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NONLINEAR REALIZATIONS OF THE (SUPER)DIFFEOMORPHISM GROUPS, GEOMETRICAL OBJECTS AND INTEGRAL INVARIANTS IN THE SUPERSPACE

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Dedicated to the memory of
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1 Introduction

As was shown in [1], gravity can be realized as a nonlinear realization of the four dimensional diffeomorphism group. The consideration was based on the fact that infinite dimensional diffeomorphism group in four dimensional space can be represented as the closure of two finite dimensional groups - conformal and affine [2]. As a consequence of such representation of the diffeomorphism group, the basic field in this consideration was the symmetric tensor field of the second rank - the metric field $g_{m,n}$, which corresponds to symmetric generators of the affine group.

The generalization of this approach to the case of superspace was given in [3].

In the present work we consider nonlinear realization of the whole infinite dimensional diffeomorphism group of the arbitrary (super)space. Among the coordinates parametrizing the group element (coset space) in such realization there are present usual coordinates of the (super)space. The (super)vielbein and (super)connection are naturally represented as the functions of other coordinates of the coset space. The structure of the connection in the purely bosonic case is such that the corresponding torsion is zero. In the superspace only some components of the torsion, namely T^a_{bc} and $T^\alpha_{\beta\gamma}$, vanish automatically.

2 Bosonic space

Firstly we consider the case of the usual D - dimensional bosonic space with the coordinates $s^m, m = 0, 1, ..., D-1$. The generators of the corresponding diffeomorphism group regular at the origin can be written in the coordinate representation as

$$P^{m_1,m_2,...,m_n}{}_m = i s^{m_1} s^{m_2} ... s^{m_n} \frac{\partial}{\partial s^m}. \tag{2.1}$$

All of this generators can be naturally ordered in accordance with their dimensionality ($dim\ s^m = +1$):

$$dim\ P_m = -1,\ dim\ P^{m_1}{}_m = 0,\ dim\ P^{m_1,m_2}{}_m = +1,\ \dots . \tag{2.2}$$

With the help of the representation (2.1) one can calculate the commutation relations between the generators of the diffeomorphism group and after that we can forget about the auxiliary coordinates s^m. The only we will need is the following algebra:

$$\begin{aligned} &[P^{m_1,m_2,...,m_n}{}_m , P^{k_1,k_2,...,k_l}{}_k] = \\ &i \sum_{i=1}^{l} \delta^{k_i}_m P^{m_1,...m_n,k_1...k_{i-1}k_{i+1}...,k_l}{}_k - i \sum_{j=1}^{n} \delta^{m_j}_k P^{m_1,...m_{j-1}m_{j+1}...m_n,k_1...,k_l}{}_m . \end{aligned} \tag{2.3}$$

Let us consider the following parametrization of the group element:

$$G = e^{ix^m P_m} e^{i\phi^l{}_{l_1,l_2} P^{l_1,l_2}{}_l} e^{i\phi^k{}_{k_1,k_2,k_3} P^{k_1,k_2,k_3}{}_k} ... e^{i\phi^n{}_{n1} P^{n1}_n} . \tag{2.4}$$

All parameters in the expression (2.4) are symmetric with respect to the permutation of lower indices as a consequence of the symmetry properties of the generators (2.1). It is convenient to take the element of the finite dimensional group $GL(D)$, generated by $P^{n_1}_n$, as the last multiplier in the expression (2.4). The rest of the factors in (2.4) are ordered with respect to the dimensionality of the generators. As a consequence, the product of factors to the right from arbitrary one form a subgroup of the diffeomorphism group. Such structure of the group element simplifies the evaluation of the variations $\delta\phi^l{}_{l_1,\ldots,l_n}$ under the infinitesimal left action

$$G' = (1 + i\epsilon)G, \tag{2.5}$$

where $\epsilon = \epsilon^m P_m + \epsilon^m{}_{m_1} P^{m_1}{}_m + \epsilon^m{}_{m_1,m_2} P^{m_1,m_2}{}_m + \ldots$ belongs to the algebra of the diffeomorphism group. The coordinates in (2.4) transform through the infinitesimal transformation parameters $\epsilon^m{}_{m_1,m_2,\ldots,m_k}$ and coordinates wich are placed to the left from given ones in parametrization (2.4):

$$\delta x^m = \delta x^m(\epsilon, x^i), \ \delta\phi^n{}_{n_1} = \delta\phi^n{}_{n_1}(\epsilon, x^m, \phi^k{}_{k_1}), \tag{2.6}$$

$$\delta\phi^l{}_{l_1,l_2} = \delta\phi^l{}_{l_1,l_2}(\epsilon, \ x^i, \ \phi^k{}_{k_1,k_2}), \ldots. \tag{2.7}$$

The only exception is the transformation law for $\phi^n_{n_1}$, which includes only ϵ, x^m and $\phi^n_{n_1}$ itself. At this stage it is natural to consider all parameters as the fields in D-dimensional space parametrized by coordinates x^m.

