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THE SOLUTION OF THE $N=2$ SUPERSYMMETRIC f-TODA CHAIN WITH FIXED ENDS

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1 Introduction

Quite recently, we introduced the minimal $N = 2$ supersymmetric extension of the Toda chain called f-Toda [1]. We have shown that the $N = 2$ supersymmetric Nonlinear Schrödinger (NLS) hierarchy in the $N = 2$ superspace [2] possesses a discrete group of integrable mappings [3] which is equivalent to the f-Toda chain. Its origin was analyzed in [4]. In contrast to other chains, the f-Toda chain is not algebraically solvable. Such type of chains has not been considered in the literature before.

The goal of the present Letter is to prove the integrability of the f-Toda chain and to construct its general solution under appropriate boundary conditions, including the condition corresponding to fixed ends. This last problem is equivalent to the problem of constructing multi-soliton solutions for each integrable system belonging to the $N = 2$ super-NLS hierarchy.

2 The $N = 2$ super-NLS and f-Toda superfield equations

In this section, following [1], we briefly describe the f-Toda chain equations and their relation to the $N = 2$ super-NLS hierarchy.

Let us proceed with the $N = 2$ super-NLS equation [2]

$$\frac{\partial f}{\partial t} = f\,'' + 2D(f\overline{f}\,\overline{D}f), \quad \frac{\partial \overline{f}}{\partial t} = -\overline{f}\,'' + 2\overline{D}(f\overline{f}D\overline{f}), \tag{2.1}$$

where $f(x,\theta,\overline{\theta})$ and $\overline{f}(x,\theta,\overline{\theta})$ are chiral and antichiral fermionic superfields,

$$Df = 0, \quad \overline{D}\,\overline{f} = 0, \tag{2.2}$$

respectively, in the $N = 2$ superspace with one bosonic z and two fermionic $\theta, \overline{\theta}$ coordinates[1]; D and $\overline{D}$ are the $N = 2$ supersymmetric fermionic covariant derivatives

$$D = \frac{\partial}{\partial\theta} - \frac{1}{2}\overline{\theta}\frac{\partial}{\partial z}, \quad \overline{D} = \frac{\partial}{\partial\overline{\theta}} - \frac{1}{2}\theta\frac{\partial}{\partial z},$$
$$D^2 = \overline{D}^2 = 0, \quad \{D,\overline{D}\} = -\frac{\partial}{\partial z} \equiv -\partial. \tag{2.3}$$

[1]The sign $'$ means the derivative with respect to z.

The chirality condition (2.2) reduces the number of independent components of the fermionic superfield f $(\overline{f})$ from four to two.

Equations (2.1), as well as each system of the equations belonging to the $N = 2$ super-NLS hierarchy, are invariant with respect to the following discrete f-Toda transformation [1]:

$$\overleftarrow{f}\,\overleftarrow{\overline{f}} - f\overline{f} = (\ln(\overline{D}\overleftarrow{f}\,D\overline{f}))\,'. \tag{2.4}$$

This relation is the definition of mapping, i.e., the rule determining the correspondence between two initial superfields f and $\overline{f}$, and two final ones, $\overleftarrow{f}$ and $\overleftarrow{\overline{f}}$. The notation $\overleftarrow{f}$ $(\overrightarrow{f})$ means that the index of the superfield f is shifted by $+1$ (-1), i.e., $\overleftarrow{f_k} \equiv f_{k+1}$ $(\overrightarrow{f_k} \equiv f_{k-1})$, and it denotes the action of the direct (inverse) f-Toda transformation applied to the superfield f. The invariance of the $N = 2$ super-NLS hierarchy with respect to the transformation (2.4) can be checked directly, and it means, in particular, that the right-hand sides of eqs. (2.1) are the solutions of the symmetry equation[2] corresponding to the mapping (2.4) [1].

3 The component form and boundary conditions

The $N = 2$ superfield form (2.4) of the f-Toda mapping is not very suitable for actual calculations and we rewrite it in terms of the superfield components defined as

$$v_k = D\overline{f}_k|, \quad \overline{\psi}_k = D\overline{f}_k|, \quad u_k = \overline{D}f_k|, \quad \psi_k = \overline{D}f_k|, \tag{3.1}$$

where the index $k \in Z$ and $|$ means the $(\theta,\overline{\theta}) \to 0$ limit. In terms of these components, eq. (2.4) becomes the infinite-dimensional chain of equations

$$(\ln(u_{k+1}v_k))\,' = \psi_{k+1}\overline{\psi}_{k+1} - \psi_k\overline{\psi}_k, \tag{3.2}$$

$$-(\frac{\psi_{k+1}}{u_{k+1}}{}')\,' = -v_{k+1}\psi_{k+1} + v_k\psi_k, \tag{3.3}$$

[2]Let us recall that the symmetry equation for a given system can be obtained by differentiation of the system with respect to an arbitrary parameter.

$$-\left(\frac{\overline{\psi}_k{}'}{v_k}\right)' = u_{k+1}\overline{\psi}_{k+1} - u_k\overline{\psi}_k, \tag{3.4}$$

$$-(\ln v_k)'' = u_{k+1}v_{k+1} - u_kv_k - \psi_{k+1}{}'\,\overline{\psi}_{k+1} + \psi_k{}'\,\overline{\psi}_k. \tag{3.5}$$

Each point of this chain contains four functions: two bosonic u_k, v_k and two fermionic $\psi_k, \overline{\psi}_k$ functions. In the bosonic limit (i.e., as $\psi_k \to 0$ and $\overline{\psi}_k \to 0$) it is equivalent to the usual Toda chain[3] [5]. This is the reason why the chain (2.4) is called the fermionic Toda or f-Toda chain [1].

For completeness, let us also present the component form of the global $N = 2$ supertranslations

$$v_k \to v_k - \varepsilon\overline{\psi}_k{}' - \frac{1}{2}\varepsilon\overline{\varepsilon}v_k{}', \quad \overline{\psi}_k \to \overline{\psi}_k + \overline{\varepsilon}v_k + \frac{1}{2}\varepsilon\overline{\varepsilon}\overline{\psi}_k{}',$$
$$u_k \to u_k - \overline{\varepsilon}\psi_k{}' + \frac{1}{2}\varepsilon\overline{\varepsilon}u_k{}', \quad \psi_k \to \psi_k + \varepsilon u_k - \frac{1}{2}\varepsilon\overline{\varepsilon}\psi_k{}' \tag{3.6}$$

under which eqs. (3.2)–(3.5) remain invariant. Here, ε and $\overline{\varepsilon}$ are constant fermionic parameters. Equations (3.2)–(3.5) also possess the inner discrete automorphism σ_m ($m \in Z$) with the properties

$$\sigma_m u_k \sigma_m^{-1} = v_{m-k}, \quad \sigma_m v_k \sigma_m^{-1} = u_{m-k},$$
$$\sigma_m \psi_k \sigma_m^{-1} = \overline{\psi}_{m-k}, \quad \sigma_m \overline{\psi}_k \sigma_m^{-1} = \psi_{m-k}, \tag{3.7}$$

which will be useful in what follows. Eqs. (3.7) can also be easily represented in terms of the superfields and superspace coordinates

$$\sigma_m f_k \sigma_m^{-1} = \overline{f}_{m-k}, \quad \sigma_m \overline{f}_k \sigma_m^{-1} = f_{m-k},$$
$$\sigma_m z \sigma_m^{-1} = z, \quad \sigma_m \theta \sigma_m^{-1} = \overline{\theta}, \quad \sigma_m \overline{\theta} \sigma_m^{-1} = \theta. \tag{3.8}$$

Let us discuss the boundary conditions which can supplement eqs. (3.2)–(3.5).

