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1. Introduction

In our preceding papers [1,2] as well as in [3] there was elaborated a special sort of canonical realizations of some real forms of classical Lie algebras. In this note, continuing in the same spirit, we give a short description of our results concerning now the Lie algebra $u(p,q)$, $p \geq q \geq 1$.

Our task is to express the generators of the given Lie algebra through a definite sort of functions of a given number of quantum canonical variables q_i and p^i. We are interested in realizations with the following properties:

(i) The Casimir operators are realized by multiples of the identity element. Such a realization we call Schur-realizations.

(ii) Under involution, induced on the set of the chosen sort of functions of q_i and p_i by the relations $p^{i+} = -p^i$ and $q_i^+ = q_i$, the realizations of the real Lie algebras are skew-Hermitian. (Note that we assume $[p^i, q_j] = \delta_j^i \mathbb{1}$).

In refs. [1,2] we had described a large class of canonical realizations of $gl(n,R)$ and $sp(2n,R)$ with these properties. The functions through which the generators are expressed in these cases were the most simple ones, polynomials. In other words we were dealing with canonical realizations in the Weyl algebra W_{2N}, N the number of canonical pairs used.

In order to get the analogous large class of canonical realizations of the Lie algebras $u(n,m)$, $n \geq m \geq 1$, the above properties demand the use of a more general class of functions, elements of $W_{2N} \otimes Mat_M$, where Mat_M denotes the algebra of complex $M \times M$ -matrices (see /3/). In the set of skew-Hermitian realizations of $u(p,q)$ presented in this note we use functions further generalized. The generators of $u(p,q)$ are realized now in the tensor product $W^r_{2N,M} = W^r_{2N} \otimes Mat_M$ where W^r_{2N} denotes a certain localization of the Weyl algebra W_{2N}, i.e., certain rational functions of canonical pairs. The realizations defined recurrently are classified by sequences of real numbers $(d; \alpha_1, \dots, \alpha_{p+q})$, $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{p+q}$, the so called si natures. For $d = 1, 2, \dots, 2q-1$ we get a $d+1$ -parameter set of realizations with the real parameters $\alpha_{p+q}, \alpha_{p+q-1}, \dots, \alpha_{p+q-d}$ (the other parameters α_i, $i = 1, \dots, p+q-d-1$ are equal to zero by definition of the signature). In dependence of d these realizations are contained in $W^{r(d)}_{2N(d)}$ with $N(d) = \frac{1}{2}(2n-1-d)$, $r(d) = \left[\frac{d+1}{2}\right]$, $n = p+q$. (We see that for $d = 1$ the number $N(d) = n-1$ equals the minimal number of canonical pairs such that a nontrivial realization of $u(p,q)$ by rational functions exists /4/ . If $p > q$ there exists furthermore a set of realizations in $W^{r(d)}_{2N(d),M}$ corresponding to the signature $(d; \alpha_1, \dots, \alpha_{p+q})$, $d = 2q$, where ($\alpha_1, \dots, \alpha_{p-q}$) is the signature of the skew-Hermitian irreducible representation of the compact Lie algebra $u(p-q)$ with dimension M. The remaining parameters $\alpha_{p-q+1}, \dots, \alpha_{p+q}$ allow any real value and the number of canonical pairs in $W^{r(d)}_{2N(d),M}$ equals $N(d) = q(2p-1)$. In the analogical sets of canonical realizations for all the other classical Lie algebra which we have studied up to now /1,2,3/ the realizations are Schur-realizations. We believe that this is true also for the presented realizations of $u(p,q)$. As to realizations characterized by signatures with $d = 1$ it is a consequence of their minimally property /4/ and in Example 2 we show that also the remaining realizations are Schur-realizations in the particular case of $u(2,1)$. In this example we briefly discuss further the question of "equivalence" among some presented realizations while Example 1 gives the connection of realization of $u(p,q)$ characterized by signatures with $d = 1$ and realizations of $gl(r, C)$ given in /2/.