Step by step one can evaluate the variations of all parameters of the coset. The general method of calculations is as follows. To find the variation $\delta\phi^l{}_{l_1,\ldots,l_n}$ we have to solve the equation

$$(1 + i\epsilon)e^{i\phi^n} = e^{i\phi^n + i\delta\phi^n}(1 + i\tilde{\epsilon}). \tag{2.8}$$

where, for the brevity, $\phi^n = \phi^l{}_{l_1,\ldots,l_n} P^{l_1,\ldots,l_n}{}_l$ and parameter ϵ contains the generators with n and more upper indices. Correspondingly, $\tilde{\epsilon}$ contains the generators with $n+1$ or more upper indices. Both of these parameters contain P^m_n.

From (2.13) it simply follows:

$$ie^{-i\phi^n}\epsilon e^{i\phi^n} = e^{-i\phi^n}\delta e^{i\phi^n} + i\tilde{\epsilon}. \tag{2.9}$$

Both right and left part of this equation can be written in terms of multiple commutators

$$e^{-i\phi^n} \wedge \epsilon = \frac{e^{-i\phi^n} - 1}{i\phi^n} \wedge \delta\phi^n + \tilde{\epsilon}, \tag{2.10}$$

where, for simplicity, we use the notation

$$e^{-i\phi^n} \wedge \epsilon = \epsilon + \frac{1}{1!}[-i\phi^n, \epsilon] + \frac{1}{2!}[-i\phi^n, [-i\phi^n, \epsilon]] + \ldots. \tag{2.11}$$

The equation (2.10) is the basic equation for $\delta\phi^n$ and $\tilde{\epsilon}$.

The simplest transformation law have the dimension-one coordinates x^m. They transform as the coordinates of the D-dimensional space under the reparametrization:

$$\delta x^m = \varepsilon^m(x) \equiv \epsilon^m + \epsilon^m_{m_1} x^{m_1} + \epsilon^m_{m_1 m_2} x^{m_1} x^{m_2} + \ldots. \tag{2.12}$$

Here $\varepsilon^m(x)$ is infinitesimal function of the coordinates x^n. This is a consequence of first among the relations (2.13):

$$(1 + i\epsilon)e^{ix^m P_m} = e^{i(x^m + \delta x^m)P_m}(1 + i\tilde{\epsilon}), \tag{2.13}$$

in which δx^m is given by (2.12) and:

$$\tilde{\epsilon} = \frac{1}{1!}\partial_{m_1}\epsilon^m P^{m_1}{}_m + \frac{1}{2!}\partial_{m_1 m_2}\epsilon^m P^{m_1,m_2}{}_m + \ldots. \tag{2.14}$$

The next parameters in the coset have three indices and transform as a Cristoffel symbol:

$$\delta\phi^m{}_{m_1 m_2} = \frac{\partial\varepsilon^m}{\partial x^n}\phi^n{}_{m_1 m_2} - \frac{\partial\varepsilon^n}{\partial x^{m_2}}\phi^m{}_{m_1 n} - \frac{\partial\varepsilon^n}{\partial x^{m_1}}\phi^m{}_{n m_2} + \frac{1}{2}\frac{\partial^2\varepsilon^m}{\partial x^{m_1}\partial x^{m_2}}. \tag{2.15}$$

In general the transformation law for parameter with n lower indices will contain the term with n-th derivative of infinitesimal parameter $\epsilon^m(x)$.

Only variations of the last parameters $\phi^n{}_{n_1}$ need the separate consideration. To find them one have to evaluate the expression

$$\delta(e^{i\phi^n{}_{n1} P^{n_1}_n}) = i\frac{\partial\varepsilon^m(x)}{\partial x^k} P^k_m \, e^{i\phi^n{}_{n1} P^{n_1}_n} \tag{2.16}$$

The simplest way to do this is to use the matrix representation for the generators of $GL(D)$ group:

$$(P^{n_1}_n)^l_k = i\delta^{n_1}_k \delta^l_n. \tag{2.17}$$

In this representation the element of $GL(D)$ group is the exponent of the matrix ϕ^m_n:

$$(e^{i\phi^n{}_{n_1} P^{n_1}_n})^l_k = (e^{-\phi})^l_k \equiv E^l_k. \tag{2.18}$$

It is convenient to consider the matrix E^l_k instead of ϕ^m_n because its transformation law is very simple:

$$\delta E^l_k = -\frac{\partial\varepsilon^m(x)}{\partial x^k} E^l_m. \tag{2.19}$$

It means that the E^l_k transforms as the covariant vector with respect to its lower index. Simultaneously, its upper index is inert. This is the transformation law of the vielbein.

