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RENORMALON CHAINS CONTRIBUTIONS TO THE NON-SINGLET EVOLUTION KERNELS IN $[\varphi^3]_6$ AND QCD

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1 Introduction

The evolution equations play an important role both in inclusive [1] and exclusive [2] hard processes. They describe the dependence of parton distribution functions of DIS and the parton wave functions on the renormalization parameter μ. The main ingredients for this evolution analysis are the kernels of evolution equations. The two-loop kernels $P(z)$ for the DGLAP evolution equation were obtained in [3], a more complicated two-loop kernel $V(x, y)$ for the Brodsky–Lepage (BL) evolution equation for meson wave functions was calculated in [4, 5]. A three-loop calculation of these kernels looks as a tremendous problem. Recently, the results of very complex calculations of the first few elements of the three-loop anomalous dimension of composite operators in DIS – $\gamma_{(2)}(N = 2, 4, 6, 8)$ that are proportional to the corresponding moments of $P_{(2)}(z)$ were presented in [6]. Sooner, these results were applied to improve the QCD analysis of the DIS experimental data [7]. Despite this important technical and phenomenological progress one may feel an insufficiency because of the absence of the whole kernel in this order or, even at least, some parts of it. Of course, the kernel possessing a physical interpretation is a more general and appreciable object from the theoretical point of view.

A method to estimate the anomalous dimensions of composite operators in the limit of a large number of flavors, N_f, was suggested by J. Gracey [8, 9]. It is based on conformal properties of the theory at the non-trivial zero, g_c, of the D-dimensional β-function, $\bar{\beta}(g_c) = 0$. The generating function has been constructed to obtain the leading part of the anomalous dimension $\gamma_{(n)}(N)$ in the parameter N_f, for any order n of perturbation theory (PT).

I suggest here a method to calculate some classes of multiloop diagrams directly for the kernel $P(z)$ of the DGLAP evolution equation as well as for the kernel $V(x, y)$ of the BL evolution equation in the $\overline{\text{MS}}$ scheme. These diagrams contain the insertion of chains of one-loop self-energy parts (renormalon chains) into the first one-loop diagrams (see fig.1 a,b) for kernels. The kernels $P_{(n)}$ are obtained in any order n of PT based on the analysis of these "dressed by chain" diagrams. Then, the kernel P (or V) for diagrams with the totally dressed propagator is calculated, the kernel appearing as a generating function to obtain the partial kernels $P_{(n)}$. It is to be stressed that the method to obtain P or V does not depend on the nature of self-energy insertions and does not appeal to the value of parameters $N_f T_R$, $C_A/2$ or C_F (for QCD case) associated with loops. Another distinctive feature of it is that the PT-improved evolution kernels are calculated in the direct and standard way by using rather elementary methods. This is possible due to the simple structure of the counterparts of considered diagrams. In this letter mainly the technical results are presented in the framework of the $[\varphi^3]_6$ model and, in part, for QCD. The solution to the BL evolution equation for large N_f is briefly discussed too.

To develop the diagrammatic analysis of the multiloop evolution kernels, let us first consider a toy model based on the scalar $L_{int} = g \sum_i^{N_f} (\psi_i^* \psi_i \varphi)_{(6)}$ model in 6 space-time dimensions (ψ_i^*, ψ_i being the charged "quark" fields and φ the "gluon" one). The number of "quark" flavors, N_f, will be considered as an arbitrary free parameter associated with the "quark" loops. This theory has much in common with the more physically interesting QCD: it is renormalizable; its β-function has a structure similar to that of β_{QCD},

$$\beta(a) = -a^2 b_0 + \ldots, \quad b_0 = \left(\frac{2}{3} - \frac{N_f}{6}\right), \quad a = \frac{g^2}{(4\pi)^3};$$

the one-loop P_0 and V_0 evolution kernels, following from the simplest triangular diagram in figs.1a and 1b, are proportional to the corresponding QCD expressions for the same diagrams

$$aP_0(z) = a(1-z), \; aV_0(x,y) = a\left(\theta(y > x)\frac{x}{y} + \theta(\bar{y} > \bar{x})\frac{\bar{x}}{\bar{y}}\right), \tag{1}$$
$$\text{where } \bar{y} = 1-y, \; \bar{x} = 1-x, \ldots$$

The similarity of elements and the structure of the whole kernels continue to the two-loop level [10]. It is not a gauge theory, nevertheless, due to the relevant Feynman integrals being simple, in this model it is much easier to study the structure of multiloop expressions.

In principle, the calculation of the evolution kernels $P(z;\alpha)$ [3] and $V(x,y;\alpha)$ [5] is quite analogous to that of the anomalous dimensions. The major modification is the change $(kn)^N \to \delta(z - kn)$ of the vertex factor corresponding to a composite operator, k being the relevant momentum related to the incoming "quark" line, see figs. 1a,b. In these figs., p ($\bar{y}p$ or p) is the external momentum of the diagrams; n is a light-like vector ($n^2 = 0$) introduced to pick out the symmetric traceless composite operator; $pn = 1$. Detailed examples of similar calculations may be found in [10].

2 First triangular diagrams for DGLAP evolution kernel

Here, the method will be demonstrated for the well-understood example [9, 11] of simple quark-loop insertions. Let us consider the triangular diagram in fig.1c with an insertion into the gluon line, $\prod_i g_i$, where subgraphs g_i are 1PI self-energy parts. The whole diagram Γ in fig.1c may be represented as a generalized product $\Gamma = G \otimes \prod_i g_i$ where G denotes the "outer triangle" (compare to the intrinsic block $\prod_i g_i$), containing a composite vertex.