We call the system (3.2)–(3.5) with additional manifest $N = 2$ supersymmetric boundary conditions

$$f_0 = 0 \Leftrightarrow u_0 = \psi_0 = 0, \quad k \geq 0, \tag{3.9}$$

or

$$\overline{f}_M = 0 \Leftrightarrow v_M = \overline{\psi}_M = 0, \quad k \leq M, \tag{3.10}$$

[3] In the bosonic limit, eq. (3.2) admits the obvious solution $u_{k+1} \sim \frac{1}{v_k}$ which, being substituted into eq. (3.5), produces the well-known representation for the usual Toda chain equations.

where $M \in Z_+$, the f-Toda chain interrupted from the left or from the right, respectively. The case where these conditions are satisfied simultaneously corresponds to the f-Toda chain with fixed ends.

Application of the transformation σ_M (3.7) to the f-Toda chain interrupted from the left transforms it into the f-Toda chain interrupted from the right, and vice versa. Due to their relation, it is sufficient to consider only one of them; for concreteness, we discuss the f-Toda chain interrupted from the left in what follows. As concerns the f-Toda chain with fixed ends, it is invariant with respect to σ_M, or, in other words, it possesses the inner automorphism σ_M.

4 The integrability of the f-Toda chain interrupted from the left

In this section, we construct the recurrent procedure for solving of the f-Toda chain equations (3.2)–(3.5) with the boundary conditions (3.9).

4.1 The conservation laws

Our first goal is to resolve the f-Toda chain interrupted from the left. In other words, we assume that the boundary condition (3.9) is satisfied and $v_0, \overline{\psi}_0$ are arbitrary functions. The problem consists in expressing, using the f-Toda chain equations, all of the functions u_k, v_k, ψ_k and $\overline{\psi}_k$ at $k \geq 1$ in terms of the functions v_0 and $\overline{\psi}_0$, as well as a number of additional constants which can arise in integrating the equations.

Let us start with a discussion of the conservation laws which are relevant to the problem under consideration.

By a direct check, one can verify that the following set of conservation laws:

$$c_{k-1} = u_k v_{k-1} - \frac{(v_{k-1}\psi_k)'}{v_{k-1}}\overline{\psi}_{k-1} + \psi_k\overline{\psi}_{k-1}{}' \tag{4.1}$$

takes place at each value of k, where c_k is an arbitrary constant. However, there is a simpler way to check this statement by rewriting (4.1) in the superfield form

$$c_{k-1} = \frac{D\overline{D}(f_k\overline{f}_{k-1}D\overline{f}_{k-1})}{D\overline{f}_{k-1}} = -\frac{\overline{D}D(f_k\overline{f}_{k-1}\overline{D}f_k)}{\overline{D}f_k} \tag{4.2}$$

using eq. (2.4) and definitions (3.1). The last equality of formula (4.2) demonstrates that c_k is a constant (i.e., it is the integral of motion), because it is both the chiral and antichiral superfield, i.e. $Dc_k = \overline{D}c_k = 0$. The action of the transformation σ_m (3.7) interchanges different integrals in accordance with the following law:

$$\sigma_m c_k \sigma_m^{-1} = c_{m-k-1}. \qquad (4.3)$$

Using the conservation law (4.1), eq. (3.3) for the fermionic function ψ_k can identically be rewritten as

$$\left(\frac{(v_{k-1}\psi_k)'}{v_{k-1}}\right)' - v_{k-1}\left[u_{k-1} - \left(\frac{\psi_{k-1}\overline{\psi}_{k-1}}{v_{k-1}}\right)'\right]\psi_k = -c_{k-1}\psi_{k-1}. \qquad (4.4)$$

From eq. (4.4), we observe that if we want to express ψ_k as a functional of the chain functions u_i, v_i, ψ_i and $\overline{\psi}_i$ defined at the previous (from the left) points of the chain (i.e., at $i \leq k-1$), it is necessary to solve a linear equation of the second order with the coefficient functions that also depend on the functions of the previous points of the chain. If we could find such an expression for ψ_k, all other independent functions $\overline{\psi}_k$, u_k and v_k of the f-Toda chain could also be obtained in a recurrent way as functionals of the previous points of the chain. Indeed, using eqs. (3.4), (3.5) and conservation law (4.1), one can easily obtain the corresponding formulae for the functions $\overline{\psi}_k$, v_k and u_k:

$$\begin{aligned}
\overline{\psi}_k =& \Big(-v_{k-1}\Big(\frac{\overline{\psi}_{k-1}{}'}{v_{k-1}}\Big)' + u_{k-1}\overline{\psi}_{k-1}\Big)\Big/\Big(c_{k-1} + \frac{(v_{k-1}\psi_k)'}{v_{k-1}}\overline{\psi}_{k-1} - \psi_k\overline{\psi}_{k-1}{}'\Big),\\
v_k =& v_{k-1}\big(-(\ln v_{k-1})'' + u_{k-1}v_{k-1} + \psi_k{}'\overline{\psi}_k\\
&-\psi_{k-1}{}'\overline{\psi}_{k-1}\big)\Big/\Big(c_{k-1} + \frac{(\psi_k v_{k-1})'}{v_{k-1}}\overline{\psi}_{k-1} - \psi_k\overline{\psi}_{k-1}{}'\Big),\\
u_k =& \Big(c_{k-1} + \frac{(\psi_k v_{k-1})'}{v_{k-1}}\overline{\psi}_{k-1} - \psi_k\overline{\psi}_{k-1}{}'\Big)\Big/v_{k-1},
\end{aligned} \qquad (4.5)$$

respectively.

Thus, the crucial problem is the solution of eq. (4.4) for ψ_k by some recurrent procedure. This task is solved in the next subsection.

4.2 The integrable factors

The aim of this subsection is to construct the recurrent procedure for solving eq. (4.4) for an arbitrary chain point.

Let us assume that the homogeneous part of the second order equation (4.4) possesses a bosonic integrable factor μ_k. This means that after multiplication by μ_k, the corresponding equation may be represented as a whole derivative. Following this line, let us rewrite eqs. (4.4) in the following equivalent form:

$$\left(\frac{\mu_k{}^2}{v_{k-1}}\left(\frac{v_{k-1}\psi_k}{\mu_k}\right)'\right)' - v_{k-1}\left\{-\left(\frac{\mu_k{}'}{v_{k-1}}\right)' + \mu_k\left[u_{k-1} - \left(\frac{\psi_{k-1}\overline{\psi}_{k-1}}{v_{k-1}}\right)'\right]\right\}\psi_k$$
$$= -c_{k-1}\mu_k\psi_{k-1}. \qquad (4.6)$$

Equating the coefficient of the function ψ_k in the l.h.s. of eq.(4.6) to zero, we obtain the following equation:

$$\left(\frac{\mu_k{}'}{v_{k-1}}\right)' = \left[u_{k-1} - \left(\frac{\psi_{k-1}\overline{\psi}_{k-1}}{v_{k-1}}\right)'\right]\mu_k \qquad (4.7)$$

for the integrable factor μ_k. At $k=1$, it has an obvious solution $\mu_1 = 1$ because, in this case, its r.h.s. becomes equal to zero due to the boundary conditions (3.9).