2. The recurrent relations

We start our considerations with the complex Weyl algebra $W_{2(2n-3)}$, $n = p + q \geq 2$. It is useful for our purposes to denote its generating elements by $q_0, p_0, \bar{a}_i, a^i, \bar{b}_i, b^i$, $i = 1, \dots, n-2$ with the non-zero commutators

$$[p_0, q_0] = 1, \quad [a^i, \bar{a}_j] = [b^i, \bar{b}_j] = \delta^i_j \quad . \tag{1}$$

We use the tensor notation, $a_i = g_{ij} a^j$, etc.
For a given metric tensor

$$(g_{ij}) = \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_{n-2})$$

with

$$\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_{p-1} = -\varepsilon_p = \dots = -\varepsilon_{n-2} = 1 \tag{2}$$

an involution on $W_{2(2n-3)}$ is induced by the relations

$$p_i' = -p_i\,,\ q_i' = q_i\,,\ (a_i)' = \bar{a}_i\,,\ (b_i)' = \bar{b}_i\ . \tag{3}$$

The localization $W'_{2(2n-3)}$ (in the element $q_i \in W_{2(2n-3)}$ see /9/) is defined by

$$W'_{2(2n-3)} = \left\{ q_i^{-l} \cdot x \mid x \in W_{2(2n-3)}\,,\ l = 0,1,2,\dots \right\} .$$

Now let A be any associative complex algebra with involution and let F_i^j, $i,j = 1, \dots, n-2$, be elements in A such that their commutators are

$$[F_i^j, F_k^l] = \delta_k^j F_i^l - \delta_i^l F_k^j\ . \tag{4}$$

Further assume that

$$F_i^{j*} = g_{ik}\, g^{jl}\, F_l^k \tag{5}$$

with g_{ij} according to (2). Then the $(n-2)^2$ skew-Hermitian elements $i(F_i^{j*} + F_i^j)$ and $(F_i^{j*} - F_i^j)$ form the basis of a skew-Hermitian realization of $u(p-1, q-1)$ in A [1]. Note that the same is valid for nondiagonal real matrix (g_{ij}) similar to the diagonal one (2).

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[1] The reader may consult in this question for instance the book of Gourdin /5/ .

Theorem 1: Let A be an associative complex algebra with involution. Assume that in A a skew-Hermitian realization of $u(p-1, q-1)$, $p \geq q \geq 1$, is given through $(n-2)^2$, $n = p+q$, elements F_i^j satisfying (4) and (5). Then the following elements

$$
\begin{aligned}
E_i^j &= \bar{a}_i a^j - \eta(b_i \bar{b}^j - F_i^j) + \frac{\alpha_{n-1}}{n} \delta_i^j , \\
E_n^{n-1} &= \varphi_0^2 , \\
E_n^i &= \varphi_0 a^i , \qquad E_i^{n-1} = \varphi_0 \bar{a}_i , \\
E_n^n &= \frac{1}{2}\left(\varphi_0 p_0 - \bar{a}a + \eta b\bar{b} + \alpha_n + \frac{1}{2}\right) + \frac{\alpha_{n-1}}{n} , \\
E_{n-1}^{n-1} &= \frac{1}{2}\left(-\varphi_0 p_0 - \bar{a}a + \eta b\bar{b} + \alpha_n - \frac{1}{2}\right) + \frac{\alpha_{n-1}}{n} , \\
E_i^n &= \varphi_0^{-1}\left\{\eta\left[(b_k - \bar{a}_k)(F_i^k - b_i \bar{b}^k) - \frac{1}{2}\alpha b_i\right] + \bar{a}_i\left(E_n^n - \frac{\alpha_{n-1}}{n}\right)\right\} , \\
E_{n-1}^i &= \varphi_0^{-1}\left\{\eta\left[(F_k^i - b_k \bar{b}^i)(\bar{b}^k - a^k) - \frac{1}{2}\alpha^* \bar{b}^i\right] + a^i\left(E_{n-1}^{n-1} + 1 - \frac{\alpha_{n-1}}{n}\right)\right\} , \\
E_{n-1}^n &= \varphi_0^{-2}\left\{\left(E_{n-1}^{n-1} + 1 - \frac{\alpha_{n-1}}{n}\right)\left(E_n^n - \frac{\alpha_{n-1}}{n}\right) + \eta\left[(b_k - \bar{a}_k)(F_l^k - b_l \bar{b}^k)(\bar{b}^l - a^l) \right.\right. \\
&\qquad \left.\left. - \frac{\alpha}{2}(a\bar{b} - ba) - \frac{\alpha^*}{2}(b\bar{b} - \bar{a}a) - \frac{1}{4}|\alpha|^2\right]\right\} , \\
&\eta = 0, 1, \qquad \alpha = \alpha_n + i\alpha_{n-1} ; \; \alpha_{n-1}, \alpha_n \in \mathbb{R}
\end{aligned}
\tag{6}
$$

