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G.N.Afanasiev*

SOME REMARKABLE CURRENT CONFIGURATIONS

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*E-mail: afanasev@thsun1.jinr.dubna.su

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1 Introduction

We study current configurations with a number of interesting properties. For example, a semi-infinite cylinder with the circular currents flowing on its surface generates the magnetic field which is very similar to that of magnetic monopole [1].

Further, we find that the semi-infinite cylinder densely covered by the toroidal solenoids with the linear rising currents in their windings produces electric field which is very alike to that of electric charge. Qualitatively, these conclusions were obtained earlier in Refs. [2].

At last, the torus densely covered by the toroidal solenoids with the linear rising currents in their windings gives static electric field which differs from zero only inside the torus. This electric field is adequately described by the electric vector potential (VP), rather than by the scalar one.

We consider the closed circular magnetic ring C encircling the cylindrical solenoid with the constant magnetic flux in it. Suppose that initially there is no current in C. Let this ring be cooled. At some temperature T_c the transition to the superconductive state occurrs. At this moment the supercurrent in C arises despite the fact that C is located in the region where magnetic field $\vec{H} = 0$. Qualitatively this was predicted in Refs. [3,4]. We evaluated the value of the supercurrent and the magnetic field produced by this current. This can be checked experimentally.

2 Currents, Magnetic Dipoles and Monopoles

The magnetic field created by the magnetic dipole $\vec{m}$ located at the origin is (see,e.g.,[5]):

$$\vec{B} = \frac{1}{r^3}[\frac{3}{r^2}\vec{r}(\vec{m}\vec{r}) - \vec{m}] + \frac{8\pi}{3}\vec{m}\delta^3(\vec{r}). \qquad (2.1)$$

Sometimes in a physical literature another representation of $\vec{B}$ is used [5,6]:

$$\vec{B} = \frac{1}{r^3}[\frac{3}{r^2}\vec{r}(\vec{m}\vec{r}) - \vec{m}] - \frac{4\pi}{3}\vec{m}\delta^3(\vec{r}). \qquad (2.2)$$

This difference is due to the following reason [7]. If we identify the magnetic dipole with the electric current flowing in the infinitely small circular current , then VP is given by

$$\vec{A} = \frac{1}{cr^2}(\vec{m}_e \times \vec{r}), \quad \vec{m}_e = \frac{1}{2}\int(\vec{r}\times\vec{j})dV. \tag{2.3}$$

Applying to $\vec{A}$ the *curl* operator and using the identity (see, e.g., [8])

$$\frac{\partial}{\partial x_j}(\frac{x_i}{r^3}) = \frac{1}{r^3}(\delta_{ij} - 3\frac{x_i x_j}{r^2}) + \frac{4\pi}{3}\delta_{ij}\delta^3(\vec{r}), \tag{2.4}$$

we get (2.1). Note that in the absence of medium $\vec{B} = \vec{H}$ and $\vec{D} = \vec{E}$.

On the other hand, if we suggest that magnetic dipoles consist of the magnetic monopoles

$$\vec{m}_m = \int \rho_m \vec{r} dV, \quad \int \rho_m dV = 0, \tag{2.5}$$

then the magnetic induction is obtained from the scalar magnetic potential:

$$\vec{B} = \vec{H} = -\vec{\nabla}\Phi_m, \quad \Phi_m = \frac{\vec{m}_m \vec{r}}{r^3}. \tag{2.6}$$

Again, using the differentiation rule (2.4) we arrive at (2.2). This means that different coefficients at $\delta^3(\vec{r})$ terms in (2.1) and (2.2) are due to different definitions of magnetic dipoles.

Consider a semi-infinite cylindrical solenoid of the radius R formed either of the circular currents or the magnetic current dipoles (Fig. 1). For $R \to 0$ the magnetic VP of a particular current lying in the $z = z_0$ plane is given by (2.3) where one should change z by $z - z_0$. Or, explicitly,

$$\vec{A} = A_\phi \vec{n}_\phi, \quad A_\phi \approx \frac{\pi R j \rho}{c\tilde{r}^3}, \quad \tilde{r} = [x^2 + y^2 + (z - z_0)^2]^{1/2}.$$