If the integrable factors are restricted by eqs. (4.7), eq. (4.6) for the function ψ_k becomes

$$\left(\frac{\mu_k{}^2}{v_{k-1}}\left(\frac{v_{k-1}\psi_k}{\mu_k}\right)'\right)' = -c_{k-1}\mu_k\psi_{k-1} \qquad (4.8)$$

and one can easily integrate it,

$$\psi_k = \frac{\mu_k}{v_{k-1}}\left[-c_{k-1}\int dz\frac{v_{k-1}}{\mu_k{}^2}\int dz\mu_k\psi_{k-1} + \beta_k\int dz\frac{v_{k-1}}{\mu_k{}^2} + \alpha_k\right], \qquad (4.9)$$

where α_k and β_k are fermionic constants of integration, and one can find the explicit expression for ψ_k if all other parts of formula (4.9) are known. The appearance of these additional constants is a qualitative difference of the f-Toda chain in comparison with the integrable chains investigated before.

Concerning eqs. (4.7) for the factors μ_k, they can be resolved in recurrent form with the following answer:

$$\mu_{k+1} = v_{k-1}\left(\frac{\mu_k}{v_{k-1}}\right)' - \frac{\mu_k{}^2}{c_{k-1}}\left(\frac{v_{k-1}\psi_k}{\mu_k}\right)'\left(\frac{\overline{\psi}_{k-1}}{v_{k-1}}\right)', \quad \mu_1 = 1. \qquad (4.10)$$

Indeed, this can be checked by direct substitution, taking into account the f-Toda chain equations (3.2)-(3.5). If the integrable factor μ_k satisfies eq.(4.7), the factor μ_{k+1} (4.10) satisfies the equation

$$(\frac{\mu_{k+1}}{v_k}{}')' = [u_k - (\frac{\psi_k\overline{\psi}_k}{v_k})'\,]\mu_{k+1}. \tag{4.11}$$

This verification can be simplified if one uses the following convenient identity:

$$(\frac{1}{v_k}(v_{k-1}(\frac{\overline{\psi}_{k-1}}{v_{k-1}})')')'$$
$$= [u_k - (\frac{\psi_k\overline{\psi}_k}{v_k})'\,]\, v_{k-1}(\frac{\overline{\psi}_{k-1}}{v_{k-1}})' - c_{k-1}(\frac{\overline{\psi}_k}{v_k})', \tag{4.12}$$

which can easily be checked.

Let us stress that the integrable factor μ_{k+1} (4.10) is the functional of the chain functions defined at the previous (from the left) points of the chain, thus, the same important property is also satisfied for the function ψ_k (4.9).

To consider the case of the f-Toda chain interrupted from the right with the boundary conditions (3.10), one can simply apply the transformation σ_M (3.7) to formulae (4.4)-(4.12). Without additional comments, let us present expressions for $\overline{\psi}_k$ and their integrable factors $\overline{\mu}_k$

$$\overline{\psi}_k = \frac{\overline{\mu}_k}{u_{k+1}}[-c_k\int dz\frac{u_{k+1}}{\overline{\mu}_k{}^2}\int dz\overline{\mu}_k\overline{\psi}_{k+1} + \overline{\beta}_k\int dz\frac{u_{k+1}}{\overline{\mu}_k{}^2} + \overline{\alpha}_k], \tag{4.13}$$

$$\overline{\mu}_{k-1} = u_{k+1}(\frac{\overline{\mu}_k}{u_{k+1}})' + \frac{\overline{\mu}_k{}^2}{c_k}(\frac{\psi_{k+1}}{u_{k+1}})'(\frac{u_{k+1}\overline{\psi}_k}{\overline{\mu}_k})', \quad \overline{\mu}_{M-1} = 1, \tag{4.14}$$

where $\overline{\alpha}_k$ and $\overline{\beta}_k$ are fermionic constants, which will be useful in what follows.

Thus, the following proposition summarizes this section.

The solution of the f-Toda chain interrupted from the left is given by recurrent relations (4.9), (4.10) and (4.5).

5 The integrability of the f-Toda chain with fixed ends

In this section, we construct the recurrent procedure for solving the f-Toda chain with the boundary conditions (3.9), (3.10).

5.1 New form of the f-Toda chain

Let us introduce the new basis q_k, r_k, ξ_k and $\overline{\xi}_k$ in the space of the f-Toda chain functions u_k, v_k, ψ_k and $\overline{\psi}_k$, defined by the following invertible transformation:

$$q_k = \prod_{s=1}^{k}(u_s v_{s-1}), \quad r_0 = v_0, \quad r_k = v_k\prod_{s=1}^{k}(u_s v_{s-1}),$$
$$\overline{\xi}_0 = \overline{\psi}_0, \quad \overline{\xi}_k = \overline{\psi}_k\prod_{s=1}^{k}(u_s v_{s-1}), \quad \xi_k = \psi_k/\prod_{s=1}^{k}(u_s v_{s-1}), \tag{5.1}$$

$$u_k = \frac{q_k}{r_{k-1}}, \quad v_0 = r_0, \quad v_k = \frac{r_k}{q_k}, \quad \overline{\psi}_0 = \overline{\xi}_0, \quad \overline{\psi}_k = \frac{\overline{\xi}_k}{q_k}, \quad \psi_k = \xi_k q_k, \tag{5.2}$$

where $k = 1, 2, ..., M$. In this case, the boundary conditions (3.9) and (3.10) are transformed into the following conditions:

$$q_0 = \xi_0 = 0, \tag{5.3}$$

$$r_M = \overline{\xi}_M = 0, \tag{5.4}$$

respectively. In the new basis, the f-Toda chain equations (3.2)-(3.5) have the following form:

$$(\ln q_j)' = \xi_j\overline{\xi}_j, \quad (\ln q_M)' = 0, \tag{5.5}$$

$$-(r_{M-1}\xi_M{}')' = r_{M-1}\xi_{M-1}, \quad -(\frac{\overline{\xi}_0{}'}{r_0})' = \frac{\overline{\xi}_1}{r_0}, \tag{5.6}$$

$$-(r_{j-1}\xi_j{}')' = -r_j\xi_j + r_{j-1}\xi_{j-1}, \tag{5.7}$$

$$-\left(\frac{\overline{\xi}_j\,'}{r_j}\right)' = \frac{\overline{\xi}_{j+1}}{r_j} - \frac{\overline{\xi}_j}{r_{j-1}}, \qquad (5.8)$$

$$-(\ln r_0)'' = \frac{r_1}{r_0} - \xi_1\,'\,\overline{\xi}_1,$$
$$-(\ln r_j)'' = \frac{r_{j+1}}{r_j} - \frac{r_j}{r_{j-1}} - \xi_{j+1}\,'\,\overline{\xi}_{j+1} - \xi_j\overline{\xi}_j\,'. \qquad (5.9)$$

Hereafter, the index j lies in the following range $1 \leq j \leq M-1$.

