where x_i and y_i are the components of three-dimensional vectors and the covariant derivative D_t is defined as

$$(D_t x)_i := \dot{x}_i + g\epsilon_{ijk} y_j x_k\,. \tag{3.2}$$

One can see that this model is nothing else than Yang-Mills theory in $0+1$ dimensional space-time and that is invariant under $SO(3)$ gauge transformations.

Performing the Legendre transformations,

$$\begin{aligned} p_y^i &= \frac{\partial L}{\partial \dot{y}_i}\,, & (3.3) \\ p^i &= \frac{\partial L}{\partial \dot{x}_i} = \dot{x}_i + g\epsilon^{ijk} y_j x_k\,, & (3.4) \end{aligned}$$

one obtains the canonical Hamiltonian

$$H_C = \frac{1}{2} p_i p_i - \epsilon_{ijk} x_j p_k y_i + V(x^2)\,, \tag{3.5}$$

and identifies the three primary constraints $p_y^i = 0$ as well as the three secondary ones

$$\Phi_i = \epsilon_{ijk} x_j p_k = 0\,, \tag{3.6}$$

obeying the $SO(3)$ algebra

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[4] In all cases, the proofs use the large freedom in the canonical description of the constrained systems. Apart from the ordinary canonical transformations there exist generalized canonical transformations [38] i.e., those which preserve the form of all constraints of the theory as well as the canonical form of the equations of motion. The Abelianization transformation (2.21) is of course non-canonical, but belongs to this class of generalized canonical transformations.

$$\{\Phi_i, \Phi_j\} = \epsilon_{ijk}\Phi_j \,. \tag{3.7}$$

One easily verifies that the secondary constraints are functionally dependent, $x_i\Phi_i = 0$. We shall now carry out the Abelianization procedure and choose

$$\Phi_1^{(0)} := x_2p_3 - x_3p_2 \,, \quad \Phi_2^{(0)} := x_3p_1 - x_1p_3 \,, \tag{3.8}$$

as the two independent constraints with the algebra

$$\{\Phi_1^{(0)}, \Phi_2^{(0)}\} = -\frac{x_1}{x_3}\Phi_1^{(0)} - \frac{x_2}{x_3}\Phi_2^{(0)} \,. \tag{3.9}$$

The general iterative scheme of the construction of Abelianization matrix [37] consists of two steps for this simple case. Let us at first exclude $\Phi_1^{(0)}$ from the right hand side of eq. (3.9). This can be achieved by performing the transformation

$$\begin{aligned} \Phi_1^{(1)} &:= \Phi_1^{(0)} \,, \\ \Phi_2^{(1)} &:= \Phi_2^{(0)} + C\,\Phi_1^{(0)} \,, \end{aligned} \tag{3.10}$$

with the function C obeying the partial differential equation

$$\{\Phi_1^{(0)}, C\} = -\frac{x_2}{x_3}C + \frac{x_1}{x_3} \,. \tag{3.11}$$

Writing down a particular solution of this equation

$$C(x) = \frac{x_1x_2}{x_2^2 + x_3^2} \,, \tag{3.12}$$

we get the algebra for new constraints

$$\{\Phi_1^{(1)}, \Phi_2^{(1)}\} = -\frac{x_2}{x_3}\Phi_2^{(1)} \,. \tag{3.13}$$

Now let us perform the second transformation

$$\begin{aligned} \Phi_1^{(2)} &:= \Phi_1^{(1)} \,, \\ \Phi_2^{(2)} &:= B\,\Phi_2^{(1)} \,, \end{aligned} \tag{3.14}$$

with the function B satisfying the equation

$$\{\Phi_1^{(2)}, B\} = \frac{x_2}{x_3}B \,. \tag{3.15}$$

A particular solution of this equation is $B(x) = \frac{1}{x_3}$. As result of the two above transformations, the Abelian constraints equivalent to the initial non-Abelian ones have the form

$$\begin{aligned} \Phi_1^{(2)} &= x_2p_3 - x_3p_2 \,, \\ \Phi_2^{(2)} &= \frac{1}{x_3}\left[(x_3p_1 - x_1p_3) + \frac{x_1x_2}{x_2^2 + x_3^2}(x_2p_3 - x_3p_2)\right] . \end{aligned} \tag{3.16}$$

A. Canonical transformation and reduced Hamiltonian

We are now ready to perform a canonical transformation to new variables so that two new momenta will coincide with the Abelian constraints (3.16) [5]

$$p_\theta := \frac{(\vec{x}\cdot\vec{p})\,x_1 - \vec{x}^2\,p_1}{\sqrt{x_2{}^2 + x_3{}^2}} \,, \quad p_\phi := x_2p_3 - x_3p_2 \tag{3.17}$$

It is easy to verify that the contact transformation from the Cartesian coordinates to the spherical ones

$$\begin{aligned} x_1 &= r\cos\theta, & r &= \sqrt{x_1{}^2 + x_2{}^2 + x_3{}^2}, \\ x_2 &= r\sin\phi\sin\theta, & \theta &= \arccos\frac{x_1}{\sqrt{x_1{}^2 + x_2{}^2 + x_3{}^2}}, \\ x_3 &= r\cos\phi\sin\theta, & \phi &= \arctan\left(\frac{x_2}{x_3}\right) , \end{aligned} \tag{3.18}$$

is just the required transformation. Indeed, using the corresponding generating function

$$F[\vec{x};\, p_r, p_\theta, p_\phi] = p_r\sqrt{x_1{}^2 + x_2{}^2 + x_3{}^2} + p_\theta\arccos\frac{x_1}{\sqrt{x_1{}^2 + x_2{}^2 + x_3{}^2}} + p_\phi\arctan\left(\frac{x_2}{x_3}\right) , \tag{3.19}$$

we get

$$p_1 = \frac{\partial F}{\partial x_1} = p_r\cos\theta - p_\theta\frac{\sin\theta}{r} \,, \tag{3.20}$$