Expressions for $P(z)$ or $V(x,y)$ via the renormalization constant Z_Γ in the $\overline{\text{MS}}$ scheme (see [12]) are given by the equations [5],

$$Z_\Gamma = 1 - \hat{K}R'(\Gamma), \qquad P = -a\partial_a \left(Z_\Gamma^{(1)}\right) = a\partial_a \left(\hat{K}_1 R'(\Gamma)\right). \tag{2}$$

Here R' is the incompleted BPHZ R-operation; $\varepsilon = (6-D)/2$ and D is the space-time dimension; $\hat{K}$ picks out poles in ε; whereis $\hat{K}_1$ projects out the first pole, and $Z^{(1)}$ is the coefficient of the first pole for the expansion of Z. If the block $\prod_i g_i$ is the chain of one-loop insertions, then Exp. (2) for P may be simplified following the definition of the R'-operation

$$\hat{K}_1 R'(\Gamma) \equiv \hat{K}_1 R'(G \otimes \prod_i g_i) = \hat{K}_1 \left[G \otimes \prod_i (1 - \hat{K}_1) g_i\right] \Rightarrow$$
$$\Rightarrow P_{(n)} = (n+1) a \hat{K}_1 \left[G \otimes \prod_i^n (1 - \hat{K}_1) g_i\right], \tag{3}$$

where the kernel $P_{(n)}$ corresponds to the chain of n one-loop insertions into the triangular diagram,

$$Rg_i(k^2) = (1 - \hat{K}_1) g_i(k^2) = -\frac{aN_f}{\varepsilon}\left(\gamma_\varphi(\varepsilon)\left(\frac{\mu^2}{k^2}\right)^\varepsilon - \gamma_\varphi(0)\right), \tag{4}$$
$$\gamma_\varphi(\varepsilon) = C(\varepsilon) B(2-\varepsilon, 2-\varepsilon), \quad B(a,b) \text{ is the Euler B-function.} \tag{5}$$

Here $\gamma_\varphi = \gamma_\varphi(0)$ is the one-loop anomalous dimension of the gluon field (at $N_f = 1$), the constant $C(\varepsilon) = \Gamma(1-\varepsilon)\Gamma(1+\varepsilon)$ reflects the concrete choice of $\overline{\text{MS}}$ scheme where every loop integral is multiplied by the scheme factor $\Gamma(D/2-1)(\mu^2/4\pi)^\varepsilon$. Substituting (4) into (3) and performing a direct calculation (see, *e.g.*, [10]), one arrives at the expression

$$P_{(n)}^{(1)}(z) = (n+1) a P_0(z) (-aN_f\gamma_\varphi)^n \cdot$$
$$\hat{K}_1 \left[\frac{C(\varepsilon)(1-\varepsilon)}{\varepsilon^{n+1}} z^{-\varepsilon} \sum_{j=0}^n \left(\frac{\gamma_\varphi(\varepsilon)}{\gamma_\varphi(0)}\right)^j \frac{\Gamma(1+(j+1)\varepsilon)}{\Gamma(1+j\varepsilon)\Gamma(1+\varepsilon)} \binom{n}{j} \frac{(-)^{n-j}}{j+1}\right]. \tag{6}$$

It is instructive to consider the properties of a set of the first eight kernels $P_{(k)}^{(1)}(z)$. Their expressions are presented in Table 1; to obtain them, the FORM 2.0 program [13] has been used [2]. The lessons are : i) the perturbation theory for kernels $P_{(n)}^{(1)}(z)$ looks as being **improved** in comparison with the corresponding set of anomalous dimensions $\gamma_{(n)}(N)$. Indeed, the term in $P_{(n)}^{(1)}(z)$ leading in z is $\sim \frac{\ln^n(z)}{n!}$ while for the corresponding contribution to $\gamma_n(N) \sim n^0$; ii) one can see the **factorial suppression** in n of all other logarithmic terms in $P_{(n)}^{(1)}(z)$; iii) the n-bubble chain generates $\zeta(n)$ (the Riemann **zeta**-function) in the **non-logarithmic** term in $P_{(n)}^{(1)}(z)$, this may point to the expansion of the Euler B-function, as their possible origin. These properties give hints about a possible resummation of the $P_{(n)}^{(1)}(z)$-series.

[2] I am greatly indebted to Dr. L. Avdeev who provided to me his brilliant FORM-based program for expansion in ε, see [14]. Note that the contents of Table 1 is limited here only by place.

Table 1 The results of the $P^{(1)}_{(n)}(z;A)$ calculations, the $\zeta(n)$ is Riemann zeta–function, note that the $\zeta(2)$ and the Euler constant γ_E does not appear in this expansion

n	The partial kernels $P^{(1)}_{(n)}(z)$ (the common factor $aP_0(z)A^n$ is dropped)
1	$\frac{[\ln(z)+\frac{8}{3}]^1}{1!}$
2	$\frac{[\ln(z)+\frac{8}{3}]^2}{2!} - \frac{20}{9}$
3	$\frac{[\ln(z)+\frac{8}{3}]^3}{3!} - \frac{20}{9}\frac{\ln(z)}{1!} - \left(\frac{256}{81}+2\zeta(3)\right)$
4	$\frac{[\ln(z)+\frac{8}{3}]^4}{4!} - \frac{20}{9}\frac{\ln^2(z)}{2!} - \left(\frac{256}{81}+2\zeta(3)\right)\frac{\ln^1(z)}{1!} - \frac{512}{243} + 3\zeta(4) - \frac{16}{3}\zeta(3)$
5	$\frac{[\ln(z)+\frac{8}{3}]^5}{5!} - \frac{20}{9}\frac{\ln^3(z)}{3!} - \left(\frac{256}{81}+2\zeta(3)\right)\frac{\ln^2(z)}{2!} - \left(\frac{512}{243}+3\zeta(4)-\frac{16}{3}\zeta(3)\right)\frac{\ln^1(z)}{1!} - \frac{4096}{3645}$ $+8\zeta(4) - 6\zeta(5) - \frac{8}{3}$
6	$\frac{[\ln(z)+\frac{8}{3}]^6}{6!} - \frac{20}{9}\frac{\ln^4(z)}{4!} - \left(\frac{256}{81}+2\zeta(3)\right)\frac{\ln^3(z)}{3!} - \left(\frac{512}{243}+3\zeta(4)-\frac{16}{3}\zeta(3)\right)\frac{\ln^2(z)}{2!} + \left(-\frac{4096}{3645}\right.$ $\left.+8\zeta(4) - 6\zeta(5) - \frac{8}{3}\right)\frac{\ln^1(z)}{1!} - \frac{16384}{32805} + 10\zeta(6) - 16\zeta(5) + 4\zeta(4) + 2\zeta^2(3)$
7	$\frac{[\ln(z)+\frac{8}{3}]^7}{7!} - \frac{20}{9}\frac{\ln^5(z)}{5!} - \left(\frac{256}{81}+2\zeta(3)\right)\frac{\ln^4(z)}{4!} - \left(\frac{512}{243}+3\zeta(4)-\frac{16}{3}\zeta(3)\right)\frac{\ln^3(z)}{3!} + \left(-\frac{4096}{3645}\right.$ $\left.+8\zeta(4) - 6\zeta(5) - \frac{8}{3}\right)\frac{\ln^2(z)}{2!} + \left(-\frac{16384}{32805} + 10\zeta(6) - 16\zeta(5) + 4\zeta(4) + 2\zeta^2(3)\right)\frac{\ln^1(z)}{1!} - 18\zeta(7)$ $+\frac{80}{3}\zeta(6) - 8\zeta(5) - 6\zeta(3)\zeta(4) + \frac{16}{3}\zeta^2(3)$
8	$\frac{[\ln(z)+\frac{8}{3}]^8}{8!} - \frac{20}{9}\frac{\ln^6(z)}{6!} - \left(\frac{256}{81}+2\zeta(3)\right)\frac{\ln^5(z)}{5!} - \left(\frac{512}{243}+3\zeta(4)-\frac{16}{3}\zeta(3)\right)\frac{\ln^4(z)}{4!} + \left(-\frac{4096}{3645}\right.$ $\left.+8\zeta(4) - 6\zeta(5) - \frac{8}{3}\right)\frac{\ln^3(z)}{3!} + \left(-\frac{16384}{32805} + 10\zeta(6) - 16\zeta(5) + 4\zeta(4) + 2\zeta^2(3)\right)\frac{\ln^2(z)}{2!} - \left(18\zeta(7)\right.$ $\left.+\frac{80}{3}\zeta(6) - 8\zeta(5) - 6\zeta(3)\zeta(4) + \frac{16}{3}\zeta^2(3)\right)\ln(z) + \frac{63}{2}\zeta(8) - 48\zeta(7) + \frac{40}{3}\zeta(6) + \frac{9}{2}\zeta^2(4)$ $+12\zeta(3)\zeta(5) - 16\zeta(3)\zeta(4) + \frac{8}{3}\zeta^2(3)$