The nonvanishing components of magnetic strength are

$$H_x = \frac{\pi R j}{c}\frac{\partial^2}{\partial x \partial z}\frac{1}{\tilde{r}}, \quad H_y = \frac{\pi R j}{c}\frac{\partial^2}{\partial y \partial z}\frac{1}{\tilde{r}},$$

$$H_z = \frac{\pi R j}{c}[\frac{\partial^2}{\partial z^2}\frac{1}{\tilde{r}} + 4\pi\delta(x)\delta(y)\delta(z - z_0)]. \tag{2.7}$$

The magnetic field of semi-infinite solenoid is obtained by integrating $\vec{H}$ from $z_0 = -\infty$ to $z_0 = 0$. This results in

$$\vec{B} = \vec{H} = \frac{\pi R j}{c}[\frac{\vec{r}}{r^3} + 4\pi\vec{n}_z\delta(x)\delta(y)\Theta(-z)]. \tag{2.8}$$

Thus, an infinitely thin semi-infinite magnetized filament generates the field of a magnetic monopole everywhere except for the position of the filament itself. Due to the presence of the δ function terms in (2.8) thus obtained monopoles are not true ones. Similar results were obtained earlier in [1].

3 Current Electrostatics

Consider a semi-infinite cylinder C densely covered by the infinitely thin toroidal solenoids. For simplicity, consider the case when the radius of C tends to zero. In the limit one obtains a semi-infinite filament composed of the toroidal moments μ_t. The VP of a particular toroidal moment lying at $z = z_0$ is [9,10]

$$A_x = \mu_t \frac{\partial^2}{\partial x \partial z}\frac{1}{\tilde{r}}, \quad A_y = \mu_t \frac{\partial^2}{\partial y \partial z}\frac{1}{\tilde{r}},$$

$$A_z = \mu_t[\frac{\partial^2}{\partial z^2}\frac{1}{\tilde{r}} + 4\pi\delta(x)\delta(x)\delta(z - z_0)], \quad \tilde{r} = \sqrt{x^2 + y^2 + (z - z_0)^2}.$$

To obtain the VP of the semi-infinite filament composed of the toroidal moments, we integrate these equations from $z_0 = -\infty$ to $z_0 = 0$:

$$A_x = \mu_t \frac{x}{r^3}, \quad A_y = \mu_t \frac{y}{r^3}, \quad A_z = \mu_t[\frac{z}{r^3} + 4\pi\delta(x)\delta(y)\Theta(-z)], \quad div\vec{A} = 0.$$

Let in the windings of toroidal solenoids covering the surface of C flows the current linearly rising with time. The VP of a particular infinitely small solenoid located at $z = z_0$ is given by [9,11,12]:

$$A_x = t\dot{\mu}_t \frac{\partial^2}{\partial x \partial z}\frac{1}{\tilde{r}}, \quad A_y = t\dot{\mu}_t \frac{\partial^2}{\partial y \partial z}\frac{1}{\tilde{r}},$$

$$A_z = t\dot{\mu}_t[\frac{\partial^2}{\partial z^2}\frac{1}{\tilde{r}} + 4\pi\delta(x)\delta(x)\delta(z - z_0)], \quad div\vec{A} = 0.$$

Here $\dot{\mu}_t$ is the constant characterizing the rate of the current change. The total VP of the semi-infinite filament densely covered by the infinitely small toroidal solenoids with time-dependent currents in their windings is obtained by integrating these equations from $z_0 = -\infty$ to $z_0 = 0$:

$$A_x = t\dot{\mu}_t \frac{x}{r^3}, \quad A_y = t\dot{\mu}_t \frac{y}{r^3}, \quad A_z = t\dot{\mu}_t[\frac{z}{r^3} + 4\pi\delta(x)\delta(y)\Theta(-z)], \quad div\vec{A} = 0..$$

To this semi-infinite filament corresponds the static electric field

$$\vec{D} = \vec{E}, \quad E_x = -\dot{\mu}_t \frac{x}{cr^3}, \quad E_y = -\dot{\mu}_t \frac{y}{cr^3},$$

$$E_z = -\frac{\dot{\mu}_t}{c}[\frac{z}{r^3} + 4\pi\delta(x)\delta(y)\Theta(-z)], \quad div\vec{E} = 0.$$

and the singular magnetic field confined to the negative z semi-axis

$$\vec{B} = \vec{H} = \vec{n}_\phi H_\phi, \quad H_\phi = -4\pi t\dot{\mu}_t\Theta(-z)\frac{d}{d\rho}\frac{\delta(\rho)}{2\pi\rho}, \quad div\vec{H} = 0..$$

The resulting electromagnetic field coincides with that of the point electric charge $e = -\dot{\mu}_t/c$ everywhere except for the semi-infinite filament (left part of Fig. 2).
The equalities $div\vec{D} = 0$, $\int D_n d\Omega = 0$ guarantee the absence of free charges.