$$p_2 = \frac{\partial F}{\partial x_2} = p_r\sin\theta\sin\phi + p_\theta\frac{\sin\phi\cos\theta}{r} + p_\phi\frac{\cos\phi}{r\sin\theta} \,, \tag{3.21}$$

$$p_3 = \frac{\partial F}{\partial x_3} = p_r\sin\theta\cos\phi + p_\theta\frac{\cos\phi\cos\theta}{r} - p_\phi\frac{\sin\phi}{r\sin\theta} \,, \tag{3.22}$$

and convince ourselves that in terms of these new variables the two independent constraints are indeed $p_\theta = 0$ and $p_\phi = 0$ in accordance with (3.17). It is worth noting here that starting with the set of reducible constraints (3.6) and performing the above transformation (3.18) one obtains the representation

$$\Phi_1 = -p_\phi \,, \tag{3.23}$$

$$\Phi_2 = -p_\theta\cos\phi + p_\phi\sin\phi\cot\theta \,, \tag{3.24}$$

$$\Phi_3 = p_\theta\sin\phi + p_\phi\cos\phi\cot\theta \,, \tag{3.25}$$

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[5] Here we introduce the compact notations for three-dimensional vectors $\vec{x}, \vec{p}$ and multiply the constraint $\Phi_2^{(2)}$ by the factor $\sqrt{x_2{}^2 + x_3{}^2}$ to deal with constraints of one and the same dimension. This multiplication conserves the Abelian character of the constraints, since $\{\Phi_1^{(2)}, \sqrt{x_2{}^2 + x_3{}^2}\} = 0$.

adapted to the Abelianization. The corresponding Abelianization matrix for the reducible set of constraints is

$$D := \frac{1}{d}\begin{pmatrix} -d_2\sin\phi - d_3\cos\phi\,, & d_1\sin\phi\,, & d_1\cos\phi\,, \\ (d_2\cos\phi - d_3\sin\phi)\cot\theta\,, & -d_3 - d_1\cos\phi\cot\theta\,, & d_2 + d_1\sin\phi\cot\theta\,. \\ \cot\theta\,, & -\sin\phi\,, & \cos\phi\,, \end{pmatrix}. \quad (3.26)$$

with arbitrary $\vec{d}$ and $d := d_1\cot\theta + d_2\sin\phi + d_3\cos\phi$. This example demonstrates two important features of the Abelianization procedure: *i)* it is not necessary to work with an irreducible set of constraints, because the Abelianization procedure leads automatically to an irreducible set of constraints, *ii)* in certain special coordinates the problem of the solution of differential equations reduces to the solution of a simple algebraic problem. In terms of the new canonical variables the canonical Hamiltonian (3.5) reads

$$H_C = \frac{1}{2}p_r^2 + \frac{1}{2r^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - p_\phi y_\phi - p_\theta y_\theta + V(r)\,, \quad (3.27)$$

with the physical momentum $p_r = \frac{(\vec{x}\cdot\vec{p})}{\sqrt{x_1{}^2+x_2{}^2+x_3{}^2}}$, and

$$y_\phi := y_1 + y_2\sin\phi + y_3\cos\phi\cot\theta\,,$$
$$y_\theta := y_2\cos\phi - y_3\sin\phi\,. \quad (3.28)$$

As a result, all the unphysical variables are separated from the physical r and p_r and their dynamics is governed by the physical Hamiltonian obtained from the canonical one by putting p_ϕ and p_θ in (3.27) equal to zero

$$H_{phys} = \frac{1}{2}p_r^2 + V(r)\,. \quad (3.29)$$

IV. SPATIALLY HOMOGENEOUS $SU(2)$ DIRAC-YANG-MILLS FIELDS IN 3+1 DIMENSIONS

A. Canonical formulation of the model

The dynamics of $SU(2)$ Yang-Mills gauge fields $A_\mu^a(x)$ minimally coupled to the isospinor fields $\Psi_\alpha(x)$ [6] in four-dimensional Minkowski space-time is defined by the Lagrange density

$$\mathcal{L} = \mathcal{L}_{Y.-M.} + \mathcal{L}_{Matter} + \mathcal{L}_I\,. \quad (4.1)$$

The first term is the kinetic term of the non-Abelian fields

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[6]The matter isospinor variables Ψ_α are treated classically as a collection of four Grassmann quantities. Detailed notations are collected in the Appendix.

$$\mathcal{L}_{Y.-M.} = \frac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu})\,, \quad (4.2)$$

the second term corresponds to the matter part

$$\mathcal{L}_{Matter} = \frac{i}{2}[\bar{\Psi}_\alpha\gamma_\mu\partial^\mu\Psi_\alpha - \partial^\mu\bar{\Psi}_\alpha\gamma_\mu\Psi_\alpha] - m\,\bar{\Psi}_\alpha\Psi_\alpha\,, \quad (4.3)$$

and the last term describes the interaction between the gauge and the matter fields

$$\mathcal{L}_I = g\,\frac{1}{2}\bar{\Psi}_\alpha\gamma^\mu(\tau^a)_{\alpha\beta}\Psi_\beta\,A_\mu^a\,, \quad (4.4)$$

with the Pauli matrices τ_a, $a = 1, 2, 3$.

After the supposition of the spatial homogeneity of the fields, (4.1) reduces to a finite dimensional model described by the Lagrangian

$$L = \frac{1}{2}(D_tA)_{ai}(D_tA)_{ai} + \frac{i}{2}\left(\bar{\Psi}_\alpha\gamma_0\dot{\Psi}_\alpha - \dot{\bar{\Psi}}_\alpha\gamma_0\Psi_\alpha\right) - m\bar{\Psi}_\alpha\Psi_\alpha - g\rho_a Y_a + gj_{ia}A_{ai} - V(A)\,, \quad (4.5)$$

where the nine spatial components A_i^a are written in the form of a 3×3 matrix A_{ai}, the time component of the gauge potential is identified with $Y_a := A_0^a$ and D_t denotes the covariant derivative,

$$(D_tA)_{ai} := \dot{A}_{ai} - g\epsilon_{abc}Y_bA_{ci}\,.$$

The part of the Lagrangian density corresponding to the selfinteraction of the gauge fields is gathered in the "potential" $V(A)$

$$V(A) := \frac{g^2}{4}\left[\mathrm{tr}^2(AA^T) - \mathrm{tr}(AA^T)^2\right]\,, \quad (4.6)$$

while their interactions with the matter fields are via the isospinor currents

$$\rho_a\,[\Psi] := \frac{1}{2}\bar{\Psi}_\alpha\gamma_0\,(\tau_a)_{\alpha\beta}\,\Psi_\beta\,,$$
$$j_{ia}\,[\Psi] := \frac{1}{2}\bar{\Psi}_\alpha\gamma_i\,(\tau_a)_{\alpha\beta}\,\Psi_\beta\,. \quad (4.7)$$