The fact that E^l_k is endeed the vielbein becomes evident if we consider the Cartan's differential form

$$\Omega = G^{-1}dG = i\Omega^a P_a + i\Omega^b_a P^a_b + i\Omega^b_{a_1 a_2} P^{a_1 a_2}_b + \ldots, \tag{2.20}$$

which simultaneously with its components (Ω^a, $\Omega^a_b, ...$) is invariant with respect to the left transformation (2.5). We emphasize the fact of invariance by using the letters from the beginning of the alphabet for indices. The explicit expressions for the components of the Ω -form are:

$$
\begin{aligned}
\Omega^a &= E^a_m dx^m, && (2.21)\\
\Omega^a_b &= -E^m_b dE^a_m - 2dx^k \phi^m_{kn} E^n_b E^a_m, && (2.22)\\
\Omega^a_{bc} &= (d\phi^m_{kn} - dx^l \phi^m_{il} \phi^i_{kn} + && (2.23)\\
&+dx^l \phi^m_{in} \phi^i_{kl} + dx^l \phi^m_{ik} \phi^i_{ln} - 3dx^l \phi^m_{knl}) E^a_m E^k_b E^n_c, \dots .
\end{aligned}
$$

The first of this forms is exactly one-form vielbein. The physical meaning of its index a becomes clear if we consider the right gauge transformation belonging to $GL(D)$

$$G' = G\{1 - ih(x)\} = G\{1 - ih^a_b(x) P^b_a\} \qquad (2.24)$$

Vielbein one -form $E^a \equiv \Omega^a$ transforms as the vector

$$\delta E^a = h^a_b E^b \qquad (2.25)$$

of this $GL(D)$, which can be considered as the gauge group in the tangent space. All Ω -forms with higher number of indices transform homogeneously as corresponding tensors. The only exception is the differential one-form (2.22) which transforms inhomogeneously:

$$\delta\Omega^a_b = h^a_c \Omega^c_b - \Omega^a_c h^c_b - dh^a_b. \qquad (2.26)$$

This is exactly the transformation law of the connection one-form and Ω^a_b is the natural candidate for the connection in the absence of any other tensors of second rank, which could be, in principle, added to the connection. So, the "minimal" one-form connection is given by (2.22) in terms of vielbein E^a_m and Cristoffel symbol ϕ^m_{kn}. The corresponding curvature two -form

$$R^a_b = d\Omega^a_b + \Omega^a_c \Omega^c_b \qquad (2.27)$$

transforms as a tensor of second rank.

Due to its definition, the Ω -form satisfy the Maurer-Cartan equation

$$d\Omega + \Omega \wedge \Omega = 0, \qquad (2.28)$$

or, in components

$$
\begin{aligned}
d\Omega^a &+ \Omega^a_b \Omega^b = 0, && (2.29)\\
d\Omega^a_b &+ \Omega^a_c \Omega^c_b + 2\Omega^a_{bc} \Omega^c = 0, && (2.30)\\
d\Omega^a_{bc} &+ \Omega^a_d \Omega^d_{bc} + \Omega^a_{bd} \Omega^d_c + \Omega^a_{cd} \Omega^d_b + 3\Omega^a_{bcd} \Omega^d = 0, \dots . && (2.31)
\end{aligned}
$$

The lefthand side of first of these equations (2.29) is the covariant differential of the vielbein with the connection Ω^a_b. The fact of its equality to zero means vanishing of

the corresponding torsion. Second equation represents the curvature two-form (2.27) in terms of vielbein and Ω-form with three indices

$$R^a_b = -2\Omega^a_{bc} \Omega^c. \qquad (2.32)$$

The rest of equations express covariant differentials of Ω-forms in terms of other Ω-forms.

The following expression for the action

$$S = \int R^{a_1}_b \eta^{b a_2} \Omega^{a_3} \dots \Omega^{a_D} \varepsilon_{a_1 a_2 \dots a_D} \qquad (2.33)$$

leads to Einstein - Gilbert action

$$S = \int d^D x \sqrt{-g} R. \qquad g^{mn} = \eta^{ab} E^m_a E^n_b. \qquad (2.34)$$

after elimination of ϕ^m_{kn} with the help of its equation of motion in terms of g^{mn}. Some comments are needed here. Up to now the gauge group in tangent space, considered as right transformations (2.24), was group $GL(D)$ of general linear transformations in
D - dimensions. In principle, one can construct action, invariant under the whole $GL(D)$ gauge group, for example $\int R^b_a R^a_b$ in four dimensions. The presence in the action (2.33) of two constant tensors - absolutely antisymmetric tensor $\varepsilon_{a_1 a_2 \dots a_D}$ and tangent space flat metric $\eta^{ab} = diag(1, 1, ..., 1, -1)$ means that the invariance group of the action (2.33) is the subgroup of $GL(D)$, namely, the group $SO(D-1,1)$. So, the choice of the gauge group in the tangent space depends on the structure constants in the action, which can break $GL(D)$ down to its subgroup.