Theorem 1 *There is the closed expression for the sum of partial kernels* $P^{(1)}(z;A) = a\sum_{n=0}^{\infty} P^{(1)}_{(n)}(z)$, *which is equal to* $P^{(1)}(z;A) = aP_0(z)z^{-A}(1-A)\left(\frac{\gamma_\varphi(0)}{\gamma_\varphi(A)}\right)$, *where* $A = aN_f\gamma_\varphi$. *The kernel* $P^{(1)}(z;A)$ *is the generating function for* $P^{(1)}_{(n)}(z)$.

This result possesses several remarkable properties.

- the $P^{(1)}(z;A)$ becomes the dominant part of the total perturbative kernel $P(z)$ when $N_f \gg 1$. Below, the result for the kernel $P^{(1)}(z;A)$ will be completed by taking into account similar corrections to "quark leg" (see fig.1d), whose contribution is proportional to $\delta(1-z)$.

- the kernel $P^{(1)}(z;A)$ is an analytic function in variable A, except for singularities at points $A = 2+k+1/2$, $k = 0, 1, \ldots$, where the function has simple poles. The nearest singularity appears at $A = 5/2$, *i.e.*, at $aN_f = 15$, determining, roughly speaking, the range of convergence of the PT series.

- the analytic properties of $P^{(1)}(z;A)$ in the variable A are determined completely by the one-loop "anomalous dimension" $\gamma_\varphi(A)$ in D dimensions (see Eq.(5)). The singularities of $P^{(1)}(z;A)$ correspond to zeros of the function $B(2-A, 2-A)$ in $\gamma_\varphi(A)$.

Proof. One could split the sum in Eq.(6) into two parts

$$(n+1)\sum_{j=0}^{n}(F(\varepsilon))^j\frac{\Gamma(1+(j+1)\varepsilon)}{\Gamma(1+j\varepsilon)\Gamma(1+\varepsilon)}\binom{n}{j}\frac{(-)^{n-j}}{j+1} = S_{(n+1)}(\varepsilon) + \frac{(-)^n}{F(\varepsilon)C(\varepsilon)} \quad (7)$$

$$S_{(n+1)}(\varepsilon) \equiv \left[\sum_{j=0}^{n+1}(F(\varepsilon))^{j-1}\frac{\Gamma(1+j\varepsilon)(-)^{n+1-j}}{\Gamma(1+(j-1)\varepsilon)\Gamma(1+\varepsilon)}\binom{n+1}{j}\right] \quad (8)$$

Here and below we use the notations: $F(\varepsilon) \equiv \frac{\gamma_\varphi(\varepsilon)}{\gamma_\varphi(0)}$, and $S_{(n+1)}$ is the sum including all "combinatorics" of the l.h.s. of (7). The first term $S_{(n+1)}$ in the r.h.s. of (7) does not contribute to the pole part, by Lemma 1 $S_{(n+1)}(\varepsilon) \le O(\varepsilon^{n+1})$. The expression for $P^{(1)}_{(n)}(z)$ can be derived by the following chain of equations

$$\begin{aligned} P^{(1)}_{(n)}(z) &= aP_0(z)(-A)^n\hat{K}_1\left[\frac{(1-\varepsilon)z^{-\varepsilon}}{\varepsilon^{n+1}}\left(C(\varepsilon)S_{(n+1)}(\varepsilon) + \frac{(-)^n}{F(\varepsilon)}\right)\right] = \\ &= aP_0(z)(A)^n\hat{K}_1\left[\frac{1}{\varepsilon^{n+1}}\frac{(1-\varepsilon)z^{-\varepsilon}}{F(\varepsilon)}\right] = aP_0(z)\frac{(A)^n}{n!}\left\{\frac{d^n}{d\varepsilon^n}\left[\frac{(1-\varepsilon)z^{-\varepsilon}}{F(\varepsilon)}\right]\right\}\Bigg|_{\varepsilon=0} \end{aligned} \quad (9)$$

From the form of Exp.(9) it follows that the generating function for $P^{(1)}_{(n)}(z)$ is $P^{(1)}(z;A)$:
$P^{(1)}(z;A) = a\sum_{n=0}^{\infty}P^{(1)}_{(n)}(z) = aP_0(z)z^{-A}(1-A)\left(\frac{\gamma_\varphi(0)}{\gamma_\varphi(A)}\right)$; and the partial kernels $P^{(1)}_{(n)}(z)$ appear in the Taylor expansion of $P^{(1)}(z;A)$ in the variable A. ■

Lemma 1 $S_{(n+1)}(\varepsilon) \leq O(\varepsilon^{n+1})$ *for any analytic function* $F(\varepsilon)$ *at point* $\varepsilon = 0$.