After Legendre transformation one obtains the canonical Hamiltonian

$$H_C = \frac{1}{2}E_{ai}E_{ai} + m\bar{\Psi}_\alpha\Psi_\alpha - g\,(\epsilon_{abc}A_{ci}E_{bi} - \rho_a)\,Y_a - g\,j_{ia}A_{ai} + V(A)\,, \quad (4.8)$$

defined on the phase space endowed with the canonical symplectic structure (see Appendix) and spanned by the bosonic and fermionic canonical variables (Y_a, P_{Y_a}), (A_{ai}, E_{ai}) and $(\Psi_\alpha, P_{\Psi_\alpha})$, $(\bar{\Psi}_\alpha, P_{\bar{\Psi}_\alpha})$, where

$$P_{Y_a} := \frac{\partial L}{\partial \dot{Y}_a} = 0\,, \quad (4.9)$$

$$E_{ai} := \frac{\partial L}{\partial \dot{A}_{ai}} = \dot{A}_{ai} - g\epsilon_{abc}Y_bA_{ci}\,, \quad (4.10)$$

$$P_{\Psi_\alpha} := L\frac{\overleftarrow{\partial}}{\partial\dot{\Psi}_\alpha} = -\frac{i}{2}\bar{\Psi}_\alpha\gamma_0\,, \quad (4.11)$$

$$P_{\bar{\Psi}_\alpha} := \frac{\overrightarrow{\partial}}{\partial\dot{\bar{\Psi}}_\alpha}L = -\frac{i}{2}\gamma_0\Psi_\alpha\,. \quad (4.12)$$

According to the definition of the canonical momenta (4.9), (4.11) and (4.12) the phase space is restricted by the three primary bosonic constraints

$$P^a_Y = 0\,, \tag{4.13}$$

and the sixteen Grassmann constraints

$$\Upsilon^1_\alpha := P_{\Psi_\alpha} + \frac{i}{2}\bar\Psi_\alpha\gamma_0\,, \quad \Upsilon^2_\alpha := P_{\bar\Psi_\alpha} + \frac{i}{2}\gamma_0\Psi_\alpha\,. \tag{4.14}$$

Thus the evolution of the system is governed by the total Hamiltonian

$$H_T := H_C + u^a_Y(t)\; P^a_Y + \Upsilon^1_\alpha\; u^1_\alpha(t) + u^2_\alpha(t)\; \Upsilon^2_\alpha\,. \tag{4.15}$$

The conservation of bosonic constraints (4.13) in time entails the following further condition on canonical variables

$$\dot P_{Y_a} = 0 \quad\longrightarrow\quad \Phi_a := \epsilon_{abc}A_{ci}E_{bi} - \rho_a[\Psi] = 0\,, \tag{4.16}$$

which is the non-Abelian Gauss law. In contrast, the maintenance of Grassmann constraints Υ^1_α and Υ^2_α in time allows to determine the Lagrange multipliers $u^1_\alpha(t)$ and $u^2_\alpha(t)$ in the expression (4.15) for the total Hamiltonian. Taking into account the Poisson brackets of constraints

$$\{\Phi_i, \Phi_j\} = \epsilon_{ijk}\Phi_j \;+ \epsilon_{ijk}\,\rho_k[\Psi]\,, \tag{4.17}$$

$$\{\Phi_a, \Upsilon^1_\alpha\} = -\bar\Psi_\beta\gamma_0\,(\tau_a\,)_{\beta\alpha}\,, \tag{4.18}$$

$$\{\Phi_a, \Upsilon^2_\alpha\} = \;\gamma_0\,(\tau_a\,)_{\alpha\beta}\,\Psi_\beta\,, \tag{4.19}$$

$$\{\Upsilon^1_\alpha, \Upsilon^2_\beta\} = -i\delta_{\alpha\beta}\;\gamma_0\,, \tag{4.20}$$

one can convince oneself that no new constraints emerge and hence that ternary constraints are absent in the theory, $\dot\Phi\Big|_{Constraint\ Shell} = 0$.

To implement the reduction procedure without using gauge fixing conditions we have to put the constraints into the canonical form discussed in the next paragraph.

B. Putting the constraints into the canonical form

1. *Separation of first and second class constraints*

The set of the 22 constraints $C_\mathcal{A} := (P_Y, \Phi, \Upsilon)$ represent a mixed system of first and second class constraints. The Poisson matrix $M_{\mathcal{AB}} := \{C_\mathcal{A}, C_\mathcal{B}\}$ is degenerate on constraint shell, $\mathrm{rank}\,||\mathcal{M}||\Big|_{C_\mathcal{A}=0} = 16$. Hence among the constraints there are six first class ones.