The same electric field may be also realized via two linearly-rising currents flowing in opposite directions along the cylindrical surfaces parallel to the z axis (right part of Fig. 2).

Qualitatively, these results were predicted earlier by M.A. Miller [2] who pointed out on the possibility to simulate the charge distributions by the time-dependent currents. He referred to it as to "current electrostatics". The present study may be viewed as a concrete realization of his ideas.

Another interesting configuration is the torus T densely covered by the toroidal solenoids (Fig. 3) [9]. Let in the windings of these solenoids flows the current linearly rising with time. Then, the static electric field $\vec{E}$ differs from zero only inside T and on its surface; the magnetic field $\vec{H}$ differs from zero only on the surface of C. The electric scalar potential is everywhere zero, the linearly rising with time magnetic vector potential differs from zero only inside T and on its surface. Vectors $\vec{E}$ and $\vec{H}$ have the same direction, the vector $\vec{H}$ is orthogonal to them. Since $div\vec{E} = 0$, $\vec{E}$ can be presented in the form $\vec{E} = curl\vec{A}_e$. Applying the Stokes theorem to the contour passing through the torus hole, one gets

$$\int\int \vec{E}d\vec{S} = \int \vec{A}_e d\vec{l}.$$

This means that outside the torus there is electric vector potential which cannot be eliminated by the gauge transformation as the gauge invariant quantity (electric field flux) $\int\int \vec{E}d\vec{S}$ differs from zero. There are known attempts (see, e.g. [13] and refs. therein) to measure the electric field

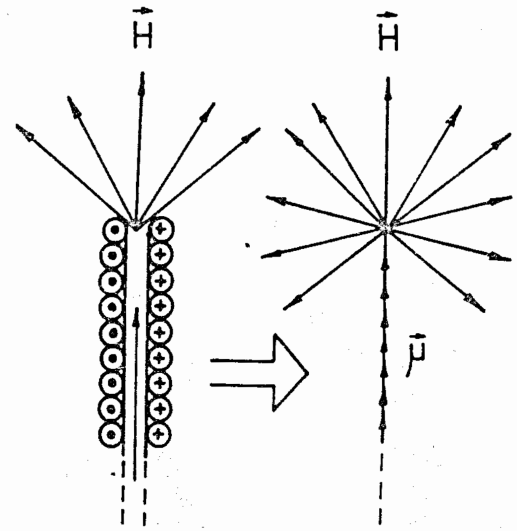

Fig.1: The magnetic fields of the semi-infinite solenoid and the magnetized filament coincide with the field of magnetic monopole everywhere except for the position of solenoid or filament.

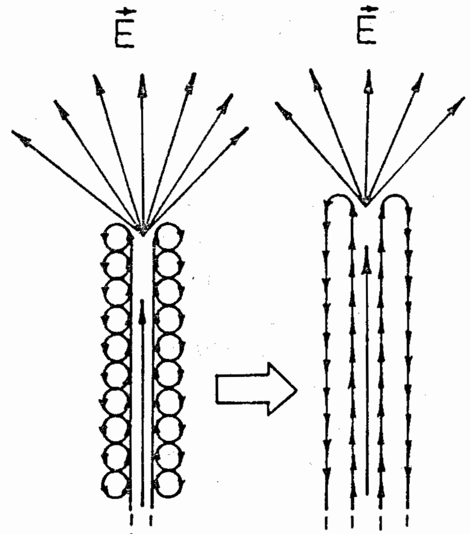

Fig.2: A semi-infinite set of infinitely thin toroidal solenoids with linear rising currents in their windings (left part of figure) and linear rising currents flowing along the semi-infinite parallel cylindrical surfaces (right part) generate the field of an electric charge everywhere except for the position of the cylinder .