Proof. Let us consider the expansion of every element of the sum in $S_{(n+1)}$

$$\frac{F(\varepsilon)^j}{F(\varepsilon)} \frac{\Gamma(1+j\varepsilon)}{\Gamma(1+(j-1)\varepsilon)\Gamma(1+\varepsilon)} \tag{10}$$

in powers of ε up to ε^n. Any power ε^m of this expansion is accompanied by powers j^l, where $l \leq m$, and a coefficient that does not depend on j. Therefore, the expansion of $S_{(n+1)}(\varepsilon)$ as a whole in the power series ε^m will generate such coefficients of the powers, which are composed only of the elements proportional to $\sum_{j=0}^{n+1} j^l(-)^{n+1-j}\binom{n+1}{j}$. All these elements are equal to 0 in virtue of the identity

$$\sum_{j=0}^{n+1} j^l(-)^{n+1-j}\binom{n+1}{j} = 0, \quad \text{if} \;\; l \leq n \tag{11}$$

The proof implies an obvious generalization of the elements (10), that can be constructed as superpositions of Γ-functions depending on j only through the arguments like $1 + j\varepsilon,\ 1 + (j-1)\varepsilon, \ldots$. ■

In a similar way one can derive an expression for the sum of diagrams in fig.1 d connected with the anomalous dimension of the quark field $\gamma_\psi^{(1)}(A)$ obtained in the "quark-loop" approximation. Collecting the results of resummation in the main approximation in A, which correspond to diagrams figs.1c and 1d, we arrive at the final expression

$$\begin{aligned} &P^{(1)}(z;A) - \delta(1-z)\gamma_\psi^{(1)}(A) = \\ &= a\left(\frac{\gamma_\varphi(0)}{\gamma_\varphi(A)}\right)\left[\bar{z}z^{-A}(1-A) - \delta(1-z)\frac{(1-A)}{(3-A)(2-A)}\right]. \end{aligned} \tag{12}$$

Integration produces the following expression for the anomalous dimension $\gamma(N, A)$ of the composite operator

$$\begin{aligned} \gamma(N,A) &= \int_0^1 z^N\left(P^{(1)}(z;A) - \delta(1-z)\gamma_\psi(A)\right)dz = \\ &= a\gamma_\varphi\frac{\Gamma(4-2A)}{\Gamma(2-A)\Gamma(1-A)^2\Gamma(1+A)}\left[\frac{\Gamma(N+1-A)}{\Gamma(N+3-A)} - \frac{\Gamma(2-A)}{\Gamma(4-A)}\right]. \end{aligned} \tag{13}$$

Formula (13) can be obtained by another method applied in [8] to the QCD case (see Eq.(30) below). Note, that the anomalous dimension $\gamma(N = 1, A)$ corresponds to the parton energy. The equality $\gamma(N = 1, A) = 0$ following from the RHS of (13) signals that the conformal symmetry is conserved.

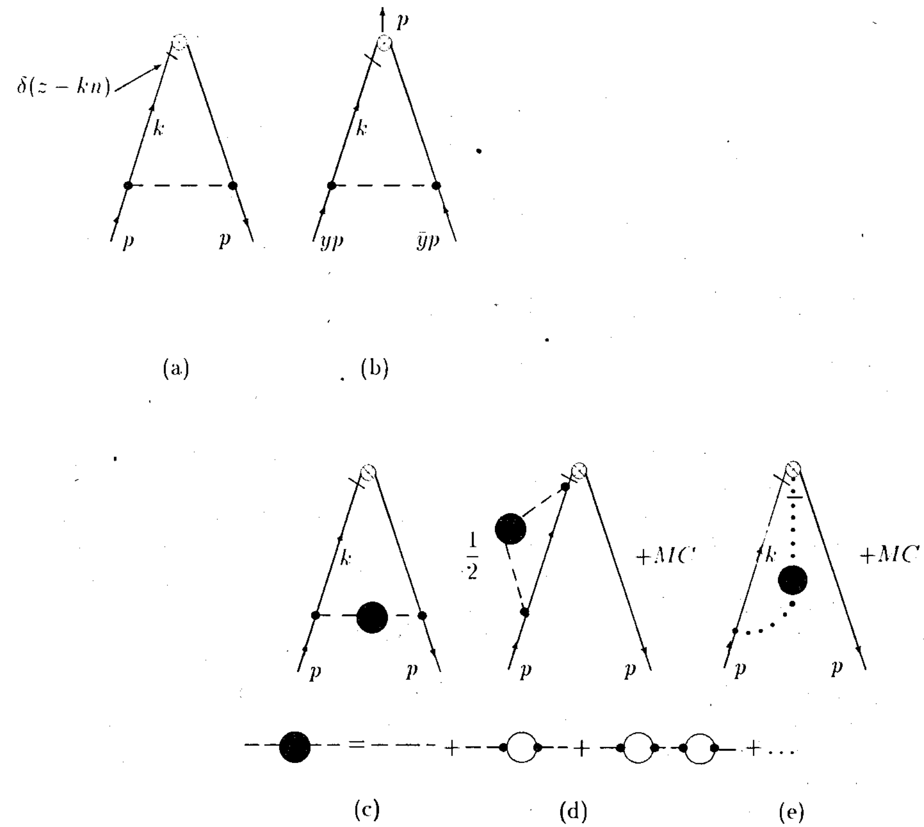

Figure 1: The diagrams in figs. 1a - 1d are for the scalar model (and QCD); dashed line for "gluons", solid line for "quarks"; the slash on line denotes the delta function $\delta(z - kn)$, see fig.1a; black circle denotes the sum of all one-loop chains; MC denotes the mirror-conjugate diagram. The diagram in fig. 1e is for QCD; solid line for quarks, dotted line for gluons.