The substitution of the transformations (5.1), (5.2) into (3.7) at $m = M$ gives the inner automorphism σ_M,

$$\sigma_M q_k \sigma_M^{-1} = \frac{q_M}{q_{M-k}}, \quad \sigma_M r_k \sigma_M^{-1} = \frac{q_M}{r_{M-k-1}},$$
$$\sigma_M \xi_k \sigma_M^{-1} = \frac{\overline{\xi}_{M-k}}{q_M}, \quad \sigma_M \overline{\xi}_k \sigma_M^{-1} = q_M \xi_{M-k}, \qquad (5.10)$$

for eqs. (5.5)–(5.9).

It is interesting to note that the functions q_k, ξ_M and $\overline{\xi}_0$ are completely decoupled from eqs. (5.7), (5.8) and (5.9), which form a closed set of equations for the functions r_0, r_j ξ_j and $\overline{\xi}_j$. Moreover, taking into account the corollary from eq. (5.5) that q_M is an arbitrary constant, one can conclude that the transformations (5.10) are also closed for them. However, this is not the case for the $N=2$ supersymmetry transformations (3.6), which, for eqs. (5.5)–(5.9), have the following infinitesimal form:

$$\delta q_k = -\prod_{s=1}^{k}(\overline{\varepsilon}\xi_s\,' r_{s-1} + \varepsilon\frac{\overline{\xi}_s\,'}{r_{s-1}})\frac{q_s}{q_{s-1}},$$
$$\delta r_k = -\varepsilon\overline{\xi}_k\,' + \frac{q_k\delta q_k}{r_{k-1}}, \quad \delta\overline{\xi}_k = \overline{\varepsilon} r_k + \frac{\overline{\xi}_k\delta q_k}{q_k}, \quad \delta\xi_k = \frac{\varepsilon}{r_{k-1}} - \frac{\xi_k\delta q_k}{q_k}, \qquad (5.11)$$

where eqs. (5.5) have been used.

We call eqs. (5.7)–(5.9) the restricted f-Toda chain. If some of their explicit solutions are known, one can also easily obtain the solutions of eqs. (5.5)–(5.6) for q_j, ξ_M and $\overline{\xi}_0$,

$$q_{j+1} = c_j q_j/(1 - \frac{(r_j\xi_{j+1})\,'}{r_j}\overline{\xi}_j + \xi_{j+1}\overline{\xi}_j\,'),$$
$$q_1 = c_0/(1 - \frac{(r_0\xi_1)\,'}{r_0}\overline{\xi}_0 + \xi_1\overline{\xi}_0\,'), \qquad (5.12)$$

$$\xi_M = -\int dz\frac{1}{r_{M-1}}\int dz r_{M-1}\xi_{M-1} + \beta_M\int dz\frac{1}{r_{M-1}} + \alpha_M,$$
$$\overline{\xi}_0 = -\int dz r_0\int dz\frac{\overline{\xi}_1}{r_0} + \overline{\beta}_0\int dz r_0 + \overline{\alpha}_0, \qquad (5.13)$$

where $\alpha_M, \overline{\alpha}_0$ and $\beta_M, \overline{\beta}_0$ are the fermionic constants of integration and we have used the conservation laws (4.1) expressed in terms of the new functions q_k, r_k, ξ_k and $\overline{\xi}_k$ (5.1), (5.2).

Let us introduce the new bosonic function x_j

$$r_0 = \exp(x_0), \quad r_j = \exp(x_j), \quad \exp(-x_{-1}) = \exp(x_M) = 0, \qquad (5.14)$$

which is usually used in the case of the bosonic Toda chain and is more suitable for the system under consideration. In this case, the restricted f-Toda chain equations (5.7)–(5.9) and their inner automorphism σ_M become

$$\xi_j\,'' + x_{j-1}\,'\,\xi_j\,' - \xi_j\exp(x_j - x_{j-1}) = -\xi_{j-1}, \qquad (5.15)$$

$$\overline{\xi}_j\,'' - x_j\,'\,\overline{\xi}_j\,' - \overline{\xi}_j\exp(x_j - x_{j-1}) = -\overline{\xi}_{j+1}, \qquad (5.16)$$

$$x_0\,'' + \exp(x_1 - x_0) - \xi_1\,'\,\overline{\xi}_1 = 0,$$
$$x_j\,'' + \exp(x_{j+1} - x_j) - \exp(x_j - x_{j-1}) - \xi_{j+1}\,'\,\overline{\xi}_{j+1} - \xi_j\overline{\xi}_j\,' = 0, \qquad (5.17)$$

$$\sigma_M x_0 \sigma_M^{-1} = -x_{M-1}, \quad \sigma_M x_j \sigma_M^{-1} = -x_{M-j-1},$$
$$\sigma_M \xi_j \sigma_M^{-1} = \overline{\xi}_{M-j}, \quad \sigma_M \overline{\xi}_j \sigma_M^{-1} = \xi_{M-j}, \qquad (5.18)$$

respectively. In what follows, we concentrate on the analysis of these equations and prove their integrability.

5.2 The general assertion

In this subsection, we briefly discuss the general problem of integrating a system of ordinary differential equations containing both unknown fermionic and bosonic functions.

Althought this is not crucial for the general discussion, for definiteness, we assume that the system consists of the second order differential equations for M bosonic and $2(M-1)$ fermionic independent functions, as takes place for eqs. (5.15)–(5.17). In this case, its general exact solution must include $2M$ bosonic and $4(M-1)$ fermionic independent constants, of course, if the system is integrable.

Let us analyze the qualitative structure of this exact solution in more detail.

Evidently, the exact solution is a polynomial with respect to fermionic constants for any function involved in the system. It is well known that the bosonic (fermionic) functions are even (odd) polynomials and any fermionic constant enters a monomial representing the product of fermionic components only once[4].

It is instructive to treat this exact solution as the result of some iteration procedure or as a result of calculations in the framework of some perturbation theory, where the role of perturbation parameters is played by the fermionic constants, and different orders of perturbation calculations are in one-to-one correspondence with different degrees of the monomials, i.e., with the different numbers of fermionic components composing them.

Now we will analyze the qualitative structure of different orders of perturbation calculations in the framework of such a perturbation theory.

The zero-order approximation to the exact solution corresponds to the general solution of a pure bosonic system, which can be derived by taking the bosonic limit of the initial system. All of the bosonic integration constants parametrising the exact solution of the initial system must arise in this order of perturbative theory.

The first order approximation corresponds to the general solution of a linear fermionic system, which can be derived by substituting the zero-order approximation for bosonic functions into the initial system linearized with respect to fermionic functions. Similarly to the case of zero-order approximation, all fermionic integration constants parametrising the exact solution of the initial system must arise in this order of perturbative theory.