define, through their skew-Hermitian combinations, a skew-Hermitian realization of $u(p, q)$ in $A \otimes W^1_{2(n-1+\eta(n-2))}$. In other words, the E_i^j-s satisfy the relations

$$[E_\lambda^\nu, E_\mu^\kappa] = \delta_\mu^\nu E_\lambda^\kappa - \delta_\lambda^\kappa E_\mu^\nu \tag{7}$$

and

$$E_\lambda^{\nu\dagger} = g_{\lambda\kappa} g^{\nu\mu} E_\mu^\kappa , \quad \lambda, \nu = 1, 2, \ldots, n \tag{8}$$

where

$$(g_{\mu\nu}) = \left(\begin{array}{c|c} g_{ij} & 0 \\ & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \\ 0 & \end{array} \right). \tag{9}$$

The proof is straightforward and therefore omitted here; the fact that E^{ν}_{μ}'s lead just to $u(p,q)$ is a consequence of the similarity of the $(g_{\mu\nu})$ to the matrix diag $(\underbrace{+1,\dots,+1}_{p\text{-times}},\underbrace{-1,\dots,-1}_{q\text{-times}})$.

The obviously recurrent character of the realization (6) for $\eta = 1$ will be used in the next section for construction of further realizations of $u(p,q)$.

3. Realizations of $u(p,q)$ denoted by signatures.

Definition: For given nonnegative integers $p,q,\ p \geq q \geq 0,\ p+q \geq 1$, the (p+q+1)-tuple of real numbers

$$\alpha_{p,q} = (d;\alpha_1,\dots,\alpha_{p+q})$$

is called signature if

$$d = 0,1,2,\dots,2q-\delta_{pq}$$

and

(i) for $d < 2q - \delta_{pq}$

$$\alpha_1 = \dots = \alpha_{p+q-d-1} = 0$$

(ii) for $d = 2q$ (i.e. $p > q$)

$(\alpha_1,\dots,\alpha_{p-q})$ from the signature [2] of an irreducible skew-Hermitian representation of $u(p,q)$.

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2) That means $\alpha_i \in \mathbb{R}$, $\alpha_{i+1} - \alpha_i = \lambda_i$, $\lambda_i = 0,1,2,\dots$ and the repr. of $u(p-q)$ is given by the repr. of $su(p-q)$ with highest weight $(\lambda_1,\lambda_2,\dots,\lambda_{p-q-1})$ according to /6/ p.548 extended by $E = E_{11} + \dots + E_{p-q\,p-q} = i(\alpha_1 + \dots + \alpha_{p-q})\cdot I$.

Note the following important property of the signature. For every signature $\alpha_{p,q} = (d; \alpha_1, \dots, \alpha_{p+q})$, $d \geq 2$ the sequence ($d-2, \alpha_1, \dots, \alpha_{p+q-2}$) is again a signature $\alpha_{p-1,q-1}$.

To every signature $\alpha_{p,q}$ we will define now a skew-Hermitian realization of $u(p,q)$ in $W^r_{2N,M}$ (See Theorem 2). The realizations which correspond to signatures with $d = 0, 1$ are given in the following table:

	$\alpha_{p,q}$	Skew-Hermitian realization of $u(p,q)$ in $W'_{2N,M} = W'_{2N} \otimes Mat_M$	N, M
$q \geq 1$	$(1; 0, \dots, 0, \alpha_{p+q-1}, \alpha_{p+q})$	given by eq. (6) with $\eta = 0$	$N = p+q-1$ $M = 1$
	$(0; 0, \dots, 0, \alpha_{p+q})$	$E^\nu_\mu = \frac{\alpha_{p+q}}{p+q} \delta^\nu_\mu$, $\alpha_{p+q} \in R$	$N = 0$ $M = 1$
$q = 0$	$(0; \alpha_1, \dots, \alpha_p)$	skew-Hermitian irrep. of $u(p)$ with signature $(\alpha_1, \dots, \alpha_p)$	$N = 0$ M = dim of this irrep.