3 Superspace

As a generalization of the approach we consider the diffeomorphism group of the superspace with coordinates s^M, from which D coordinates s^m. $m = 0, 1, ..., D-1$ are bosonic and D_G coordinates η^μ, $\mu = 1, 2, ..., D_G$ are grassmann. Both numbers D and D_G from the very beginning are arbitrary. The grassmann grading of the coordinates $g(s^m) = 0$, $g(s^\mu) = 1$ means the standard commutation relations: $s^M s^N - (-1)^{g(s^M) g(s^N)} s^N s^M = 0$, or, for the brevity, $s^M s^N - (-1)^{MN} s^N s^M = 0$. The generators of the algebra

$$P^{M_1, M_2, ..., M_n}{}_M = i s^{M_1} s^{M_2} \dots s^{M_n} \frac{\partial}{\partial s^M}. \qquad (3.1)$$

have the following dimensionalities:

$$
\begin{aligned}
&dim\ P_m = -1,\ dim\ P_\mu = dim\ P^\mu{}_m = -\frac{1}{2},\\
&dim\ P^{m_1}{}_m = dim\ P^{\mu_1}{}_\mu = dim\ P^{\mu_1 \mu_2}{}_m = 0, \qquad (3.2)\\
&dim\ P^{\mu_1 \mu_2}{}_\mu = dim\ P^{m_1 \mu}{}_m = +\frac{1}{2},\ dim\ P^{m_1 \mu_1}{}_\mu = dim\ P^{m_1 m_2}{}_m = +1, \dots .
\end{aligned}
$$

Some of the generators are bosonic and others (with halfinteger dimensionality) - fermionic with grassmann grading 0 or 1 correspondingly. The same grading 0 or 1 corresponds to bosonic m or fermionic μ indices. The algebra of the generators (3.1) is graded algebra:

$$P^{M_1,M_2,\dots,M_n}{}_M P^{N_1,N_2,\dots,N_k}{}_N - (-1)^{(M_1+\dots+M_n+M)(N_1+\dots+N_k+N)} P^{N_1,\dots,N_k}{}_N P^{M_1,\dots,M_n}{}_M =$$
$$i\sum_{l=1}^{k} \delta_{M}{}^{N}_{l}(-1)^{M(N_1+\dots+N_{l-1})} P^{M_1\dots M_n N_1\dots N_{l-1}N_{l+1}\dots N_k}{}_N - \tag{3.3}$$
$$-i\sum_{l=1}^{n} \delta_N{}^{M_l}(-1)^{(M_1+\dots+M_n+M)(N_1+\dots+N_k+N)+N(M_1+\dots+M_{l-1})} P^{N_1\dots N_k M_1\dots M_{l-1}M_{l+1}\dots M_n}{}_M.$$

It is convenient to parametrize the group element in the form

$$G = KH, \tag{3.4}$$

where

$$K = e^{ix^m P_m} e^{i\theta^\mu P_\mu} e^{iu^M{}_{M_1M_2} P^{M_2M_1}{}_M} e^{iu^N{}_{N_1N_2N_3} P^{N_3N_2N_1}{}_N} \dots, \tag{3.5}$$

$$H = e^{i\psi^m{}_\mu P^\mu{}_m} e^{i\phi^\nu{}_n P^n{}_\nu} e^{iu^k{}_l P^l{}_k} e^{iv^\rho{}_\sigma P^\sigma{}_\rho}. \tag{3.6}$$

The element H belongs to the finite - dimensional subgroup $GL(D, D_G)$ of the superdiffeomorphism group and its parameters have dimensions: $dim\ \psi^m{}_\mu = 1/2$, $dim\ \phi^\nu{}_n = -1/2$, $dim\ u^k{}_l = dim\ v^\rho{}_\sigma = 0$. The coset $K = G/H$ is parametrized by infinite number of the parameters with dimensions: $dim\ x^m = 1$, $dim\ \theta^\mu = 1/2$, $dim\ u^N{}_{N_1N_2}$ run from 0 to $-3/2$ etc.

Consider the element of the diffeomorphism algebra

$$\epsilon = \epsilon^M P_M + \epsilon^M{}_{M_1} P^{M_1}{}_M + \epsilon^M{}_{M_1M_2} P^{M_2M_1}{}_M + \dots. \tag{3.7}$$

with the constant infinitesimal coefficients. Under the left action

$$G' = (1 + i\epsilon)G, \tag{3.8}$$

the parameters $x^M = (x^m,\ \theta^\mu)$ transform as the coordinates of the (D, D_G) - dimensional superspace: $\delta x^M = \varepsilon^M(x)$, where

$$\varepsilon^M(x) = \epsilon^M + \epsilon^M{}_{M_1} x^{M_1} + \epsilon^M{}_{M_1M_2} x^{M_2} x^{M_1} + \dots. \tag{3.9}$$

The rest of the parameters in (3.6) transform in a more complicated way. Exactly as in the bosonic case the transformation laws of the parameters with n lower indices includes all parameters up to n lower indices and all derivatives of $\varepsilon^M(x)$ up to n -th order. The calculation of this transformation laws is complitely analogous to the purely bosonic case.