The second-order correction corresponds to the solution of a linear bosonic system with the following structure: its homogeneous part is the symmetry equation[2] of a zero-order bosonic system and its inhomogeneous part is what remains after the substitution of the zero and first orders of perturbative calculations into the initial system and rejection of the monomials of all degrees but monomials of the second degree.

[4]Let us recall that each polynomial is the sum of monomials composed of the products of different fermionic components and, by definition, its degree is the number of components of its maximum monomial.

The homogeneous part of a fermionic system, corresponding to the third order-correction to the exact solution, coincides with the fermionic system of the first order-approximation and its inhomogeneous part is given by perturbative decomposition of the initial system using the functions of the first and second orders of the perturbative calculations, and so on.

To close this general discussion, let us stress that the homogeneous part of a system corresponding to any even (odd) order-correction to the exact solution coincides with the system of the second (first) order-approximation and neither new bosonic nor fermionic independent parameters appear in any order starting with the second order of the perturbation calculations[5]. We would also like to emphasize that by construction, such a perturbation theory is convergent and it gives the exact solution: the perturbation series is interrupted due to the fermionic nature of the perturbative parameters because a limited number of monomials can be composed using a finite set of fermionic constants (i.e., the solutions of the first-order approximation).

Now let us formulate the assertion with respect to the initial system:

if the equations of the zero and first orders of the perturbation theory are exactly integrable, the initial system is also integrable, at least in the quadratures.

The proof is given in a few words.

If the equations of the zero (first) order of the perturbative theory are exactly integrable, the corresponding symmetry equation is also exactly integrable. Indeed, one can derive its $2M$ ($4(M-1)$) linear-independent solutions by taking the derivatives of the general solution of the zero (first) order equations with respect to its $2M$ ($4(M-1)$) bosonic (fermionic) independent constants. As follows from the previous discussion, the problem of resolving equations corresponding to any other order of the perturbative calculations reduces to the problem of resolving the inhomogeneous symmetry equation, but the last problem is an exactly solvable one. Indeed, if the general solution of some linear homogeneous system is known, like in the case under consideration, then, by applying the well-known method of varying the arbitrary constants, one can algorithmically construct the solution of the corresponding inhomogeneous equation, at least, in the quadratures. Thus, all the orders of the above-discussed perturbation cal-

[5]Clearly, having arisen, they can always be eliminated by redefining the parameters of the zero and first orders of the perturbation calculations.

culations can be resolved in explicit form.

5.3 Solution of the restricted f-Toda chain

The purpose of this subsection is to apply the above-developed regular algorithm of integrating a system of ordinary differential equations containing both unknown fermionic and bosonic functions to the restricted f-Toda chain equations (5.15)-(5.17).

Following the line of section 5.2, let us represent the functions x_0, x_j, ξ_j and $\overline{\xi}_j$ as a perturbation series,

$$x_0 = \sum_{l=0}^{2(M-1)} x_0^{(2l)}, \quad x_j = \sum_{l=0}^{2(M-1)} x_j^{(2l)},$$
$$\xi_j = \sum_{l=1}^{2(M-1)} \xi_j^{(2l-1)}, \quad \overline{\xi}_j = \sum_{l=1}^{2(M-1)} \overline{\xi}_j^{(2l-1)}, \tag{5.19}$$

where the perturbation parameters coincide with the $4(M-1)$ fermionic constants of the first-order approximation to the general solution. Here, the functions $x_0^{(2l)}$ and $x_j^{(2l)}$ ($\xi_j^{(2l+1)}$ and $\overline{\xi}_j^{(2l+1)}$) are the $2l$ $(2l+1)$ order corrections to the zero (first) order approximation. The fermionic character of the decomposition parameters guarantees that this series is interrupted, starting with the $4(M-1)+1$ order. As result, such a perturbation theory is convergent and gives an exact result.

Substituting decompositions (5.19) into eqs. (5.15)-(5.17), extracting terms of the same order, and equating their algebraic sum to zero independently for different orders, one can obtain the complete set of perturbative equations corresponding to the presented perturbation theory.

Here, we demonstrate that the conditions of the general assertion in the previous subsection is satisfied for the restricted f-Toda chain, i.e., it is exactly integrable.

The zero order bosonic system coincides with the usual one-dimensional Toda chain [5]

$$x_0^{(0)\,\prime\prime} + \exp(x_1^{(0)} - x_0^{(0)}) = 0,$$
$$x_j^{(0)\,\prime\prime} + \exp(x_{j+1}^{(0)} - x_j^{(0)}) - \exp(x_j^{(0)} - x_{j-1}^{(0)}) = 0, \tag{5.20}$$

which is exactly integrable. Its general solution is well known and can be

represented in the following form [6]:

$$r_0^{(0)} = \exp(x_0^{(0)}) = \sum_{i=1}^{M} a_i \exp(b_i x), \quad \exp(x_j^{(0)}) = (-1)^j \frac{Det_{j+1}}{Det_j}, \tag{5.21}$$

where a_i and b_i are arbitrary constants, and Det_j is the jth principal minor of the matrix

$$\begin{pmatrix} r_0^{(0)} & r_0^{(0)\,\prime} & r_0^{(0)\,\prime\prime} & \dots \\ r_0^{(0)\,\prime} & r_0^{(0)\,\prime\prime} & r_0^{(0)\,\prime\prime\prime} & \dots \\ r_0^{(0)\,\prime\prime} & r_0^{(0)\,\prime\prime\prime} & r_0^{(0)\,\prime\prime\prime\prime} & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}. \tag{5.22}$$

The first order fermionic systems have the following form:

$$\xi_j^{(1)\,\prime\prime} + x_{j-1}^{(0)\,\prime}\, \xi_j^{(1)\,\prime} - \xi_j^{(1)} \exp(x_j^{(0)} - x_{j-1}^{(0)}) + \xi_{j-1}^{(1)} = 0, \tag{5.23}$$

$$\overline{\xi}_j^{(1)\,\prime\prime} - x_j^{(0)\,\prime}\, \overline{\xi}_j^{(1)\,\prime} - \overline{\xi}_j^{(1)} \exp(x_j^{(0)} - x_{j-1}^{(0)}) + \overline{\xi}_{j+1}^{(1)} = 0. \tag{5.24}$$

These equations are also exactly integrable and using the results of section 4.2, one can easily generate their general solutions. Thus, substituting transformations (5.2), (5.14), as well as solutions (5.12), into relations (4.9), (4.10), (4.13), and (4.14), and rejecting terms nonlinear with respect to the fermionic fields, we get the following recurrent formulae[6]:

$$\xi_j^{(1)} = \exp(-x_{j-1}^{(0)})\mu_j^{(0)}[-\int dz \frac{\exp(x_{j-1}^{(0)})}{\mu_j^{(0)2}} \int dz \mu_j^{(0)} \xi_{j-1}^{(1)}$$
$$+\beta_j \int dz \frac{\exp(x_{j-1}^{(0)})}{\mu_j^{(0)2}} + \alpha_j],$$
$$\mu_{j+1}^{(0)} = \mu_j^{(0)\,\prime} - x_{j-1}^{(0)\,\prime}\, \mu_j^{(0)}, \quad \mu_1^{(0)} = 1, \tag{5.25}$$

$$\overline{\xi}_j^{(1)} = \exp(x_j^{(0)})\overline{\mu}_j^{(0)}[-\int dz \frac{\exp(-x_j^{(0)})}{\overline{\mu}_j^{(0)2}} \int dz \overline{\mu}_j^{(0)}\, \overline{\xi}_{j+1}^{(1)}$$
$$+\overline{\beta}_j \int dz \frac{\exp(-x_j^{(0)})}{\overline{\mu}_j^{(0)2}} + \overline{\alpha}_j],$$
$$\overline{\mu}_{j-1}^{(0)} = \mu_j^{(0)\,\prime} + x_j^{(0)\,\prime}\, \overline{\mu}_j^{(0)}, \quad \overline{\mu}_{M-1}^{(0)} = 1 \tag{5.26}$$

[6]Here, we have rescaled the fermionic constants of expressions (4.9) and (4.13).

for the exact solutions of eqs. (5.23)–(5.24). Thus, following the general assertion of section 5.2, we conclude that the restricted f-Toda chain is also integrable.