Table : Realizations with signatures $\alpha_{p,q} = (d; \alpha_1, \dots, \alpha_{p+q})$, $d = 0, 1$.

Further, the skew-Hermitian realization of $u(p,q)$ with signatures $\alpha_{p,q} = (d; \alpha_1, \dots, \alpha_{p+q})$ $d \geq 2$ is given by eq. (6) with $\eta = 1$, where $F^{\,j}_i$ is the realization of $u(p-1, q-1)$ with signature $\alpha_{p-1,q-1} = (d-2, \alpha_1, \dots, \alpha_{p+q-2})$.
The following theorem summarizes and completes these results.

Theorem 2: (i) To every signature $\alpha_{p,q}$ there corresponds a skew-Hermitian realization of the Lie algebra $u(p,q)$ defined in the above described way by means of eq. (6).

(ii) The realization is contained in $W^{r(d)}_{2N(d),M} = W^{r(d)}_{2N(d)} \otimes Mat_M$ where

$$W^{r(d)}_{2N(d)} = \begin{cases} \bigotimes_{k=1}^{\frac{d}{2}} W'_{2(2n-4k+1)} , & d - \text{even} \\ \bigotimes_{k=1}^{\left[\frac{d}{2}\right]} W'_{2(2n-4k+1)} \otimes W'_{2(n-d)} , & d - \text{odd} \end{cases}$$

and where $M = 1$ with exception of the case $p-q \geq 2$ and $d = 2q$ where M is the dimension of the skew-Hermitian irreducible representation of $u(p,q)$ with signature $(\alpha_1, \dots, \alpha_{p-q})$. The number $N(d) = \frac{d}{2}(2n-d-1)$ gives the whole number of canonical pairs used, and $r(d) = \left[\frac{d+1}{2}\right]$.

3. Examples

(1) Realizations of $u(p,q)$ with signature $(1, 0, \dots {}_{p-q-1}, \alpha_{p-q})$

As we pointed out in Introduction these realizations are Schur-realizations due to the minimal number of canonical pairs used: $N(d) = N(1) = p+q-1$.

We show now that these realizations are connected with the subset of skew-Hermitian Schur-realizations of $gl(n,R)$, $n = p+q$, described in our paper [2] . Realizations of $gl(n,R)$ denoted here by the signature $(1, 0, \ldots, 0, \hat{\alpha}_{n-1}, \hat{\alpha}_n)$, $\hat{\alpha}_{n-1}, \hat{\alpha}_n \in R$ have the following form:

$$\hat{E}^b_a = q_a p^b + \left(\frac{i\hat{\alpha}_{n-1}}{n-1} + \frac{1}{2}\right)\delta^b_a \,, \quad \hat{E}^n_n = -q_a p^a + \left(i\hat{\alpha}_n - \frac{n}{2}\right),$$

$$\hat{E}^a_n = -p^a \,, \quad \hat{E}^n_a = q_a\left(q_b p^b + \frac{n}{2} - i\hat{\alpha}_n + \frac{i\hat{\alpha}_{n-1}}{n-1}\right), \tag{10}$$

$a, b = 1, 2, \ldots, n-1$, i.e., they are contained in the Weyl algebra $W_{2(n-1)} \equiv W_{2(n-1)}(q_1, p^1, \ldots, q_{n-1}, p^{n-1})$.

The complex linear combinations

$$i(\hat{E}^\nu_\mu + g^{\nu\rho} g_{\mu\tau} \hat{E}^\tau_\rho), \quad (\hat{E}^\nu_\mu - g^{\nu\rho} g_{\mu\tau} \hat{E}^\tau_\rho)$$

define a non-skew-Hermitian Schur-realisation of $u(p,q)$.