3 Other triangular diagrams contributions

Theorem 2 *There are the closed expressions for sums of partial kernels*

$$P^{(2a)}(z;B) = a\sum_{m=0}^{\infty} P^{(2)}_{(m)}(z) = aP_0(z)\left((\bar z)^{-B}\frac{\gamma_\psi(0)}{\gamma_\psi(B)}\right), \; diag.\ in\ fig.2a\ , \quad (14)$$

$$\tilde P^{(2a)}(z;B) = a\sum_{m=0}^{\infty}(2-\delta_{0,n})P^{(2)}_{(m)}(z) = aP_0(z)\left(2(\bar z)^{-B}\frac{\gamma_\psi(0)}{\gamma_\psi(B)} - 1\right), \quad (15)$$

diag. in fig.2a + MC,

$$P^{(2b)}(z;B) = a\sum_{m=0}^{\infty}(m+1)P^{(2)}_{(m)}(z) = aP_0(z)\left(1+B\frac{d}{dB}\right)\left((\bar z)^{-B}\frac{\gamma_\psi(0)}{\gamma_\psi(B)}\right) \quad (16)$$

diag. in fig.2b ,

$$\delta(1-z)\gamma^{(2)}_\psi = \delta(1-z)a\left(\frac{\gamma_\psi(0)}{\gamma_\psi(A)}\right)\frac{(1-A)}{(3-A)(2-A)}, \; diagr.\ in\ fig.2c\ , \quad (17)$$

the functions $P(z;B)$ appear as generating functions for the corresponding partial kernels.

Here $P^{(2)}_{(m)}(z)$ is the partial kernel with m insertions into one of the quark lines; $\gamma_\psi(\varepsilon)$ is the one-loop anomalous dimension of the quark field in D-dimension; for this model $\gamma_\psi(\varepsilon) = \gamma_\varphi(\varepsilon)$, $\gamma_\psi \equiv \gamma_\psi(0) = \gamma_\varphi(0)$, $B = a\gamma_\psi$; MC denotes a mirror-conjugate diagram. The first Eq.(14) corresponds to the diagrams in fig. 2a, where the chain of quark self-energy parts is substituted only into the left quark line of the triangle. To prove it, one has to repeat the way similar to theorem 1. Equation (15) corresponds to the sum of the diagrams in fig.2a and its MC diagrams. This result will be used to restore the corresponding kernel $V(x,y)$ in the next section. The analytic properties of the functions $P^{(2a)}(z;B)$, $\tilde P^{(2a)}(z;B)$ in the parameter B are the same as for the kernel $P^{(1)}(z;B)$, they are determined by the function $\gamma_\psi(B)$. Equation (16) corresponds to substitutions of the chains into both quark lines of the triangle in fig.2b. At least, Eq.(17) corresponds to contributions to the anomalous dimension of the quark field from the diagram in fig.2c. The contribution $P^{(2)}(z;B)$ will be suppressed in the parameter B in comparison with $P^{(1)}(z;A)$, if $N_f \gg 1$, *i.e.*, $A \gg B$. Note, however, that the N-moments of the kernels $P^{(2a,b)}$ - $\gamma^{(2)}(N,B)$ decrease in N more slowly than $\gamma^{(1)}(N,A)$ corresponding to the kernel $P^{(1)}$. Therefore, at sufficiently large N, $\gamma^{(2)}(N,B) > \gamma^{(1)}(N,A)$ for any fixed parameters A and B.

The expression for the kernel $P_{(n,m)}(z;A,B)$ corresponding to insertions of different one-loop parts both in gluon (n–bubble insertions) and quark (m–self-energy part insertions) lines of the triangular diagram (see fig. 2d), is obtained as well. This formula is similar to Eq.(6) for $P_{(n)}$, but looks more cumbersome and is not shown here. The partial kernels $P_{(n,m)}(z;A,B)$ can be obtained by using the FORM program, in principle, for any given n and m. For illustration, we demonstrate here

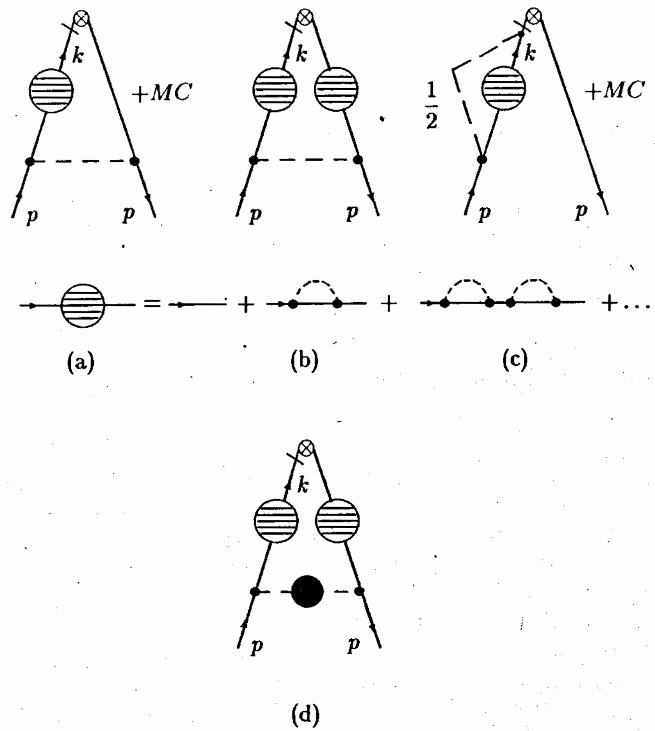

Figure 2: The dashed circle in figs.2 denotes the sum of chains of one-loop self energy "quark" parts.

the first nontrivial kernel $aP_{(1,1)}$

$$P_{(1,1)} = P_0(z)AB\left(\frac{1}{2}\left[\ln(z)+\frac{8}{3}\right]^2 - \frac{20}{9} + \frac{1}{2}\left[\ln(\bar{z})+\frac{5}{3}\right]^2 - \frac{31}{18}\right.$$
$$\left. -2\left[\ln(z)+\frac{8}{3}\right]\left[\ln(\bar{z})+\frac{5}{3}\right] + \frac{80}{9} - 2\zeta(2)\right)$$

in comparison with partial kernels of the same order in a, $P^{(1)}_{(2)}$ $(\equiv P_{(2,0)})$ following from the expression for $P^{(1)}$ and $3P^{(2)}_{(2)}$ $(\equiv P_{(0,2)})$, from $P^{(2b)}$

$$P^{(1)}_{(2)} = P_0(z)A^2\left(\frac{1}{2!}\left[\ln(z)+\frac{8}{3}\right]^2 - \frac{20}{9}\right),\ P^{(2)}_{(2)} = 3P_0(z)B^2\left(\frac{1}{2!}\left[\ln(\bar{z})+\frac{5}{3}\right]^2 - \frac{31}{18}\right).$$

To complete the section, we conclude that the contributions to kernel P from any one-loop insertions into the lines of the triangular diagram are available now.