In order to perform the reduction procedure let us start with the separation of the first and second class constraints. The primary constraints P_Y "commute" with all the other

constraints and thus we should deal only with constraints $C'_\mathcal{A} := (\Phi, \Upsilon)$. The separation of constraints is achieved by a transformation to an equivalent set of constraints $\tilde C'_\mathcal{A} := (\tilde\Phi, \tilde\Upsilon)$

$$\tilde C'_\mathcal{A} := D'_{\mathcal{AB}}\,C'_\mathcal{B}\,, \tag{4.21}$$

so that the first class constraints $\tilde\Phi$ form the ideal of the algebra

$$\{\tilde\Phi, \tilde\Upsilon\} = 0\,, \qquad \{\tilde\Phi, \tilde\Phi\}\big|_{\Phi=0} = 0\,, \tag{4.22}$$

and the pairs of second class constraint satisfy the canonical algebra

$$\{\tilde\Upsilon^1_\alpha, \tilde\Upsilon^2_\beta\} = -\,\delta_{\alpha\beta}\,. \tag{4.23}$$

In order to transform the algebra of constraints to the canonical form let us at first perform the equivalence transformation

$$\Phi'_a := \Phi_a + \Upsilon^1_\alpha\frac{i}{2}\,(\tau_a)_{\alpha\beta}\,\Psi_\beta + \frac{i}{2}\,\bar\Psi_\beta\,(\tau_a)_{\beta\alpha}\,\Upsilon^2_\alpha\,, \tag{4.24}$$

on the bosonic constraints Φ_a and the equivalence transformation

$$\tilde\Upsilon^1_\alpha := -i\Upsilon^1_\alpha\gamma_0\,, \quad \tilde\Upsilon^2_\alpha := \;\Upsilon^2_\alpha\,, \tag{4.25}$$

on the Grassmann constraints. The Poisson brackets of the new constraints

$$\{\Phi'_a, \Phi'_b\} = \epsilon_{abc}\Phi'_c\,, \tag{4.26}$$

$$\{\tilde\Upsilon^1_\alpha, \Phi'_a\} = \;\frac{i}{2}\tilde\Upsilon^1_\beta\,(\tau_a)_{\beta\alpha}\,, \tag{4.27}$$

$$\{\tilde\Upsilon^2_\alpha, \Phi'_a\} = -\frac{i}{2}\,(\tau_a)_{\alpha\beta}\,\tilde\Upsilon^2_\beta\,, \tag{4.28}$$

$$\{\tilde\Upsilon^1_\alpha, \tilde\Upsilon^2_\beta\} = -i\delta_{\alpha\beta}\,, \tag{4.29}$$

show the separation of the first class constraints on the surface $\Upsilon = 0$ defined by the second class constraints. To achieve this separation on the whole phase space it is necessary to apply the additional transformation

$$\tilde\Phi_a := \Phi'_a - \tilde\Upsilon^1_\alpha\,(\tau_a)_{\alpha\beta}\,\tilde\Upsilon^2_\beta\,. \tag{4.30}$$

One can verify that the first class constraints form the ideal of the total set of constraints

$$\{\tilde\Phi_a, \tilde\Phi_b\} = \epsilon_{abc}\tilde\Phi_c, \tag{4.31}$$

$$\{\tilde\Upsilon^1_\alpha, \tilde\Phi_a\} = 0, \tag{4.32}$$

$$\{\tilde\Upsilon^2_\alpha, \tilde\Phi_a\} = 0 \tag{4.33}$$

and the second class constraints obey the canonical algebra (4.29). The explicit form of the resulting set of constraints is

$$\tilde\Upsilon^1_\alpha := -iP_{\Psi_\alpha}\gamma_0 + \frac{1}{2}\,\Psi = 0\,, \tag{4.34}$$

$$\tilde\Upsilon^2_\alpha := P_{\bar\Psi_\alpha} + \frac{1}{2}\,\gamma_0\Psi = 0\,, \tag{4.35}$$

$$\tilde\Phi_a := \epsilon_{abc}A_{bi}E_{ci} + \frac{1}{8}\bar\Psi\tau_a\gamma_0\Psi - \frac{1}{2}P_\Psi\tau_a\gamma_0 P_{\bar\Psi} + \frac{i}{4}\left(P_\Psi\tau_a\Psi + \bar\Psi\tau_a P_{\bar\Psi}\right) = 0\,. \tag{4.36}$$

In order to implement the reduction due to the second class constraints (4.34) and (4.35) let us introduce the new canonical variables $(Q^*_{\Psi_\alpha}, \bar{Q}_{\Psi_\alpha})$ and $(\Pi^*_{\Psi_\alpha}, \bar{\Pi}_{\Psi_\alpha})$ via

$$\Psi_\alpha =: i\gamma_0 \left(Q^*_{\Psi_\alpha} - \bar{Q}_{\Psi_\alpha}\right) , \qquad \bar{\Psi}_\alpha =: \Pi^*_{\Psi_\alpha} - \bar{\Pi}_{\Psi_\alpha} , \tag{4.37}$$

$$P_{\Psi_\alpha} =: \frac{i}{2}\left(\bar{\Pi}_{\Psi_\alpha} - \Pi^*_{\Psi_\alpha}\right)\gamma_0 , \qquad P_{\bar{\Psi}_\alpha} =: \frac{1}{2}\left(\bar{Q}_{\Psi_\alpha} + Q^*_{\Psi_\alpha}\right) . \tag{4.38}$$

In terms of the new variables the constraints read

$$\tilde{\Upsilon}^1_\alpha = \bar{\Pi}_{\Psi_\alpha} = 0 , \tag{4.39}$$

$$\tilde{\Upsilon}^2_\alpha = \bar{Q}_{\Psi_\alpha} = 0 , \tag{4.40}$$

$$\tilde{\Phi}_a = \epsilon_{abc} A_{bi} E_{ci} - \frac{i}{2} \Pi^*_{\Psi_\alpha} \tau_a Q^*_{\Psi_\alpha} = 0 . \tag{4.41}$$