As we pointed out in Conclusion of Ref. [2] the generators $\hat{E}^\nu_\mu$ form a Schur-realization of $gl(n,R)$ also if $\hat{\alpha}$'s are chosen complex. Let us use this possibility and substitute

$$\hat{\alpha}_{n-1} = -i(n-1)\left(\frac{\alpha_{n-1}}{n} - \frac{1}{2}\right), \quad \hat{\alpha}_n = -i\left(\alpha_n + \frac{\alpha_{n-1}}{n} + \frac{n-1}{2}\right)$$

$\alpha_{n-1}, \alpha_n \in R$.

Further we shall define two mappings: the isomorphism

$$b: W_{2(n-1)}(q_1, p^1, \ldots, q_{n-1}, p^{n-1}) \longleftrightarrow W_{2(n-1)}(a^i, \bar{a}_i, q_c, p_c)$$

$$b(q_i) = -\bar{a}_i \,, \quad b(p^i) = -a^i, \quad b(q_{n-1}) = p_c, \quad b(p^{n-1}) = -q_c$$

$(i = 1, 2, \ldots, n-2)$

and $\vartheta_c \in End\ W^{1}_{2(n-1)}(a^i, \bar{a}_i, q_0, p_c)$:

$$\vartheta_c(a^i) = q_c a^i, \quad \vartheta_c(\bar{a}_i) = q_c^{-1}\bar{a}_i, \quad \vartheta_c(q_c) = q_c^2$$

$$\vartheta_c(p_c) = \frac{1}{2} q_c^{-2}(q_c p_c + \bar{a}a + c), \quad c \in \mathbb{C}.$$

(Both mappings b and ϑ_c do not conserve the involution). If we choose $c = -\alpha_n - \frac{3}{2}$ then $b(\hat{E}^{\nu}_{\mu})$ is related to E^{ν}_{μ} (given by eq. (6), η =0), i.e.,

$$\vartheta_c\, b(\hat{E}^{\nu}_{\mu}) = E^{\nu}_{\mu} \quad .$$

As the elements $\hat{E}^{\nu}_{\mu}$ are polynomials in canonical variables they are simpler than the rational functions E^{ν}_{μ}. This more complicated form of the generators E^{ν}_{μ} can be understand as the price for the skew-Hermiticity property. The question is of course whether it is necessary, i.e., whether skew-Hermitian Schur-realization of $u(p,q)$ in the Weyl algebra exists. Due to the known isomorphism $u(1,1) \simeq gl(2,\mathbb{R})$ the realizations (10) give such an example for $u(1,1)$.

(ii) Realizations of $u(2,1)$ with signature $(2; \alpha_1, \alpha_2, \alpha_3)$

These realizations are given through 3 canonical pairs by eq. (6) with η =1 if we put $F^{j}_{i} = F^{1}_{1} = \alpha_1 \mathbb{1}$. Generating Casimir operators of the center of the enveloping algebra $U[u(2,1)]$ are $iC^{(1)}, C^{(2)}, iC^{(3)}$ where

$$C^{(1)} = E^{\mu}_{\mu}, \quad C^{(2)} = E^{\nu}_{\mu} E^{\mu}_{\nu}, \quad C^{(3)} = E^{\nu}_{\mu} E^{\rho}_{\nu} E^{\mu}_{\rho} \ .$$

By direct calculation we obtain

$$
\begin{aligned}
C^{(1)} &= \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 \\
C^{(2)} &= \tilde{\alpha}_1^2 + \tilde{\alpha}_2^2 + \tilde{\alpha}_3^2 - 2 \\
C^{(3)} &= \tilde{\alpha}_1^3 + \tilde{\alpha}_2^3 + \tilde{\alpha}_3^3 + \tfrac{1}{2}\, C^{(1)} \ldots [illegible]
\end{aligned}
\qquad (11)
$$

where

$$
\begin{aligned}
\tilde{\alpha}_1 &= \alpha_1 + \tfrac{1}{3}\alpha_2 - 1 \\
\tilde{\alpha}_2 &= -\left(\tfrac{1}{3} + \tfrac{1}{2}\right)\alpha_2 + \tfrac{1}{2}\alpha_3 + \tfrac{1}{2} \ldots [illegible]
\end{aligned}
\qquad (12)
$$

This shows that the considered realizations are Schur-realization.