Now we show how to construct the solutions of the perturbation equations for all other orders of the perturbation theory with respect to the $4(M-1)$ fermionic constants α_j, β_j, $\overline{\alpha}_j$, and $\overline{\beta}_j$ of the first order solutions (5.25), (5.26).

For the perturbation equations of the $2l$ $(l \geq 1)$ and $2l-1$ $(l \geq 2)$ orders, we have

$$\begin{aligned} &x_0^{(2l)\,\prime\prime} + (x_1^{(2l)} - x_0^{(2l)})\exp(x_1^{(0)} - x_0^{(0)}) = X_0^{(2l)}, \\ &x_j^{(2l)\,\prime\prime} + (x_{j+1}^{(2l)} - x_j^{(2l)})\exp(x_{j+1}^{(0)} - x_j^{(0)}) \\ &-(x_j^{(2l)} - x_{j-1}^{(2l)})\exp(x_j^{(0)} - x_{j-1}^{(0)}) = X_j^{(2l)}, \end{aligned} \quad (5.27)$$

$$\begin{aligned} &\xi_j^{(2l-1)\,\prime\prime} + x_{j-1}^{(0)\,\prime}\,\xi_j^{(2l-1)\,\prime} \\ &-\xi_j^{(2l-1)}\exp(x_j^{(0)} - x_{j-1}^{(0)}) + \xi_{j-1}^{(2l-1)} = \Xi_j^{(2l-1)}, \end{aligned} \quad (5.28)$$

$$\begin{aligned} &\overline{\xi}_j^{(2l-1)\,\prime\prime} - x_j^{(0)\,\prime}\,\overline{\xi}_j^{(2l-1)\,\prime} \\ &-\overline{\xi}_j^{(2l-1)}\exp(x_j^{(0)} - x_{j-1}^{(0)}) + \overline{\xi}_{j+1}^{(2l-1)} = \overline{\Xi}_j^{(2l-1)}, \end{aligned} \quad (5.29)$$

respectively. Here, the functions $X_0^{(2l)}$ and $X_j^{(2l)}$ ($\Xi_j^{(2l-1)}$ and $\overline{\Xi}_j^{(2l-1)}$) are defined by the following relations:

$$\begin{aligned} X_0 &\equiv \sum_{l=1}^{2(M-1)} X_0^{(2l)} = -\exp(x_1 - x_0) + \xi_1{}'\,\overline{\xi}_1 \\ &+(1 + x_1 - x_0 - x_1^{(0)} + x_0^{(0)})\exp(x_1^{(0)} - x_0^{(0)}), \\ X_j &\equiv \sum_{l=1}^{2(M-1)} X_j^{(2l)} = -\exp(x_{j+1} - x_j) + \exp(x_j - x_{j-1}) \\ &+\xi_{j+1}{}'\,\overline{\xi}_{j+1} + \xi_j\,\overline{\xi}_j{}' \\ &+(1 + x_{j+1} - x_j - x_{j+1}^{(0)} + x_j^{(0)})\exp(x_{j+1}^{(0)} - x_j^{(0)}) \\ &-(1 + x_j - x_{j-1} - x_j^{(0)} + x_{j-1}^{(0)})\exp(x_j^{(0)} - x_{j-1}^{(0)}), \\ \Xi_j &\equiv \sum_{l=2}^{2(M-1)} \Xi_j^{(2l-1)} = (x_{j-1}^{(0)} - x_{j-1})\,'\,\xi_j{}' \\ &+\xi_j(\exp(x_j - x_{j-1}) - \exp(x_j^{(0)} - x_{j-1}^{(0)})), \\ \overline{\Xi}_j &\equiv \sum_{l=2}^{2(M-1)} \overline{\Xi}_j^{(2l-1)} = -(x_j^{(0)} - x_j)\,'\,\overline{\xi}_j{}' \\ &+\overline{\xi}_j(\exp(x_j - x_{j-1}) - \exp(x_j^{(0)} - x_{j-1}^{(0)})), \end{aligned} \quad (5.30)$$

respectively, and they are the sum of the monomials of $2l$ $(2l-1)$ degree[7]. Let us stress that the inhomogeneous parts of perturbative equations (5.27)-(5.29), corresponding to a given order n, depend only on the functions $x_0^{(2m)}$, $x_j^{(2m)}$, $\xi_j^{(2l+1)}$ and $\overline{\xi}_j^{(2l+1)}$ of the previous orders of perturbative calculations (i.e., at $m \leq (n-1)/2$ and $l \leq (n-2)/2$). Thus, we have a consistent perturbative theory.

If we set the inhomogeneous parts of perturbative equations (5.27) equal to zero for any even order of the perturbative theory, they coincide with the symmetry equations[2] for the Toda chain (5.20). Thus, in this case, their $2M$ linear-independent solutions y_k^{Λ} at each point k $(k = 0, 1, ..., M-1)$ of the chain are given by

$$y_k^{\Lambda} = \frac{\partial x_k^{(0)}}{\partial A_{\Lambda}}, \quad (5.31)$$

where the capital Greek letter-indices run over the range $\Lambda, \Phi = 0, 1, ..., 2M-1$, and $A_{\Lambda} \equiv \{a_1, ..., a_M, b_1, ..., b_M\}$ (see eq. (5.21)). To use the formalism of varied constants for the solution of inhomogeneous equations (5.27), it is necessary to solve the linear system of the first order differential equations

$$\sum_{\Lambda=0}^{2M-1} y_k^{\Lambda} c_{\Lambda}^{(2l)}(z)\,' = 0, \quad \sum_{\Lambda=0}^{2M-1} y_k^{\Lambda\,\prime}\, c_{\Lambda}^{(2l)}(z)\,' = X_k^{(2l)} \quad (5.32)$$

with respect to $c_{\Lambda}^{(2l)}(z)$, where $c_{\Lambda}^{(2l)}(z)$ are $2M$ bosonic parameter-functions. Then, the solution of eqs. (5.27) for the function $x_k^{(2l)}$ has the following form:

$$x_k^{(2l)} = \sum_{\Lambda=0}^{2M-1} y_k^{\Lambda} c_{\Lambda}^{(2l)}(z). \quad (5.33)$$

---

[7] It is implied that the perturbative decompositions (5.19) of the functions x_0, x_j, ξ_j and $\overline{\xi}_j$ must be substituted into the r.h.s. of relations (5.30).