The next to the right after x^M are parameters with three indices: $u^M{}_{M_1M_2}$. They transform inhomogeneously as the Cristoffel symbols in the superspace with coordinates x^M

$$\delta u^M{}_{M_1M_2} = (-1)^{N(M+1)}\frac{\partial}{\partial x^N}\varepsilon^M u^N{}_{M_1M_2} - (-1)^{M_2(N+1)} u^M{}_{M_1N}\frac{\partial}{\partial x^{M_2}}\varepsilon^N -$$
$$(-1)^{(M_1+M_2)(M_1+N)} u^M{}_{NM_2}\frac{\partial}{\partial x^{M_1}}\varepsilon^N + \frac{1}{2}(-1)^{(M+1)(M_1+M_2)}\frac{\partial}{\partial x^{M_1}}\frac{\partial}{\partial x^{M_2}}\varepsilon^M. \tag{3.10}$$

The transformation laws of the components of the supervielbein are as follows:

$$\begin{aligned}
\delta\psi^m{}_\mu &= -\partial_\mu\epsilon^m + \partial_n\epsilon^m\psi^n{}_\mu - \partial_\mu\epsilon^\nu\psi^m{}_\nu - \partial_n\epsilon^\nu\psi^n{}_\mu\psi^m{}_\nu = &(3.11)\\
& -D_\mu\epsilon^m - D_\mu\epsilon^\nu\psi^m{}_\nu,\ \ D_\mu = \partial_\mu - \psi^m{}_\mu\partial_m, &\\
\delta\phi^\mu{}_m &= \partial_m\epsilon^\mu - (\partial_m\epsilon^n + \partial_m\epsilon^\nu\psi^n{}_\nu)\phi^\mu{}_n + +(\partial_\nu\epsilon^\mu + \partial_n\epsilon^\mu\psi^n{}_\nu)\phi^\nu{}_m, &(3.12)\\
\delta\mathcal{E}_k{}^a &= -(\partial_k\epsilon^m + \partial_k\epsilon^\nu\psi^m{}_\nu)\mathcal{E}_m{}^a, &(3.13)\\
\delta\mathcal{E}_\mu{}^\alpha &= -(\partial_\mu\epsilon^\rho + \partial_m\epsilon^\rho\psi^m{}_\nu)\mathcal{E}_\rho{}^\alpha = -D_\mu\epsilon^\rho\mathcal{E}_\rho{}^\alpha. &(3.14)
\end{aligned}$$

In analogy with (2.18) we denote $\mathcal{E}^a_m = (e^{-u})^a_m$, $\mathcal{E}^\alpha_\mu = (e^{-v})^\alpha_\mu$. The next step is to consider all parameters as the fields in the superspace with D bosonic and D_G grassmann coordinates x^M and construct invariant differential forms in terms of these fields.

4 Differential Ω - forms in the superspace

Along with grading of the coordinates x^M, their differentials dx^M have their own grading. There exist two different gradings of the differentials of the coordinates. One of them corresponds to the independent grassmann grading and grading of the differential d[4]. It leads to the following commutation relations:

$$\begin{aligned}
[x^m, x^n] &= [x^m, dx^n] = [x^m, d\theta^\mu] = [d\theta^\mu, d\theta^\nu] = [\theta^\mu, dx^m] = 0,\\
\{dx^m, dx^n\} &= \{\theta^\mu, \theta^\nu\} = \{dx^m, d\theta^\mu\} = \{\theta^\mu, d\theta^\nu\} = 0.
\end{aligned} \tag{4.1}$$

More simple commutation relations take place when grading of the differential d coincides with grassmann grading [5]. It means that the differential changes the grading of the coordinates to the opposite one:

$$g(dx^M) = g(x^M) + 1. \tag{4.2}$$

As a result there are equal numbers $D + D_G$ of bosonic $(x^m,\ d\theta^\mu)$ and grassmann $(dx^m,\ \theta^\mu)$ variables.

The left - invariant differential Ω -form

$$\Omega = G^{-1}dG \tag{4.3}$$

belongs to the algebra of the superdiffeomorphism group

$$\Omega = i\Omega^A P_A + i\Omega^A{}_{A_1} P^{A_1}{}_A + i\Omega^A{}_{A_1 A_2} P^{A_2 A_1}{}_A + \ldots \tag{4.4}$$

and its coefficients Ω^A, $\Omega^A{}_{A_1}$, $\Omega^A{}_{A_1A_2}$ are invariant under the transformation (3.8). We emphasize this fact by using the letters from the beginning of the alphabet for indices. We will use latin letters $a, b, c, \ldots$ for bosonic and greek letters $\alpha, \beta, \gamma, \ldots$ for grassmann indices. Note, that according to the grading rule (4.2) Ω_a and Ω^α are, correspondingly, anticommuting and commuting objects.