4 Triangular diagrams for the Brodsky–Lepage evolution kernel

Here, some partial results of the bubble resummation for the BL kernels V are presented. We obtain them as a "byproduct" of our previous results for the kernel DGLAP P. The results, obtained here, may be used, in particular, to check the regular calculation of the kernel V in high orders of PT.

Note, the diagrams for BL kernels differ from the DGLAP diagrams only by the "exclusive" kinematics of the input momentum, compare diagrams in fig.1a and 1b. So, one can repeat the proofs of theorems 1,2 for this case. There is another far more elegant way – to use exact relations between V and P kernels for triangular diagrams that were established in any order of perturbation theory in [5]. These relations work both for the $[\varphi^3]_6$ model and QCD. I quote these propositions without proofs.

4.1 Let the diagram in fig.1c has a contribution to the DGLAP kernel in the form $P(z) = p(z) + \delta(1-z)\cdot C$, then its contribution to the BL kernel is

$$V(x,y) = C\theta(y>x)\int_0^{\frac{x}{y}} \frac{p(z)}{\bar{z}}dz + \delta(y-x)\cdot C, \qquad (18)$$

where $C \equiv 1 + (x \to \bar{x}, y \to \bar{y})$. From relation (18) and Eq. (12) for $P^{(1)}$ we immediately derive the expression for $V^{(1)}$

$$V^{(1)}(x,y;A) = Ca\left(\frac{\gamma_\varphi(0)}{\gamma_\varphi(A)}\right)\left[\theta(y>x)\left(\frac{x}{y}\right)^{1-A} - \frac{1}{2}\delta(y-x)\frac{(1-A)}{(3-A)(2-A)}\right]. \qquad (19)$$

That may be independently verified by other relations reducing any V to P [5, 15] ($V \to P$ reduction)

$$V^{(1)}(x,y;A) = C\theta(y>x)F(\frac{x}{y};A) - \delta(y-x)C(A) \to P^{(1)}(z;A)$$
$$P^{(1)}(z;A) = \theta(1-z)\bar{z}\frac{d}{dz}F(z;A) - \delta(1-z)C(A), \qquad (20)$$

indeed, substituting Eq.(19) in (20) we return to the same Eq.(12) for $P^{(1)}$. Moreover, the first term of the Taylor expansion of $V^{(1)}(x,y;A)$ in A coincides with the results of the two-loop calculation in [10]. The contribution $V^{(1)}$ should dominate for $N_f \gg 1$ in the whole kernel V, and moreover, the function $V^{(1)}$ possesses an important symmetry. Really, the function $\mathcal{V}(x,y;A) = V^{(1)}(x,y;A)\cdot(\bar{y}y)^{1-A}$ is symmetric under the change $x \leftrightarrow y$, $\mathcal{V}(x,y) = \mathcal{V}(y,x)$. That symmetry allows us to obtain the eigenfunctions $\psi_n(x)$ of the equation

$$\int_0^1 V^{(1)}(x,y;A)\psi_n(y)dy = \gamma(n;A)\psi_n(x) \qquad (21)$$

$$\psi_n(y) = (\bar{y}y)^{\alpha-\frac{1}{2}}C^{(\alpha)}_n(y-\bar{y}),\ \text{here}\ \ \alpha = D_A/2 - 3/2,\ \ D_A = 6-2A. \qquad (22)$$

Here $C^{(\alpha)}_n(z)$ are the Gegenbauer polynomials of order α. This form of $\{\psi_n(y)\}$ eigenfunctions as well as the $x \leftrightarrow y$ symmetry of function $\mathcal{V}(x,y)$ are the consequences of conformal symmetry conservation for the sum of diagrams under consideration (see the end of sec.2). Equation (21) is tightly connected with the BL evolution equation in a special case when the β-function $\beta = 0$. In this case the BL equation is simplified, the variables are separated, and the equation reduces to Eq. (21). The partial solution to the BL equation turns out to be proportional to $\psi_n(y)$ in (22).

4.2 For the diagrams in fig. 2a plus its MC diagrams, the relation between $\tilde{V}^{(2a)}$ and $\tilde{P}^{(2a)}$ is [5]

$$\tilde{V}^{(2a)}(x,y) = C\theta(y>x)\frac{1}{2}\left[\frac{1}{\bar{y}}\tilde{P}^{(2a)}(x) + \frac{1}{y}\tilde{P}^{(2a)}(\bar{x}) - \frac{1}{\bar{y}}\tilde{P}^{(2a)}\left(\frac{x}{y}\right)\right]. \qquad (23)$$

Substituting (15) into (23) one arrives at the expression for $\tilde{V}^{(2a)}(x,y)$

$$\tilde{V}^{(2a)}(x,y) = Ca\theta(y>x)\left\{\left[\frac{\bar{x}^{1-B}}{\bar{y}} + \frac{x^{1-B}}{y} - \frac{1}{\bar{y}}\left(1-\frac{x}{y}\right)^{1-B}\right]\left(\frac{\gamma_\psi(0)}{\gamma_\psi(B)}\right) - \frac{x}{y}\right\}, \qquad (24)$$

which satisfies the same check as in the previous case. Of course, this formula does not cover the diagrams in fig. 2b, where both the quark lines are dressed.

5 The results for QCD evolution kernels

Assertions similar to the above theorems 1,2 are also valid for the QCD diagrams, their proof does not contain essentially new elements, but looks rather cumbersome.