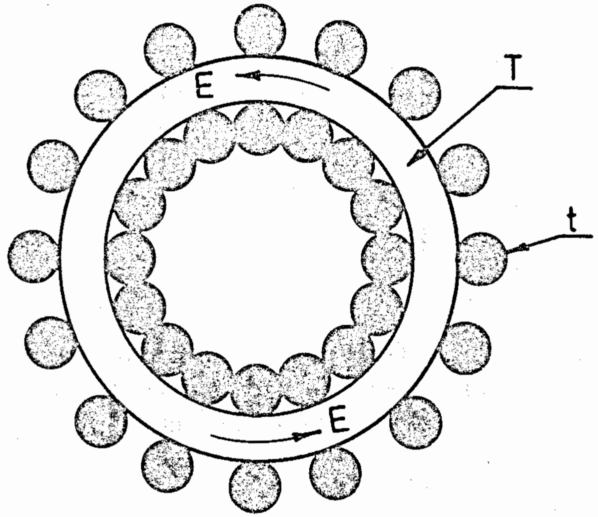

Fig.3: The torus T is densely covered by the infinitely thin toroidal solenoids t (only few of them are shown) in the windings of which flows the current linearly rising with time. The magnetic field $\vec{H}$ differs from zero only inside t (that is, on the surface of T in the limit of infinitely thin t), while independent of time electric field $\vec{E}$ differs from zero inside T. The scalar electric potential is everywhere zero. It turns out that $\vec{E} = curl\vec{A}_e$, where electric vector potential $\vec{A}_e \neq 0$ everywhere. The Stokes theorem (see the text) ensures us that $\vec{A}_e$ can not be removed by the gauge transformation.

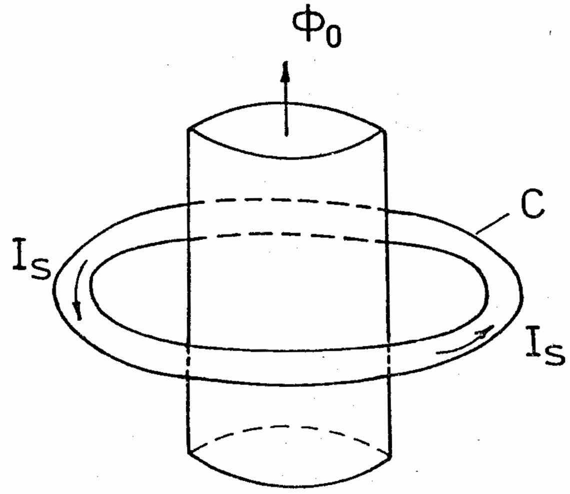

Fig.4: The cylindrical solenoid with magnetic flux Φ_0 is encircled by the metallic ring C. When C becomes superconductive, the supercurrent I_s arises on its surface.

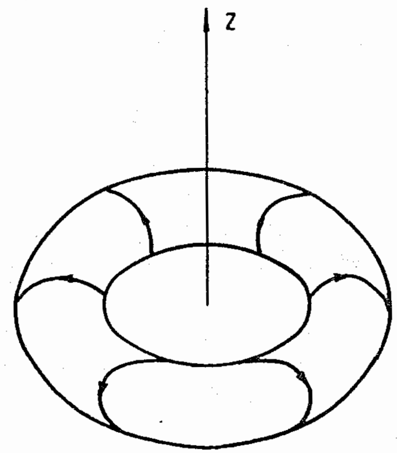

Fig.5: The lines with arrows mean the poloidal current flowing on the torus surface.

arising from the stationary currents. Maxwell's theory negates the existence of this field. On the other hand, we have seen that there exist non-static current configurations generating the static electric field. Excellent measurements of the static electric fields produced by the time-dependent currents have been reported in book [14].

4 On the supercurrent arising in a superconducting ring

Consider the closed circular metallic ring C encircling the infinite cylindrical solenoid with a constant flux Φ_0 in it (Fig.4). Suppose that initially there is no current in C. Let the ring C be cooled. At some temperature T_c the transition to the superconductive state occurs. The following two properties were observed experimentally [15-17] and explained theoretically [18-20] :
1) Magnetic field $\vec{H}$ vanishes inside C (it is, therefore, assumed that penetration depth is zero);
2) The total magnetic flux trapped by C turns out to be integer (in units $hc/2e$).
The appearance of the supercurrent flowing on the surface of C (despite its location in a field-free region where $\vec{E} = \vec{H} = 0$) for $T < T_c$ was predicted in refs. [3, 4]. Indeed, as the flux inside the cylindrical solenoid is not in general integer, the supercurrent in C arises making the total flux to be integer.
This supercurrent was, in fact, observed in Tonomura experiments (see refs.[17,20] where this fact was clearly stated). It is our aim to evaluate explicitly the distribution of supercurrent on the surface of C and the arising magnetic field.
The density of the current J_s flowing on the torus C surface and providing $\vec{H} = 0$ inside C was obtained in [11]. Let the surface of C be given by