2. Canonical transformation to adapted coordinates

The example of the Christ and Lee model in Section III shows that the realization of constraints by Abelianization is immediate if one performs a canonical transformation to a new set of variables containing the gauge invariant ones as a subset. Hence in order to simplify the Abelianization of constraints let us single out the part of the gauge potentials A_{ai}, which is invariant under gauge transformations. Because under a homogeneous gauge transformation the gauge potentials transforms homogeneously one can achieve the separation of gauge deegrees of freedom by the following simple transformation

$$A_{ai}\left(\bar{Q}, Q^*\right) = O_{ak}\left(\bar{Q}\right) Q^*_{ki} , \tag{4.42}$$

where O is an orthogonal matrix, $O \in SO(3)$, and Q^* is a positive definite symmetric matrix. This transformation induces a point canonical transformation linear in the new canonical momenta. The new canonical momenta $(P^*_{ik}, \bar{P}_i)$ can be obtained using the generating function

$$F_4\left(E; \bar{Q}, Q^*\right) = \sum_{a,i}^{3} E_{ai} A_{ai}\left(\bar{Q}, Q^*\right) = \operatorname{tr}\left(O(\bar{Q}) Q^* E^T\right) , \tag{4.43}$$

as

$$\bar{P}_j = \frac{\partial F_4}{\partial \bar{Q}_j} = \sum_{a,s,i}^{3} E_{ai} \frac{\partial O_{as}}{\partial \bar{Q}_j} Q^*_{si} = \operatorname{tr}\left[E^T \frac{\partial O}{\partial \bar{Q}_j} Q^*\right] , \tag{4.44}$$

$$P^*_{ik} = \frac{\partial F_4}{\partial Q^*_{ik}} = \frac{1}{2}\left(OE^T + EO^T\right)_{ik} . \tag{4.45}$$

In order to express the Hamiltonian and the Gauss law constraints in terms of these new canonical pairs let us write the field strength E_{ai} in the form

$$E_{ai} = O_{ak}\left(\bar{Q}\right) L_{ki}\left(\bar{P}, P^*; \bar{Q}, Q^*\right) \tag{4.46}$$

with a 3×3 matrix L_{ki} to be determined. One can immediately see that the symmetric part of the matrix L is equal to the new momenta P^*

$$P^*_{ik} = \frac{1}{2}\left(L_{ik} + L_{ki}\right) \tag{4.47}$$

and a straightforward calculation shows that its antisymmetric part is

$$\frac{1}{2}\left(L_{ik} - L_{ki}\right) = \epsilon_{ilk}(\gamma^{-1})_{ls}\left[\left(\Omega^{-1}\right)_{sj} \bar{P}_j - \epsilon_{msn}\left(P^* Q^*\right)_{mn}\right] . \tag{4.48}$$

with

$$\Omega_{ij} := \frac{1}{2}\epsilon_{min}\left[\frac{\partial O^T(\bar{Q})}{\partial \bar{Q}_j} O(\bar{Q})\right]_{mn} , \tag{4.49}$$

and

$$\gamma_{ik} := Q^*_{ik} - \delta_{ik} \operatorname{tr}(Q^*) . \tag{4.50}$$

Thus the final expressions for field strength E_{ai} in terms of the new canonical variables are

$$E_{ai} = O(\bar{Q})_{ak}\left[P^*_{ki} + \epsilon_{kli}(\gamma^{-1})_{ls}\left[\left(\Omega^{-1}\bar{P}\right)_s - \epsilon_{msn}\left(P^* Q^*\right)_{mn}\right]\right] . \tag{4.51}$$

3. Abelianization of first class constraints

The formulation of the theory in terms of the new variables is adapted to the procedure of Abelianization. Using the representations (4.42) and (4.51) one can easily convince oneself that the variables Q^* and P^* make no contribution to the secondary constraints (4.41) and $\bar{Q}$, $\bar{P}$ enter well-separated from the physical matter variables

$$\tilde{\Phi}_a := O_{as}(\bar{Q})\Omega^{-1}_{sj}\bar{P}_j - \frac{i}{2}\Pi^*_{\Psi_\alpha}(\tau_a)_{\alpha\beta} Q^*_{\Psi_\beta} = 0 . \tag{4.52}$$

In order to deal with the Abelianization it is useful to perform the following canonical transformation on the Grassmann variables

$$\Pi^*_{\Psi_\alpha} =: \mathcal{P}^*_{\Psi_\beta} U_{\beta\alpha}(\bar{Q}) , \tag{4.53}$$

$$Q^*_{\Psi_\alpha}{}' =: U^{-1}_{\alpha\beta}(\bar{Q})\, \mathcal{Q}^*_{\Psi_\beta} . \tag{4.54}$$

with the unitary matrix U in the two dimensional representation of $SO(3)$ chosen such that

$$O_{ab} = \frac{1}{2}\operatorname{tr}\left(U^+ \tau_a U \tau_b\right) . \tag{4.55}$$

As a result, the Gauss law constraints (4.52) take the form

$$\tilde{\Phi}'_a := \Omega^{-1}_{sj} P_j - \frac{i}{2} \mathcal{P}^*_{\Psi_\alpha} (\tau_a)_{\alpha\beta} \mathcal{Q}^*_{\Psi_\beta} = 0 . \quad (4.56)$$

Hence it is clear that the matrix Ω^{-1} is just the matrix of Abelianization D in (2.21). Hence, after performing the Dirac transformation with the matrix $D := \Omega(Q)$ on the constraints $\tilde{\Phi}'_a$ the equivalent set of Abelian constraints is

$$P_a - \Omega_{\ as} \Theta_s = 0 , \quad (4.57)$$

with

$$\Theta_a := \frac{i}{2} \mathcal{P}^*_{\Psi_\alpha} (\tau_a)_{\alpha\beta} \mathcal{Q}^*_{\Psi_\beta} . \quad (4.58)$$