The complete results for the QCD evolution kernels will be presented in a subsequent paper. Here, at first, the QCD results for triangular diagrams in fig. 1 c,d in the Feynman gauge are discussed. Based on theorem 1 in the QCD version, one can derive the result for the sum of diagrams in fig.1c in QCD:

$$P^{(1c)}(z;A) = a_s C_F 2\bar{z} \cdot (1-A)^2 \frac{\Pi(0)}{\Pi(A)} z^{-A} - a_s C_F \cdot \delta(1-z) \left(\frac{\Pi(0)}{\Pi(A)} - 1 \right). \quad (25)$$

Here $\Pi(\varepsilon)$ is twice the contribution to a one-loop D-dimensional anomalous dimension of the gluon field; $\varepsilon = (4-D)/2$; $\Pi(0)$ is the contribution to the standard anomalous dimension; $a_s = \frac{\alpha_s}{4\pi}$ and $A = -a_s \Pi(0)$. It should be emphasized again that the form of Exp.(25) does not depend on the nature of self-energy insertions into the gluon line. One can use it for the resummation both of the quark ($\sim N_f T_R$) and gluon ($\sim C_A/2$) loops. Substituting into the general formula (25), the well-known expressions for $\Pi(\varepsilon)$ from the quark or gluon (the ghost loop is also added) loops

$$\begin{aligned} \Pi_q(\varepsilon) &= -8 N_f T_R B(D/2, D/2) C(\varepsilon), & (26) \\ \Pi_g(\varepsilon) &= \frac{C_A}{2} B(D/2-1, D/2-1) \left(\frac{3D-2}{D-1} \right) C(\varepsilon), & (27) \end{aligned}$$

one obtains $P_q^{(1)}(z;\delta)$ for the quark-loop insertions

$$\begin{aligned} P_q^{(1c)}(z;\delta) &= a_s C_F 2\bar{z} z^{-\delta} \frac{(D_q/2-1)^2 B(2,2)}{B(D_q/2, D_q/2) C(\delta)} \\ &\quad - a_s C_F \delta(1-z) \cdot \left(\frac{B(2,2)}{B(D_q/2, D_q/2) C(\delta)} - 1 \right), \quad (28) \end{aligned}$$

Here $D_q = 4 - 2\delta$, $\delta = -a_s \Pi_q(0) = a_s N_f T_R \frac{4}{3}$,

and $P_g^{(1c)}(z;\epsilon)$ for the gluon-loop insertions

$$\begin{aligned} P_g^{(1c)}(z;\epsilon) &= a_s C_F 2\bar{z} z^{-\epsilon} \frac{10}{3} \frac{(D_g/2-1)^2 (D_g-1)}{(3D_g-2) B(D_g/2-1, D_g/2-1) C(\epsilon)} \\ &- a_s C_F \delta(1-z) \left(\frac{10}{3} \frac{(D_g-1)}{(3D_g-2) B(D_g/2-1, D_g/2-1) C(\epsilon)} - 1 \right) (29) \end{aligned}$$

Here $D_g = 4 - 2\epsilon$, $\epsilon = -a_s \Pi_g(0) = -a_s C_A \frac{5}{3}$.

The latter expression for $P_g^{(1c)}(z;\epsilon)$ has no any singularity in the parameter $\epsilon = -a_s C_A \frac{5}{3} < 0$ due to the asymptotic freedom.

Consider Eq.(28) for $P_q^{(1c)}(z;\varepsilon)$ in detail. By adding the contributions from the diagrams in fig.1d to Exp.(28), one can find that the second term there cancels in part and the final expression turns out to be the expected "cross form" (see, *e.g.* [3])

$$P_q^{(1c,d)}(z;\delta) = a_s C_F 2 \left(\bar{z} z^{-\delta} \frac{(D_q/2-1)^2 \Pi_q(0)}{\Pi_q(\delta)} \right)_+, \quad i.e., \int_0^1 P_q^{(1c,d)}(z;\delta) dz = 0 \quad (30)$$

due to the current conservation. The analytic properties of $P_q^{(1c,d)}(z;\delta)$ in δ are the same as for its scalar analogy $P^{(1)}(z;A)$ (see theorem 1); they are determined by the behavior of the function $\Pi_q(\delta)$ in δ, see Eq.(26). The nearest singularity of $P_q^{(1)}(z;\delta)$ in δ appears at $a_s N_f = 15/4$. The moments of Eq. (30) agree with the corresponding part of the generating function in [8]

$$\gamma_q^{(1)}(N,\delta) = -a_s C_F \frac{2}{3} \frac{N(D_q+N-1)}{(D_q/2+N)(D_q/2-1+N)} \left[\frac{(D_q/2-1)}{B(D_q/2, D_q/2) C(\delta) D_q} \right], \quad (31)$$

see the first term in Eq.(14) in [8], and ref. [11] (note that our moments differ in sign from the definition of the anomalous dimension of composite operators used there).

To complete the QCD calculations of $P^{(1)}$, we need the contribution from the last diagram in fig.1e with the chain in the gluon line that is inserted into the composite operator. It can be obtained in a similar way as in the previous QCD-calculations for Eq.(25)

$$P^{(1e)}(z;A) = a_s C_F 2 \cdot \left(\frac{2z^{1-A}}{1-z} \frac{\Pi(0)}{\Pi(A)} \right)_+, \quad (32)$$

and the expression has the "cross form" automatically, see [5]. Collecting Eq.(30) and Eq.(32) one easily arrives at the complete QCD expression for $P_q^{(1)}(z;\delta)$ with the main quark-loop insertions

$$P_q^{(1)}(z;\delta) = a_s C_F 2 \cdot \left[\bar{z} z^{-\delta} (1-\delta)^2 + \frac{2z^{1-\delta}}{1-z} \right]_+ \frac{\Pi_q(0)}{\Pi_q(\delta)}. \quad (33)$$

The contribution $P_q^{(1)}(z;\delta)$ is gauge invariant. The moments of $P_q^{(1)}(z;\delta)$ agree again with the complete generating function obtained in [8] (see Eq.(14) there).

6 Conclusion

A method of calculating some classes of multiloop diagrams for the kernel $P(z)$ of the non-singlet DGLAP evolution equation is presented. These multiloop diagrams appear due to the insertion of chains of one-loop self-energy parts (renormalon chains) into the lines of the first one-loop diagrams for the kernel. Closed expressions $P^{(1,2)}(z,a)$ are found for sums of all the diagrams which belong to two of the diagram classes, see theorems 1–2. These assertions are based on a simple algebraic structure of the counterparts for the diagrams under consideration. Besides, the kernels $P^{(i)}(z, a_i)$ are generating functions for the partial kernels $P^{(i)}_{(n)}(z)$ in any order n of perturbation theory. The contribution $P^{(1)}$ from one of the diagram classes would dominate in P for $N_f \gg 1$. The analytic properties of the function $P^{(i)}(z, a_i)$ in the variable a_i are briefly discussed. The expressions for partial kernels $P_{(n,m)}(z; a_1, a_2)$ for the diagrams of a "mixed class", in any order of perturbation theory can also be obtained by using the FORM program.