$$(\rho - d)^2 + z^2 = R^2$$

. It is convenient to introduce toroidal coordinates

$$\rho = a\frac{\sinh\mu}{\cosh\mu - \cos\theta}, \qquad z = a\frac{\sin\theta}{\cosh\mu - \cos\theta}, \qquad \phi = \phi. \tag{4.1}$$

For a given value of μ the points ρ, z, ϕ (where ρ, z, ϕ are defined in (4.1)) fill the surface of the torus with the parameters $d = a\coth\mu$, $R = a/\sinh\mu$.

Let $\mu = \mu_0$ corresponds to the surface of C. Then, the surface current providing the vanishing of $\vec{H}$ inside C is given by [65]:

$$\vec{J}_s = \delta(\mu - \mu_0) j(\theta) \vec{n}_\phi$$

$$j(\theta) = -\frac{C_0}{2\sqrt{2}\pi^2 a^2}\frac{(\cosh\mu_0 - \cos\theta)^{5/2}}{\sinh\mu_0}\sum\frac{\cos n\theta}{1+\delta_{n0}}[P^1_{n-1/2}(\cosh\mu_0)]^{-1}.$$

This current gives the following VP

$$A_\phi = C_0\frac{\cosh\mu - \cos\theta}{\sinh\mu}$$

inside $C(\mu > \mu_0)$ and

$$A_\phi = C_0\frac{\sqrt{2}}{\pi}(\cosh\mu - \cos\theta)^{1/2}\times$$

$$\times\sum\frac{\cos n\theta}{1+\delta_{n0}}\frac{1}{n^2 - 1/4}\frac{Q^1_{n-1/2}(\cosh\mu_0)}{P^1_{n-1/2}(\cosh\mu_0)}P^1_{n-1/2}(\cosh\mu).$$

outside $C(\mu < \mu_0)$. In particular, on the circle $z = 0, \rho = d - R$ (that is, for $\mu = \mu_0, \theta = \pi$) one gets

$$A_\phi = C_0\frac{1 + \cosh\mu_0}{\sinh\mu_0}.$$

The integral

$$\oint A_\phi dl = 2\pi C_0 a$$

taken along the same circle coincides with the flux Φ_s of the magnetic field produced by the supercurrent J_s. The total magnetic flux trapped by the superconducting ring is the sum of the cylinder solenoid flux Φ_0 and the supercurrent flux Φ_s:

$$2\pi C_0 a + \Phi_0 = \frac{hcn}{2e}$$

where n is the integer nearest to $2e\Phi_0/hc$. From this we find C_0

$$C_0 = -(\Phi_0 - \frac{hcn}{2e})/2\pi a$$

The corresponding magnetic field is given by

$$H_\mu = \frac{(\cosh\mu - \cos\theta)^2}{a\sinh\mu}\frac{\partial}{\partial\theta}(\frac{\sinh\mu A_\phi}{\cosh\mu - \cos\theta}),$$

$$H_\theta = -\frac{(\cosh\mu - \cos\theta)^2}{a\sinh\mu}\frac{\partial}{\partial\mu}\left(\frac{\sinh\mu A_\phi}{\cosh\mu - \cos\theta}\right)$$

At large distances VP and field strengths fall like r^{-2} and r^{-3}, resp.:

$$A_\phi \sim \frac{2a^2}{\pi r^2}\sin\theta_s C_1, \quad C_1 = C_0 \sum \frac{1}{1+\delta_{n0}} \frac{Q_{n-1/2}(\cosh\mu_0)}{P_{n-1/2}(\cosh\mu_0)}$$

$$H_r \sim \frac{4a^2}{\pi r^3}\cos\theta_s \cdot C_1, \quad H_\theta \sim \frac{2a^2}{\pi r^3}\sin\theta_s \cdot C_1.$$

It turns out that the cooling of the ring C below the critical temperature T_c inevitably leads to the appearance of the magnetic field in a space surrounding C.