Explicit expressions for components of Ω^A are:

$$\Omega^a = (dx^m + d\theta^\mu \psi^m{}_\mu)\mathcal{E}_m{}^a \equiv dx^M E_M{}^a, \tag{4.5}$$

$$\Omega^\alpha = \{d\theta^\mu - (dx^m + d\theta^\nu \psi^m{}_\nu)\phi^\mu{}_m\}\mathcal{E}_\mu{}^\alpha \equiv dx^M E_M{}^\alpha. \tag{4.6}$$

The expresions (4.5) and (4.6) represent the one - form supervielbein $E^A \equiv \Omega^A = dx^M E_M{}^A$ with components

$$E_M{}^A = \left| \begin{array}{ll} E_m{}^a = \mathcal{E}_m{}^a & E_m{}^\alpha = -\phi^\mu{}_m \mathcal{E}_\mu{}^\alpha \\ & \\ E_\mu{}^a = \mathcal{E}_m{}^a \psi^m{}_\mu & E_\mu{}^\alpha = \mathcal{E}_\mu{}^\alpha + \mathcal{E}_\nu{}^\alpha \phi^\nu{}_n \psi^m{}_\mu \end{array} \right| \tag{4.7}$$

From (3.11)-(3.14) it follows

$$\delta E_M{}^A = -\epsilon^N \partial_N E_M{}^A - \partial_M \epsilon^N E_N{}^A. \tag{4.8}$$

The components of the inverse supervielbein are

$$E_A{}^M = \left| \begin{array}{ll} E_a{}^m = \mathcal{E}_a{}^m - \mathcal{E}_a{}^n \phi^\nu{}_n \psi^m{}_\nu & E_a{}^\mu = \phi^\mu{}_n \mathcal{E}_a{}^n \\ & \\ E_\alpha{}^m = -\mathcal{E}_\alpha{}^\nu \psi^m{}_\nu & E_\alpha{}^\mu = \mathcal{E}_\alpha{}^\mu \end{array} \right| \tag{4.9}$$

where $\mathcal{E}_a{}^m$ and $\mathcal{E}_\alpha{}^\mu$ are inverse matrices to $\mathcal{E}_m{}^a$ and $\mathcal{E}_\mu{}^\alpha$ correspondingly. Straightforward computation shows very simple form of the superdeterminant $Ber E_M^A$

$$Ber E_M^A = \frac{det \mathcal{E}_m{}^a}{det \mathcal{E}_\mu{}^\alpha}. \tag{4.10}$$

One can show that arbitrary nonsingular graded matrix E_M^A can be parametrized in the form (4.7) for which (4.10) is valid. So, in some sense such parametrization of graded matrices is natural.

Due to its definition (4.3) Ω -form satisfy the Maurer - Cartan equation

$$d\Omega + \Omega \wedge \Omega = 0. \tag{4.11}$$

Two first components of this equation are as follows:

$$d\Omega^A + (-1)^{A+B} \Omega^A{}_B \Omega^B, \tag{4.12}$$

$$d\Omega^A{}_B + (-1)^{A+C} \Omega^A{}_C \Omega^C{}_B + 2(-1)^{A+B+C} \Omega^A{}_{BC} \Omega^C. \tag{4.13}$$

Under the right gauge transformations from the group $GL(D) \times GL(D_G)$:

$$G' = G\{1 - ih^a_b(x)P^b_a - ih^\alpha_\beta(x)P^\beta_\alpha\} \tag{4.14}$$

Ω^a_b and Ω^α_β transform as corresponding connections:

$$\delta\Omega^a_b = h^a_c \Omega^c_b - \Omega^a_c h^c_b - dh^a_b \tag{4.15}$$

$$\delta\Omega^\alpha_\beta = h^\alpha_\gamma \Omega^\gamma_\beta - \Omega^\alpha_\gamma h^\gamma_\beta - dh^\alpha_\beta. \tag{4.16}$$

Taking them as a "minimal" connectiions, the equation (4.12) expresses the covariant differentials of Ω^a and Ω^α:

$$D\Omega^a \equiv d\Omega^a + \Omega^a_b \Omega^b = \Omega^a_\alpha \Omega^\alpha = T^a_{B\alpha} \Omega^B \Omega^\alpha. \tag{4.17}$$

$$D\Omega^\alpha \equiv d\Omega^\alpha + \Omega^\alpha_\beta \Omega^\beta = \Omega^\alpha_a \Omega^a = T^\alpha_{Ba} \Omega^B \Omega^a. \tag{4.18}$$

where we expanded one forms Ω^a_α and Ω^α_a in terms of the basic system of one-forms Ω^B. The form of the right hand sides of these equations shows, that T^a_{bc} and $T^\alpha_{\beta\gamma}$ components of the torsion vanish identically.