C. Reduction due to the Gauss law and the second class constraints

In the previous section, in accordance with the general scheme of reduction formulated in subsection (II D), the new set of constraints in canonical form have been obtained and the adapted canonical pairs been chosen for the explicit implementation of the Gauss laws (4.16) and the second class constraints (4.29). After having rewritten the model in this form, the construction of the unconstrained Hamiltonian system is straightforward. In all expressions we can simply put $P = \Omega\Theta$ and $\Pi_{\Psi_\alpha} = Q_{\Psi_\alpha} = 0$. In particular, in terms of the "physical" electric field strength $\mathcal{E}_{ai}$

$$E_{ai}\Big|_{P=\Omega\Theta} =: O_{ak}(Q)\, \mathcal{E}_{ki}(Q^*, P^*) , \quad (4.59)$$

the physical unconstrained Hamiltonian

$$H_{phys} := H_C(P,Q)\Big|_{constraint\ shell}$$

may be written as

$$H^{D.-Y.-M.}_{phys} = \frac{1}{2}\mathrm{tr}(\mathcal{E}^2) + \frac{g^2}{4}\left[\mathrm{tr}^2(Q^*)^2 - \mathrm{tr}(Q^*)^4\right] - g\,\mathrm{tr}\,(j^* Q^*) + im\left(\mathcal{P}^*_{\Psi_\alpha}\gamma_0 \mathcal{Q}^*_{\Psi_\alpha}\right) . \quad (4.60)$$

where j^* is the isospin current in terms of the new Grassmann variables

$$j^*_{ia} := \frac{i}{2} \mathcal{P}^*_{\Psi_\alpha} \gamma_i \gamma_0 (\tau_a)_{\alpha\beta} \mathcal{Q}^*_{\Psi_\beta} . \quad (4.61)$$

With the aid of the identity $\det\gamma\ \epsilon_{isk} (\gamma^{-1})_{sl} = \epsilon_{alb} \gamma_{ia} \gamma_{kb}$ and representation (4.51) for the field strengths, we find the explicit form for the "physical" electric field strength in terms of P^* and Q^*

$$\mathcal{E}_{ki}(Q^*, P^*) = P^*_{ik} + \frac{1}{\det\gamma} \epsilon_{ilk} \mathrm{tr}\,(\gamma \mathcal{M} \gamma J_l) , \quad (4.62)$$

where $\mathcal{M}$ denotes the isospin angular momentum tensor

$$\mathcal{M}_{mn} := \epsilon_{msn} \mathcal{J}_s . \quad (4.63)$$

Here,

$$\mathcal{J}_s := \Theta_s + T_s . \quad (4.64)$$

is the sum of the gauge field isospin vector $T_s := \frac{1}{2}\epsilon_{msn}(Q^* P^*)_{mn}$ and the matter field Θ_s defined in (4.58). With (4.62) the unconstrained Dirac-Yang-Mills Hamiltonian reads

$$H^{D.-Y.-M.}_{phys} = \frac{1}{2}\mathrm{tr}(P^*)^2 + \frac{1}{2\det^2\gamma}\mathrm{tr}\,(\gamma\mathcal{M}\gamma)^2 + \frac{g^2}{4}\left[\mathrm{tr}^2(Q^*)^2 - \mathrm{tr}(Q^*)^4\right] - g\,\mathrm{tr}\,(j^* Q^*) + im\left(\mathcal{P}^*_{\Psi_\alpha}\gamma_0 \mathcal{Q}^*_{\Psi_\alpha}\right) . \quad (4.65)$$

In order to achieve a more transparent form for the reduced Dirac-Yang-Mills system (4.65) one can perform a canonical transformation expressing the physical coordinates Q^* and P^* in terms of new variables adapted for the analysis of the rigid symmetry possessed by the reduced Hamiltonian system (4.65). It is convenient to decompose the nondegenerate symmetric matrix Q^* in the following way:

$$Q^* = \mathcal{R}^T(\psi,\theta,\phi)\, \mathcal{D}\, \mathcal{R}(\psi,\theta,\phi) , \quad (4.66)$$

with the $SO(3)$ matrix $\mathcal{R}$ parametrized by the three Euler angles $\chi_i := (\psi,\theta,\phi)$, (see Appendix) and with the diagonal matrix $\mathcal{D} := \mathrm{diag}\,(x_1, x_2, x_3)$. The corresponding canonical conjugate coordinates $(p_\psi, p_\theta, p_\phi, p_i)$ can be found by using the generating function

$$F[x_i, \psi, \theta, \phi;\ P^*] := \mathrm{tr}\,(Q^* P^*) = \mathrm{tr}\,\left(\mathcal{R}^T(\chi) D(x) \mathcal{R}(\chi) P^*\right) \quad (4.67)$$

as

$$p_i = \frac{\partial F}{\partial x_i} = \mathrm{tr}\,(P^* \mathcal{R}^T \overline{\alpha}_i \mathcal{R}) ,$$
$$p_{\chi_i} = \frac{\partial F}{\partial \chi_i} = \mathrm{tr}\left(\frac{\partial \mathcal{R}^T}{\partial \chi_i}\mathcal{R}\,[P^* Q^* - Q^* P^*]\right), \quad (4.68)$$

where $\overline{\alpha}_i$ are the diagonal members of the orthogonal basis for symmetric matricies $\alpha_A = (\overline{\alpha}_i,\ \alpha_i)$ $i = 1,2,3$ given explicitly in the Appendix. The original physical momenta P^*_{ik} can then be expressed in terms of the new canonical variables as

$$P^* = \mathcal{R}^T \left(\sum_{s=1}^{3} p_s \overline{\alpha}_s + \sum_{s=1}^{3} \mathcal{P}_s \alpha_s\right) \mathcal{R} \quad (4.69)$$

with

$$\mathcal{P}_i := \frac{\xi_i}{x_j - x_k}, \quad (cyclic\ permutation\ i \neq j \neq k) \quad (4.70)$$

and the $SO(3)$ left-invariant Killing vectors

$$\xi_1 := \frac{\sin\psi}{\sin\theta}\, p_\phi + \cos\psi\, p_\theta - \sin\psi \cot\theta\, p_\psi\,, \tag{4.71}$$

$$\xi_2 := -\frac{\cos\psi}{\sin\theta}\, p_\phi + \sin\psi\, p_\theta + \cos\psi \cot\theta\, p_\psi\,, \tag{4.72}$$

$$\xi_3 := p_\psi\,. \tag{4.73}$$

Representing the physical electric field strength $\mathcal{E}_{ai}$ in the alternative form

$$\mathcal{E}_{ik} = P^*{}_{ik} + \frac{1}{\det\gamma}\,(\gamma J_s \gamma)_{ik}\, \mathcal{J}_s\,, \tag{4.74}$$

with the SO(3) generators J_s given in explicit form in the Appendix, we finally get the following physical Hamiltonian defined on the unconstrained phase space

$$H^{D.-Y.-M.}_{phys} = \frac{1}{2}\sum_{s=1}^{3} p_s^2 + \frac{1}{4}\sum_{s=1}^{3}\mathcal{P}_s^2 + \frac{1}{4}\sum_{cyclic}\left(\frac{\xi_i+\Theta_i}{x_j+x_k}\right)^2 + \frac{g^2}{2}\sum_{i<j} x_i^2 x_j^2 - g\sum_{s=1}^{3} j^*_{ss} x_s + im\left(\mathcal{P}^*_{\Psi_\alpha}\gamma_0 \mathcal{Q}^*_{\Psi_\alpha}\right)\,. \tag{4.75}$$