It would be interesting to observe this supercurrent experimentally. Theoretically, in Tonomura experiments the reason for the quantization of the total magnetic flux trapped by the toroidal solenoid is the appearance (for $T < T_c$) of the poloidal supercurrent on the torus surface. But the poloidal supercurrent (Fig.5) produces no magnetic field outside the toroidal solenoid. Thus, the magnetic flux quantization observed in Tonomura experiments is only indirect evidence of the supercurrent existence. On the other hand, the supercurrent arising in a circular turn embracing either cylindrical or toroidal solenoids may be observed by the detection of the magnetic field (4.3) created by this supercurrent. There are many experiments in which the dependence of the physical parameters (e.g., resistivity) of the multi-connected sample embracing the magnetic flux (but lying outside the region where $H = 0$) was studied as a function of magnetic flux value (see, e.g., Resource Letter QIMS-1: Quantum interference in macroscopic samples [21]). Like in Tonomura experiments, the arising supercurrent is not measured directly, but its existence is needed for the explanation of experimental data.

5 References

1. Goddard P. and Olive D.I., 1978, Rep. Prog.Phys., 41, 1361.
2. Miller M.A., 1986, Izvestya Vysch. Uchebnuh Zavedenej, ser. Radiofizika, 29, 991.
Miller M.A., 1984, Uspekhi Fiz.Nauk, 142, 147;
3. Liang J.Q. and Ding X.X, 1988, Phys. Rev. Lett., 9, 1987.
4. Dubovik V.M. and Shabanov S.V., 1989, Phys.Lett.A, 142, 211.
5. Jackson J.D., 1975, Classical Electrodynamics (John Wiley, New York).
6. Harrison M.J. and Spence R.D., 1994, Amer. J. Phys., 62, 828.
7. Griffiths D.J., 1992, Amer.J.Phys., 60, 979.
8. Frahn C.P., 1983, Amer.J.Phys., 51, 826.
9. Afanasiev G.N., Nelhiebel M. and Stepanovsky Yu.P., 1994, JINR Preprint E2-94-297; Afanasiev G.N., Nelhiebel M. and Stepanovsky Yu.P., 1996, Physica Scripta, 54, 417.
10. Dubovik V.M. and Kurbatov V.M., 1995, 'Multipole interaction of dipole and spin systems with external fields', In: Proc. Int. Workshop on Quantum Systems. New Trends and Methods, p.117-124, Minsk, May 23-28, 1994
11. Afanasiev G.N., 1990, J. Phys. A, 23, 5755.
12. Afanasiev G.N. and Stepanovsky Yu.P., 1995, J.Phys.A, 28, 4565.
13. Kenyon C.S. and Edwards D.F., 1991, Phys. Lett. A, 156, 391; Lemon D.K., Edwards W.F. and Kenyon C.S., 1992, Phys.Lett. A, 162, 105.
14. Ryazanov G.A., 1969, Electric simulation using solenoidal fields, (Nauka, Moscow).
15. Deaver B.S., Fairbank W.M., 1961, Phys.Rev.Lett., 7, 43-46.
16. Doll R. and Nabauer M., 1961, Phys.Rev.Lett., 7, 51.
17. Tonomura A., 1995, Nuovo Cimento B, vol.110, No 5-6, p.571-584.
18. Byers N. and Yang C.N., 1961, Phys.Rev.Lett., 7, 46.
19. Onsager L., 1961, Phys.Rev.Lett., 7, 50.
20. Tonomura A. and Fukuhara A., 1989, Phys.Rev.Lett., 62, 113.
21. Das Sarma S., Kawamura T. and Washburn S., 1995, Amer.J.Phys., vol.63, p.683.

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Афанасьев Г.Н. E2-97-166

О некоторых замечательных распределениях токов

Изучаются распределения токов, имитирующих поле магнитного монополя, электрического заряда и статического электрического поля, заключенного в тороидальной полости.

Найдены величина тока, возникающего в кольцевом сверхпроводящем проводнике, охватывающем магнитный поток, и сопутствующее этому току магнитное поле.

Работа выполнена в Лаборатории теоретической физики им.Н.Н.Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1997

Afanasiev G.N. E2-97-166

Some Remarkable Current Configurations

We investigate the current configurations imitating the field of the magnetic monopole, of the electric charge and the electrostatic field filling the toroidal cavity.

The values of the supercurrent arising in the superconducting coil embracing the magnetic flux and of the associated magnetic field are evaluated.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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