5 Integral invariants in the superspace

Having constructed invariant differential Ω - forms we have to be able to build from them the integral invariants like action. The problem lies in the fact that there are two types of differentials of grassmann coordinates. We will denote them $d\theta^\mu$ and $\underline{d}\theta^\mu$. First of these differentials are used in invariant differential Ω - forms. Unlike dx^m, which are anticommuting objects, $d\theta^\mu$ commute and one can construct from the Ω -forms the differential form of arbitrary order. Contrarily, so called Berezin pseudodifferentials $\underline{d}\theta^\mu$ anticommute. One can take integrals over the superspace

$$I = \int d^D x \underline{d}^{D_G} \theta F(x, \theta) \tag{5.1}$$

with the help of Berezin integration rules

$$\int \underline{d}\theta^\mu = 0, \int \theta^\mu \underline{d}\theta^\nu = \delta^{\mu\nu}. \tag{5.2}$$

The integration volume $dv = d^D x \underline{d}^{D_G} \theta$ transforms under the general diffeomorphism $x'^M = x'^M(x^N)$ as

$$dv' = Ber \left(\frac{\partial x'^M}{\partial x^N} \right) dv. \tag{5.3}$$

If the function $F(x, \theta)$ in (5.1) transforms as

$$F(x', \theta') = Ber^{-1} \left(\frac{\partial x'^M}{\partial x^N} \right) F(x, \theta), \tag{5.4}$$

the integral I is reparametrization invariant.

The following prescription for building of integral invariants from invariant differential forms was formulated by Bernstein and Leites [6]. Taking arbitrary invariant differential form $F(x,\theta,dx,d\theta)$ one can introduce new auxiliary variables η^m and y^μ in one to one correspondence to dx^m and $d\theta^\mu$ .These new variables η^M have the same grassmann properties as dx^m and $d\theta^\mu$ (opposite to ones of x^m and θ^μ) and the following transformation laws

$$\eta'^m = \frac{\partial x'^m}{\partial x^n}\eta^n + \frac{\partial x'^m}{\partial \theta^\mu}y^\mu, \tag{5.5}$$

$$y'^\mu = \frac{\partial \theta'^\mu}{\partial x^m}\eta^m + \frac{\partial \theta^\mu}{\partial \theta^\nu}y^\nu. \tag{5.6}$$

The transformation (5.5)-(5.6) has very important property. Its superdeterminant (superJacobian)

$$\frac{\partial\{\eta'^m, y'^\mu\}}{\partial\{\eta^n, y^\nu\}} = Ber^{-1}\left(\frac{\partial x'^M}{\partial x^N}\right) \tag{5.7}$$

is inverse to the superJacobian for the transformation of the "old" coordinates x^M. This fact is based on the following property: the superdeterminant of the graded matrix is inverse to the superdeterminant of the same matrix with opposite grading. It means the invariance of the product $dV = d^D x \underline{d}^{D_G}\theta d^{D_G} y \underline{d}^D \eta$ with respect to the general coordinate reparametrization in the superspace. In turn this fact leads to the invariance of the following integral

$$I = \int dV F(x,\theta,\eta,y). \tag{5.8}$$

If the integral over η and y exists, the result of such integration

$$I = \int d^D x \underline{d}^{D_G}\theta \tilde{F}(x,\theta) \tag{5.9}$$

will be invariant as well.

It is convenient at this stage to introduce additional set of auxiliary variables C_A, transforming as the vector of the $GL(D,D_G)$ and having the grassmann grading $g(C_a) = 1$, $g(C_\alpha = 0)$. With the help of this ghosts it is easy to construct invariants in a covariant manner under $SL(D,D_G)$. The additional volume element $dC = \underline{d}C^D dC^{D_G}$ does not transforms due to condition $Ber h = 1$ for h belonging to $SL(D,D_G)$.

The simplest invariant has the form

$$I_0 = \int dV dC e^{iE^A C_A} = \int dV dC e^{i\eta^M E_M{}^A C_A}. \tag{5.10}$$

As was shown in [7] the result of integration over η^M and C_A in (5.10) is proportional to the superdeterminant of the $E_M{}^A$

$$I_0 = \int d^D x \underline{d}^{D_G}\theta Ber E_M{}^A. \tag{5.11}$$

The wide class of invariants can be obtained from the expressions of the form

$$I = \int dV dC F(\Omega, C) e^{iE^A C_A}. \tag{5.12}$$

where $F(\Omega,C)$ is an arbitrary function of C_A and Cartan's Ω -forms, in which differentials of coordinates dx^M are changed to η^M. Due to completeness of the one-form vielbein, the function $F(\Omega,C)$ can be represented as the series in powers of vielbein and C_A

$$F(\Omega,C) = \sum F^{B_k \ldots B_1}{}_{A_n \ldots A_1} E^{A_1} \ldots E^{A_n} C_{B_1} \ldots C_{B_k} \tag{5.13}$$

with coefficients $F_{B_k \ldots B_1}{}^{A_n \ldots A_1}$ depending on the functions $u^M{}_{M_1 \ldots M_n}(x,\theta)$. Thus, the evaluation of the integral (5.12) reduces to the evaluation of the basic integrals

$$I^{A_1 \ldots A_n}{}_{B_1 \ldots B_k} = \int d\eta dC E^{A_1} \ldots E^{A_n} C_{B_1} \ldots C_{B_k} e^{iE^A C_A}. \tag{5.14}$$