The contributions $V^{(i)}(x, y; a_i)$ to the Brodsky–Lepage kernel are obtained for the same classes of diagrams as a "byproduct" of the previous technique. When $N_f \gg 1$, a special solution to the Brodsky–Lepage equation is derived. We emphasize, that the method of calculating the evolution kernels $P^{(i)}$ or $V^{(i)}$ does not depend on the nature of self-energy insertions and does not appeal to the value of parameters $N_f T_R$, $C_A/2$ or C_F (for QCD case) associated with loops.

The method and results are exemplified with a simple $[\varphi^3]_6$ model; some QCD results are presented too. In particular, the kernel $P_q^{(1)}(z; \delta)$ that corresponds to the diagram dressed by the main quark-loop chain is derived. The anomalous dimension $\gamma_q^{(1)}(N, \delta)$ corresponding to this kernel agrees with the generating function obtained earlier by Gracey [8, 9]. The contribution $P_g^{(1)}(z; \epsilon)$ from the diagrams with the gluon-loop chain is derived in the same way. It is clear, that all the results obtained above in the framework of the scalar model have a wider meaning and apply to the QCD case. The latter will be considered in detail in a subsequent paper.

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References

[1] V.N. Gribov and L.N. Lipatov, Sov.J. Nucl.Phys. **15** (1972) 438; 675; L.N. Lipatov, Sov.J. Nucl.Phys. **20** (1975) 94; Y.L. Dokshitser, JETP **46** (1977) 641; G. Altarelli and G. Parisi, Nucl.Phys. **126** (1977) 298.

[2] S.J. Brodsky, and G.P. Lepage, Phys.Lett **B87** (1979) 359; Phys. Rev **D22** (1980) 2157.

[3] E.G. Floratos, R. Lacaze and C.Kounnas, Phys. Lett **B98** (1981) 89; 285.

[4] F.M. Dittes and A.V. Radyushkin, Phys.Lett **B134** (1984) 359; M.H. Sarmadi, Phys.Lett **B143** (1984) 471; S.V. Mikhailov and A.V.Radyushkin, *"Evolution kernel for the pion wave function: two loop QCD calculation in Feynman gauge."*. Dubna preprint JINR P2-83-721 (1983).

[5] S.V. Mikhailov and A.V.Radyushkin, Nucl.Phys. **B254** (1985) 89.

[6] S.A. Larin, T. van Ritbergen, J.A.M. Vermaseren, Nucl.Phys. **B427** (1994) 41; S.A. Larin, P. Nogueira,T. van Ritbergen, J.A.M. Vermaseren, *The three loop QCD calculation of the moments of deep inelastic structure functions.* NIKHEF-96-010, hep-ph/9605317

[7] G. Parente, A.V. Kotikov, V.G. Krivokhizhin, Phys. Lett. **B333** (1994) 190; A. L. Kataev, A. V. Kotikov, G. Parente and A.V. Sidorov, Phys. Lett. **B388** (1996) 179;

[8] J. A. Gracey, Phys. Lett. **B322** (1994) 141;

[9] J. A. Gracey, *Renormalization group functions of QCD in large-N_f*, talk presented at Third International Conference on the Renormalization Group, JINR, Dubna, Russia, 26-31 August, 1996, hep-th/9609164.

[10] S.V. Mikhailov and A.V.Radyushkin, Nucl.Phys. **B273** (1986) 297.

[11] L. Mankiewicz, M. Maul and E. Stein, *Perturbative Part of the Non-Singlet Structure Function F_2 in the Large-N_F Limit* TUM/T39-97-6, hep-ph/9703356

[12] A.A. Vladimirov, Theor.Math. Phys. **36** (1978) 732; **43** (1980) 417.

[13] J.A.M. Vermaseren, Symbolic Manipulation with FORM, Version 2, Tutorial and Reference Manual (Computer Algebra Nederland, Amsterdam,1991).

[14] L. V. Avdeev, Comp.Phys.Commun. **98** (1996) 15; L.V. Avdeev, J. Fleischer, S. Mikhailov, O. Tarasov, Phys. Lett **B336** (1994) 560; L.V. Avdeev, J. Fleischer, M. Yu. Kalmykov, M. Tentyukov, *"Towards automatic analytic evaluation of massive Feynman diagrams"*, hep-ph/9610467.

[15] F.M. Dittes, D. Müller, D. Robaschik, B. Geyer and J. Horejsi, Phys. Lett **B209** (1988) 325.

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Вклад ренормалонных цепочек
в несинглетные ядра эволюции в модели $[\varphi^3]_6$ и КХД

Вычислены вклады в несинглетные ядра эволюции: $P(z)$ — для уравнений ДГЛАП и $V(x,y)$ — для уравнений Бродского–Лепажа, происходящие от определенных классов диаграмм, включающих ренормалонные цепочки. Получены замкнутые выражения для вкладов двух классов диаграмм. Вычисления проведены в $\overline{\text{MS}}$ схеме в модели $[\varphi^3]_6$ и КХД. Вклады для одного из классов диаграмм доминируют при большом числе ароматов $N_f >> 1$. Для вклада этого класса диаграмм в $V(x,y)$ получено простое решение уравнения Бродского–Лепажа.

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Mikhailov S.V. E2-97-188

Renormalon Chains Contributions
to the Non-Singlet Evolution Kernels in $[\varphi^3]_6$ and QCD

The contributions to non-singlet evolution kernels $P(z)$ for the DGLAP equation and $V(x,y)$ for the Brodsky–Lepage evolution equation are calculated for certain classes of diagrams which include the renormalon chains. Closed expressions are obtained for the sums of contributions associated with these diagram classes. Calculations are performed in the $[\varphi^3]_6$ model and QCD in the $\overline{\text{MS}}$ scheme. The contribution for one of the classes of diagrams dominates for a number of flavors $N_f >> 1$. For the latter case, a simple solution to the Brodsky–Lepage evolution equation is obtained.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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