Let us introduce the $2M \times 2M$ matrix $\mathcal{P}_\Phi^\Lambda$ defined as

$$\mathcal{P}_k^\Lambda = y_k^{\Lambda\,\prime}, \quad \mathcal{P}_{M+k}^\Lambda = y_k^\Lambda. \tag{5.34}$$

Then the solution of the system (5.32) can be represented in the form

$$c_\Lambda^{(2l)}(z) = \int dz \sum_{m=0}^{M-1} (\mathcal{P}^{-1})_\Lambda^m X_m^{(2l)}, \tag{5.35}$$

where $\mathcal{P}^{-1}$ is the inverse matrix for the matrix $\mathcal{P}$ ($\mathcal{P}^{-1}\mathcal{P} = \mathcal{P}\mathcal{P}^{-1} = I$) and expression (5.33) for $x_k^{(2l)}$ becomes

$$x_k^{(2l)} = \sum_{\Lambda=0}^{2M-1} \mathcal{P}_{M+k}^\Lambda \int dz \sum_{m=0}^{M-1} (\mathcal{P}^{-1})_\Lambda^m X_m^{(2l)}. \tag{5.36}$$

Taking the sum over all orders of the perturbative theory and using relations (5.19) and (5.30), we have the following exact expression for x_k:

$$x_k = x_k^{(0)} + \sum_{\Lambda=0}^{2M-1} \mathcal{P}_{M+k}^\Lambda \int dz \sum_{m=0}^{M-1} (\mathcal{P}^{-1})_\Lambda^m X_m. \tag{5.37}$$

One can invert the above-developed perturbation scheme and consider the relation (5.37) as the equation for x_k, and take it as a starting point. Then its iteration (5.19), together with the iteration of the corresponding equations for the fermionic functions ξ_j and $\overline{\xi}_j$ (see below), is interrupted starting with the $4(M-1)+1$ order and gives the exact solution for x_k.

The odd-order perturbative equations (5.28), (5.29) can be integrated in the same way as described for the case of the even-order equations (5.27). Without going into detail, let us present only the expressions for the functions $\xi_j^{(2l-1)}$ and $\overline{\xi}_j^{(2l-1)}$,

$$\xi_j^{(2l-1)} = \sum_{\Omega=1}^{2(M-1)} \frac{\partial \xi_j^{(1)}}{\partial \Gamma_\Omega} c_\Omega^{(2l-1)}(z), \quad \overline{\xi}_j^{(2l-1)} = \sum_{\Omega=1}^{2(M-1)} \frac{\partial \overline{\xi}_j^{(1)}}{\partial \overline{\Gamma}_\Omega} \overline{c}_\Omega^{(2l-1)}(z), \tag{5.38}$$

where $\Gamma_\Omega = (\alpha_1, ..., \alpha_{M-1}, \beta_1, ..., \beta_{M-1})$, $\overline{\Gamma}_\Omega = (\overline{\alpha}_1, ..., \overline{\alpha}_{M-1}, \overline{\beta}_1, ..., \overline{\beta}_{M-1})$ ($\Omega = 1, 2, ..., 2(M-1)$); $c_\Omega^{(2l-1)}(z)$ and $\overline{c}_\Omega^{(2l-1)}(z)$ are $4(M-1)$ fermionic parameter-functions that are solutions of the following linear system:

$$\sum_{\Omega=1}^{2(M-1)} \frac{\partial \xi_j^{(1)}}{\partial \Gamma_\Omega} c_\Omega^{(2l-1)}(z)\,' = 0, \quad \sum_{\Omega=1}^{2(M-1)} (\frac{\partial \xi_j^{(1)}}{\partial \Gamma_\Omega})\,'\, c_\Omega^{(2l-1)}(z)\,' = \Xi_j^{(2l-1)},$$
$$\sum_{\Omega=1}^{2(M-1)} \frac{\partial \overline{\xi}_j^{(1)}}{\partial \overline{\Gamma}_\Omega} \overline{c}_\Omega^{(2l-1)}(z)\,' = 0, \quad \sum_{\Omega=1}^{2(M-1)} (\frac{\partial \overline{\xi}_j^{(1)}}{\partial \overline{\Gamma}_\Omega})\,'\, \overline{c}_\Omega^{(2l-1)}(z)\,' = \overline{\Xi}_j^{(2l-1)}. \tag{5.39}$$

To close this subsection, we would like briefly discuss a slightly modified version of the perturbative theory for the odd-order perturbative equations (5.28) and (5.29).

To do this, let us represent the differential equations (5.28) and (5.29) in integral form. Thus, multiplying them by the integrable factors $\mu_j^{(0)}$ and $\overline{\mu}_j^{(0)}$, respectively, using the equations

$$(\exp(-x_{j-1}^{(0)})\, \mu_j^{(0)\,\prime}\,)\,' = \exp(-x_{j-2}^{(0)})\, \mu_j^{(0)},$$
$$(\exp(x_j^{(0)})\, \overline{\mu}_j^{(0)\,\prime}\,)\,' = \exp(x_{j-1}^{(0)})\, \overline{\mu}_j^{(0)} \tag{5.40}$$

for the factors, taking the sum of all perturbative orders, and using relations (5.30), eqs. (5.28)–(5.29) can be identically rewritten in the form

$$(\exp(-x_{j-1}^{(0)})\, \mu_j^{(0)^2}\, (\frac{\exp(x_{j-1}^{(0)})}{\mu_j^{(0)}} \xi_j)\,'\,)\,' = \mu_j^{(0)}(-\xi_{j-1} + \Xi_j),$$
$$(\exp(x_j^{(0)})\, \overline{\mu}_j^{(0)^2}\, (\frac{\exp(-x_j^{(0)})}{\overline{\mu}_j^{(0)}} \overline{\xi}_j)\,'\,)\,' = \overline{\mu}_j^{(0)}(-\overline{\xi}_{j+1} + \overline{\Xi}_j) \tag{5.41}$$

which can be easily integrated. As a result, we have the desirable integral form of eqs. (5.28)–(5.29) given by

$$\xi_j = \exp(-x_{j-1}^{(0)})\mu_j^{(0)}[\int dz \frac{\exp(x_{j-1}^{(0)})}{\mu_j^{(0)^2}} \int dz \mu_j^{(0)}(-\xi_{j-1} + \Xi_j)$$
$$+\beta_j \int dz \frac{\exp(x_{j-1}^{(0)})}{\mu_j^{(0)^2}} + \alpha_j],$$
$$\overline{\xi}_j = \exp(x_j^{(0)})\overline{\mu}_j^{(0)}[\int dz \frac{\exp(-x_j^{(0)})}{\overline{\mu}_j^{(0)^2}} \int dz \overline{\mu}_j^{(0)}(-\overline{\xi}_{j+1} + \overline{\Xi}_j)$$
$$+\overline{\beta}_j \int dz \frac{\exp(-x_j^{(0)})}{\overline{\mu}_j^{(0)^2}} + \overline{\alpha}_j]. \tag{5.42}$$

Simple inspection of eqs. (5.42) shows that their solutions can be consistently obtained by iterating with respect to the fermionic constants of integration α_j, β_j, $\overline{\alpha}_j$ and $\overline{\beta}_j$ in the framework of the above-discussed perturbative scheme.