Note that for the pure Yang-Mills system (4.75) reduces to

$$H^{Y.-M.}_{phys} = \frac{1}{2}\sum_{s=1}^{3} p_s^2 + \frac{1}{2}\sum_{cyclic} \xi_i^2 \frac{x_j^2 + x_k^2}{(x_j^2 - x_k^2)^2} + \frac{g^2}{2}\sum_{i<j} x_i^2 x_j^2\,. \tag{4.76}$$

This completes our reduction of the spatially homogeneous constraint Dirac-Yang-Mills system to the equivalent unconstraint system describing the dynamics of the physical dynamical degrees of freedom. However, apart from this reduction due to the underlying gauge symmetry, there is the possibility to realize another type of reduction connected with the rigid symmetry admitted by the unconstrained system (4.65). For simplicity, the discussion in the next section will be restricted to the pure Yang-Mills system and we shall show how to further reduce the obtained 12-dimensional system (4.76) to an 8-dimensional one in general and, for a special case to a 6-dimensional one using the corresponding first integrals.

D. Further reduction using first integrals

The reduced Yang-Mills theory (4.76) has a rigid symmetry connected with the existence of the first integrals

$$I_i = \epsilon_{ijk} E_{aj} A_{ak}\,. \tag{4.77}$$

For the subsequent reduction in the number of degrees of freedom we shall use the integrals of motion (4.77). One can verify that in terms of the new variables they read $I_i = \mathcal{R}_{ik}\xi_k$. In contrast to the reduction due to first class constraints the values of first integrals are arbitrary and depend on the initial conditions. In this case reduction means to consider the subspaces of phase space which are the levels of fixed values for these first integrals

$$I_i = c_i\,, \tag{4.78}$$

and the subsequent construction of the quotient space with respect to the rigid symmetry group. Therefore, in contrast to the reduction that we have done before, the first integrals in general are a mixed system of first and second class constraints. By "in general" we mean that not all constants of motion c_i are zero. In this case the rank of the Poisson matrix is rank $\|\{I_i, I_k\}\||_{I_i=c_i} = 1$, which means that there is one first class constraint and one pair of second class constraints [7]. The problem is now to separate the algebra of constraints and to find the equivalent set of constraints $\Psi_i = 0$, so that the first class constraint Ψ_1 forms the center of the algebra

$$\{\Psi_1, \Psi_i\} = 0 \tag{4.79}$$

and the pair of second class constraints obey the canonical algebra

$$\{\Psi_2, \Psi_3\} = 1\,. \tag{4.80}$$

After having passed to new variables in the last section in order to isolate the gauge degrees of freedom from the physical ones, we shall now perform another canonical transformation from the physical variables to new physical variables so that one of the new momenta coincides with the first class constraint Ψ_1 and another pair of new canonical variables coincides with the pair of second class constraints Ψ_2 and Ψ_3. In terms of these new canonical variables the reduced system is obtained by reducing the Hamiltonian (4.65) to the integral surface (4.78). As result the new Hamiltonian will depend on 4 canonical pairs and one parameter which reflects the existence of the integrals of motion. To demonstrate this let us choose the integral constants as $c_i = (0, 0, c)$ without loss of generality. One can then write down the needed new set of constraints, describing the surface (4.78), in the form

$$\Psi_1 := I_1^2 + I_2^2 + I_3^2 - c^2 = 0\,, \qquad \Psi_2 := \arctan\left(\frac{I_2}{I_3}\right) = 0\,, \qquad \Psi_3 := I_1 = 0\,. \tag{4.81}$$

We are now ready to perform the transformation to special canonical variables so that the pair of second class constraints is equal to the one pair of the canonical variables and one equal to the new momentum [8].

$$\Pi_0 := \Psi_1\,,\quad \Pi_1 := \Psi_3\,,\quad X_1 := \Psi_2\,. \tag{4.82}$$

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[7] In the exceptional case when $c_i = 0$ we can consider the three integrals as first class constraints, and this circumstance leads to a further reduction of our system.

[8] This type of variables are well-known from rigid body theory as Depri [41] or Andoyer [42,29] variables used in celestial mechanics.

and complete the set of canonical coordinates by the following pair

$$X_2 := \arctan\left(\frac{\xi_1}{\xi_2}\right), \quad \Pi_2 := p_\psi . \tag{4.83}$$

The canonical conjugate coordinate X_0 can be determined with the help of the generating function

$$F[\psi,\theta,\phi,\Pi_i] = \Pi_1\phi + \Pi_2\psi + \int^\theta \frac{d\alpha}{\sin\alpha}\sqrt{\Pi_0^2\,\sin^2\alpha - \Pi_1^2 - \Pi_2^2 + 2\Pi_1\Pi_2\cos\alpha}\,. \tag{4.84}$$

Due to the symmetry the X_0 is a cyclic coordinate and the reduced Hamiltonian depends only on the canonical pair

$$\Pi_2 = p_\psi, \quad X_2 := \arctan\left(\frac{\xi_1}{\xi_2}\right)\Big|_{I_i=c_i} = \psi\,.$$