One can show that such integrals are zero when numbers n and k are different. The nonzero answers for two simplest cases are

$$\int d\eta dC E^A C_B e^{iE^A C_A} = (-1)^A \delta^A{}_B Ber(E_M{}^C), \tag{5.15}$$

$$\int d\eta dC E^{A_1} E^{A_2} C_{B_1} C_{B_2} e^{iE^A C_A} = \tag{5.16}$$
$$\{(-1)^{A_1+A_2}\delta^{A_2}{}_{B_1}\delta^{A_1}{}_{B_2} + (-1)^{A_1 A_2 + 1}\delta^{A_2}{}_{B_2}\delta^{A_1}{}_{B_1}\} Ber E_M{}^A.$$

The general expression for the basic integrals (5.14) can be evaluated from the relation

$$\int d\eta dC e^{iE^A \Sigma_A{}^B C_B} = Ber(E_M{}^A \Sigma_A{}^B) = Ber E_M{}^A Ber \Sigma_D{}^B \tag{5.17}$$

by varying in it $\Sigma_A{}^B$ in the neighborhood of the identity. The resulting $I^{A_1 \ldots A_n}{}_{B_1 \ldots B_k}$ look like expression (5.16) with correspondingly symmetrized production of appropriate number of $\delta^{A_i}_{B_k}$ multiplied by $Ber E_M{}^A$. This gives the answer for the invariant I (5.12) in terms of the superspace integral

$$I = \int d^D x \underline{d}^{D_G}\theta \sum F^{B_k \ldots B_1}{}_{A_n \ldots A_1} I^{A_1 \ldots A_n}{}_{B_1 \ldots B_k}. \tag{5.18}$$

The question of finding the integral invariant of such type in the superspace for the action of supergravity, like (2.33) for gravity, is open. If such action does exist, it contains among the structure constants the Dirac gamma matrices, $\gamma^a_{\alpha\beta}$, which break the tangent space gauge group $GL(D) \times GL(D_G)$ down to its subgroup (SL(2) in the case of four dimensions) and establish the connection between bosonic and fermionic dimensionalities D and D_G.

6 Conclusions

We have considered the nonlinear realizations of infinite - dimensional diffeomorphism groups of any (super)space. The parameters of coset space in a very natural manner include the coordinates, vielbeins and connections of the corresponding (super)space. The geometrical and physical meaning of higher parameters $u^M{}_{M_1 \ldots M_n}$ with $n \geq 3$ is still unclear. Construction of invariant under the action of diffeomorphism group differential Ω -forms is straightforward in any (super)space. At the same time the $GL(D, D_G)$ gauge group, considered as the right action on the group element, plays the role of gauge group in the tangent space. The most of the Ω-forms transform as tensors of this $GL(D, D_G)$. The only Ω-form with two tangent indices Ω_b^a plays the role of connection which automatically is torsionless in the bosonic case. In the superspace only T_{bc}^a and $T_{\beta\gamma}^\alpha$ components of the torsion are vanishing identically.

Such an invariant differential Ω - forms can be considered as building blocks for construction of integral invariants like action. In purely bosonic space (2.33) gives the expression for the gravity action. In the case of superspase the method of constructing integral invariants is described in the Chapter 5. The existing of the action for supergravity of such type is an open question.

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Пашнев А. E2-97-122
Нелинейные реализации групп (супер)диффеоморфизмов, геометрические объекты и интегральные инварианты в суперпространстве

Показано, что тетрады и связности любого (супер)пространства естественным образом могут быть описаны в терминах нелинейных реализаций бесконечномерных групп диффеоморфизмов соответствующих (супер)пространств. Метод построения интегральных инвариантов из инвариантных дифференциальных Ω-форм Картана обобщен на случай суперпространств.

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Pashnev A. E2-97-122
Nonlinear Realizations of the (Super)diffeomorphism Groups, Geometrical Objects and Integral Invariants in the Superspace

It is shown that vielbeins and connections of any (super)space are naturally described in terms of nonlinear realizations of infinite-dimensional diffeomorphism groups of the corresponding (super)space. The method of construction of integral invariants from the invariant Cartan's differential Ω-forms is generalized to the case of superspace.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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