Thus, all the orders of the perturbation calculations can be resolved in explicit form.

5.4 Example: the $M = 2$ case

To illustrate the general formulae of the previous subsection, here we consider the simplest example of the restricted f-Toda chain with fixed ends at $M = 2$.

In this case, eqs. (5.15)-(5.17), boundary conditions (5.3), (5.4), (5.14), as well as perturbative decompositions (5.19), have the following form:

$$\begin{aligned}
&\xi_1 '' + x_0 ' \xi_1 ' - \xi_1 \exp(x_1 - x_0) = 0, \\
&\overline{\xi}_1 '' - x_1 ' \overline{\xi}_1 ' - \overline{\xi}_1 \exp(x_1 - x_0) = 0, \\
&x_0 '' + \exp(x_1 - x_0) - \xi_1 ' \overline{\xi}_1 = 0, \\
&x_1 '' - \exp(x_1 - x_0) - \xi_1 \overline{\xi}_1 ' = 0;
\end{aligned} \qquad (5.43)$$

$$\xi_0 = \overline{\xi}_2 = \exp(-x_{-1}) = \exp(x_2) = 0; \qquad (5.44)$$

$$\begin{aligned}
&x_0 = x_0^{(0)} + x_0^{(2)} + x_0^{(4)}, \quad x_1 = x_1^{(0)} + x_1^{(2)} + x_1^{(4)}, \\
&\xi_1 = \xi_1^{(1)} + \xi_1^{(3)}, \quad \overline{\xi}_1 = \overline{\xi}_1^{(1)} + \overline{\xi}_1^{(3)},
\end{aligned} \qquad (5.45)$$

respectively.

According to formulae (5.21), the zero-order functions $x_0^{(0)}$ and $x_1^{(0)}$ are given by

$$\begin{aligned}
&\exp(x_0^{(0)}) = a_1 \exp(b_1 z) + a_2 \exp(b_2 z), \\
&\exp(x_1^{(0)}) = -a_1 a_2 (b_1 - b_2)^2 \exp(z(b_1 + b_2) - x_0^{(0)}).
\end{aligned} \qquad (5.46)$$

Substituting them into eqs. (5.25)-(5.26), taking into account eqs. (5.44) and integrating the obtained expressions, we obtain the following results:

$$\begin{aligned}
&\xi_1^{(1)} = \beta_1 x_1^{(0)} ' + \alpha_1 \exp(-x_0^{(0)}), \\
&\overline{\xi}_1^{(1)} = \overline{\beta}_1 x_0^{(0)} ' + \overline{\alpha}_1 \exp(x_1^{(0)})
\end{aligned} \qquad (5.47)$$

for the first-order functions $\xi_1^{(1)}$ and $\overline{\xi}_1^{(1)}$.

According to formulae (5.30), we have

$$\begin{aligned}
&X_0^{(2)} = \xi_1^{(1)} ' \overline{\xi}_1^{(1)}, \quad X_1^{(2)} = \xi_1^{(1)} \overline{\xi}_1^{(1)} ', \\
&\Xi_1^{(3)} = -\xi_1^{(1)} ' x_0^{(2)} ' + \xi_1^{(1)} (x_1^{(2)} - x_0^{(2)}) \exp(x_1^{(0)} - x_0^{(0)}), \\
&\overline{\Xi}_1^{(3)} = \overline{\xi}_1^{(1)} ' x_1^{(2)} ' + \overline{\xi}_1^{(1)} (x_1^{(2)} - x_0^{(2)}) \exp(x_1^{(0)} - x_0^{(0)}), \\
&X_0^{(4)} = -\frac{1}{2}(x_1^{(2)} - x_0^{(2)})^2 \exp(x_1^{(0)} - x_0^{(0)}) + \xi_1^{(1)} ' \overline{\xi}_1^{(3)} + \xi_1^{(3)} ' \overline{\xi}_1^{(1)}, \\
&X_1^{(4)} = \frac{1}{2}(x_1^{(2)} - x_0^{(2)})^2 \exp(x_1^{(0)} - x_0^{(0)}) + \xi_1^{(1)} \overline{\xi}_1^{(3)} ' + \xi_1^{(3)} \overline{\xi}_1^{(1)} '
\end{aligned} \qquad (5.48)$$

for the inhomogeneous parts of perturbative equations (5.27)-(5.29), corresponding to the higher order corrections. Iterating eqs. (5.42) and (5.37) in consecutive order, step by step, we obtain the following expressions:

$$\begin{aligned}
&x_0^{(2)} = -\frac{1}{2}(\beta_1 \overline{\beta}_1 - \alpha_1 \overline{\alpha}_1) x_1^{(0)} ' - \alpha_1 \overline{\beta}_1 \exp(-x_0^{(0)}), \\
&x_1^{(2)} = \frac{1}{2}(\beta_1 \overline{\beta}_1 - \alpha_1 \overline{\alpha}_1) x_0^{(0)} ' + \beta_1 \overline{\alpha}_1 \exp(x_1^{(0)})
\end{aligned} \qquad (5.49)$$

for the second-order functions,

$$\begin{aligned}
&\xi_1^{(3)} = -\frac{1}{2}\alpha_1 \beta_1 \overline{\beta}_1 (\exp(-x_0^{(0)})) ' + \alpha_1 \beta_1 \overline{\alpha}_1 (\frac{1}{2}\exp(x_1^{(0)} - x_0^{(0)}) - b_1 b_2), \\
&\overline{\xi}_1^{(3)} = -\alpha_1 \overline{\alpha}_1 \overline{\beta}_1 (\frac{1}{2}\exp(x_1^{(0)} - x_0^{(0)}) - b_1 b_2) - \frac{1}{2}\beta_1 \overline{\alpha}_1 \overline{\beta}_1 (\exp(x_1^{(0)})) '
\end{aligned} \qquad (5.50)$$

for the third order functions, and, at last,

$$x_0^{(4)} = -x_1^{(4)} = \frac{1}{2}\alpha_1 \beta_1 \overline{\alpha}_1 \overline{\beta}_1 (\frac{1}{2}\exp(x_1^{(0)} - x_0^{(0)}) - b_1 b_2) \qquad (5.51)$$

for the fourth order functions.

6 Conclusion

In the present Letter, we proved the integrability of the f-Toda chain with fixed ends or interrupted from the left (right) and proposed an algorithmic method for constructing its explicit solutions. Many interesting questions arise in this connection. What is the group-theoretical foundation of this result and its connection with the representation theory of supergroups (superalgebras)? Is it possible to construct a superintegrable two-dimensional generalization of the f-Toda chain? Is it possible to represent the f-Toda chain with fixed ends in the Lax-pair or Hamiltonian forms? This last question is a very important in connection with the problem of its quantization. In the case of the usual Toda chain, all these questions have the answers and the authors hope to find their solutions for the case of the f-Toda chain in future publications.

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