Hence, using the first integrals (4.77), the pure Yang-Mills Hamiltonian (4.76) can be further reduced to

$$H^{Y.-M.}_{phys} = \frac{1}{2}\sum_{s=1}^{3} p_s^2 + \frac{1}{2}p_\psi^2\left[\frac{x_1^2+x_2^2}{(x_1^2-x_2^2)^2} - \sin^2\psi\frac{x_2^2+x_3^2}{(x_2^2-x_3^2)^2} - \cos^2\psi\frac{x_3^2+x_1^2}{(x_3^2-x_1^2)^2}\right]$$
$$+\frac{g^2}{2}\sum_{i<j} x_i^2x_j^2 + V_C\,, \tag{4.85}$$

where in accordance with the general scheme of reduction there arises the additional so-called reduced potential term

$$V_C := c^2\left[\sin^2\psi\frac{x_2^2+x_3^2}{(x_2^2-x_3^2)^2} + \cos^2\psi\frac{x_3^2+x_1^2}{(x_3^2-x_1^2)^2}\right]. \tag{4.86}$$

It is interesting to point out the difference between the reduced Yang-Mills Hamiltonian (4.76) and the corresponding one in the recent work by B. Dahmen and B. Raabe. In contrast with their representation for the gauge potentials, in which the gauge degrees of freedom are mixed with the rigid rotational cyclic coordinates, we have started with the explicit separation of all physical degrees, including the rotational ones. And only after the reduction in the number of degrees of freedom due to the rigid symmetry the obtained Hamiltonian (4.85) coincides with the one obtained in the work by B. Dahmen and B. Raabe [24] for pure Yang-Mills mechanics.

V. CONCLUDING REMARKS

As mentioned in the introduction our investigation has pursued two goals. One is pure theoretical interest. Due to the homogeneity condition SU(2) Dirac-Yang-Mills field theory has greatly simplified to a finite dimensional mechanical system, for which one can describe the equivalent unconstrained system in an explicit way. However, apart from this reason, there is also an interesting application of this model. It has been known for a long

time, that, if one considers the Euclidean QCD effective action as a function of the nonabelian electric and magnetic fields E and B, one finds that there are field configurations, corresponding to nonvanishing E and B fields, for which the value of the effective action is lower than that for $E = 0$ and $B = 0$ [40]. This observation indicates a drastic difference between the true ground state of QCD and the corresponding perturbative vacuum and constitutes the basis of all models of condensates. One of the main reasons to study the dynamics of spatially constant Yang-Mills fields, is the faith that the corresponding zero momentum quantum operators are very important for the description of the QCD ground state due to the presence of the IR singularity. There are many attempts to exploit the homogeneity approximation for gluon fields with the aim to shed light on the vacuum structure of QCD. We also adhere to this position and our task in this note was to prepare the classical description of Yang-Mills mechanics in a form that we are going to exploit for the description of squeezed vacuum [43].

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APPENDIX A: NOTATIONS AND SOME FORMULAE

1. Definition of configuration variables

$SU(2)$ Dirac-Yang-Mills theory considered in this paper includes as dynamical variables the set of spin-1 gauge fields $A_\mu := A_\mu^a\tau_a/2$, $a = 1,2,3$ in the adjoint representation of SU(2), with the corresponding field strengths

$$F_{\mu\nu} := F^a_{\mu\nu}\,\tau^a/2, \tag{A1}$$
$$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu\,, \tag{A2}$$

and the matter spinor (Dirac conjugate spinor) field variables $\Psi(\bar\Psi)$ in the fundamental representation of SU(2) with values $\Psi_\alpha := (\Psi^1_\alpha, \dots, \Psi^4_\alpha)$ obeying the Grassmann algebra

$$\Psi^i_\alpha\Psi^j_\beta + \Psi^j_\beta\Psi^i_\alpha = 0\,. \tag{A3}$$

2. Hamiltonian structures

Generalized Poisson brackets for functions on a phase space spanned by both even and odd coordinates $Z_A := ((Y, P_Y), (A_i, E_i); (\Psi_\alpha, P_{\Psi_\alpha}))$ are defined as

$$\{F(Z), G(Z)\} := \sum_{A,B} F \frac{\overleftarrow{\partial}}{\partial Z_A} \omega_{AB} \frac{\overrightarrow{\partial}}{\partial Z_B} G . \qquad (A4)$$

The nonvanishing components of the canonical symplectic form $\omega_{AB} := \{Z_A, Z_B\}$ read explicitly

$$\{Y_a, P_Y^b\} = \delta_a^b , \qquad \{A_{ai}, E^{bj}\} = \delta_i^j \delta_a^b \qquad (A5)$$

for bosonic degrees of freedom

$$\{\Psi_\alpha, P_{\Psi_\beta}\} = \{P_{\Psi_\beta}, \Psi_\alpha\} = -\delta_{\alpha\beta} ,$$
$$\{\bar\Psi_\alpha, P_{\bar\Psi_\beta}\} = \{P_{\bar\Psi_\beta}, \bar\Psi_\alpha\} = -\delta_{\alpha\beta} \qquad (A6)$$

for fermionic degrees of freedom.

3. The Euler parametrization for S0(3) group

The conventional representation of $SO(3)$ group elements in terms of Euler angles

$$\mathcal{R}(\psi, \theta, \phi) = e^{\psi J_3} e^{\theta J_1} e^{\phi J_3} \qquad (A7)$$

has been used in main text with the following matrix realization for the generators J_i obeing the $SO(3)$ algebra $[J_i, J_j] = \epsilon_{ijk} J_k$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (A8)$$

4. Basis for symmetric matricies

We use the orthogonal basis $\alpha_A = (\overline{\alpha}_i, \alpha^i)$ for symmetric matrices. They read explicitly

$$\overline{\alpha}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \overline{\alpha}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \overline{\alpha}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (A9)$$

$$\alpha^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (A10)$$

They obey the following orthonormality relations:

$$\mathrm{tr}\,(\overline{\alpha}_i \overline{\alpha}_j) = \delta_{ij} , \quad \mathrm{tr}\,(\alpha_i \alpha_j) = 2\delta_{ij} ; \quad \mathrm{tr}\,(\overline{\alpha}_i \alpha_j) = 0 . \qquad (A11)$$

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