As we see the Casimir operators $C^{(i)}$ are symmetric polynomials in the variables $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3$; this fact can be used for the following considerations. It is a well-known property of symmetric polynomials that for fixed values of $C^{(1)}, C^{(2)}, C^{(3)}$ three complex numbers $\tilde{\alpha}_1', \tilde{\alpha}_2', \tilde{\alpha}_3'$ fulfil eqs. (11) if and only if they are roots of a 3-rd order equation $x^3 + c_2 x^2 + c_1 x + c_0 = 0$ where the coefficients c_0, c_1, c_2 are in one-to-one polynomial correspondence with $C^{(1)}, C^{(2)}, C^{(3)}$.
Therefore a second family $\tilde{\alpha}_1'', \tilde{\alpha}_2'', \tilde{\alpha}_3''$ can give the same values of $C^{(1)}, C^{(2)}, C^{(3)}$ as the family $\tilde{\alpha}_1', \tilde{\alpha}_2', \tilde{\alpha}_3'$ if and only if $\tilde{\alpha}_i'' = \tilde{\alpha}_{k_i}'$, i =1,2,3, where (k_1, k_2, k_3) is some permutation of the naturals 1,2,3. Since $\tilde{\alpha}_i'$ and $\tilde{\alpha}_i''$ must be contained in the range of values of $\tilde{\alpha}_i$ (given by eqs. (12) for real $\alpha_1, \alpha_2, \alpha_3$) there are two cases:

(i) if $Im\,\tilde{\alpha}_2' = 0$ then $\tilde{\alpha}_i''$ do not differ from $\tilde{\alpha}_i'$, $\tilde{\alpha}_i' = \tilde{\alpha}_i''$, i =1,2,3.

(ii) If $Im\,\tilde{\alpha}_2' \neq 0$ there exists one and only one different family $\tilde{\alpha}_i''$:

$$\tilde{\alpha}_1'' = \tilde{\alpha}_1' , \quad \tilde{\alpha}_2'' = \tilde{\alpha}_3' , \quad \tilde{\alpha}_3'' = \tilde{\alpha}_2' .$$

For both the cases we get that the signature which gives according to (12) the values α_1'', α_2'', α_3'' must be $(2;\ \alpha_1' + \frac{2}{3}\alpha_2',\ -\alpha_2',\ \alpha_1' + \frac{4}{3}\alpha_2')$. This signature denotes the only one realization, in the class of realizations with signatures (2; $\alpha_1, \alpha_2, \alpha_3$), in which the eigenvalue of any Casimir operator is the same as in the realization with the signature (2; α_1', α_2', α_3'). So, if $\alpha_2' \neq 0$ there are just two realizations with the same eigenvalues of Casimir operators while for $\alpha_2' = 0$ no further realization leads to the same eigenvalues.

As, generally, two Schur-realizations of a Lie algebra which differ in eigenvalue of some Casimir operator cannot be related (inequivalent mod $\mathrm{End}\, W$) we have proved that in the class of realizations of $u(2,1)$ with signatures (2; α_1, α_2, α_3) any pair of realizations are non-related realizations with the possible exception of pairs with signatures (2; α_1', α_2', α_3') and (2; $\alpha_1' + \frac{2}{3}\alpha_2'$, $-\alpha_2'$, $\alpha_1' + \frac{4}{3}\alpha_2'$) $\alpha_1', \alpha_2', \alpha_3' \in \mathbb{R}$, $\alpha_2' \neq 0$.

This case remains for an independent discussion.

4.Conclusion

The generators of the Lie algebra $u(p,q)$ in the realizations described in this paper are matrices the elements of which are the most simple rational functions of canonical variables. As we already said there is a question if negative powers of the q_c's are, without loss of skew-Hermiticity, necessary or not. The similar situation arises as to the use of matrices in our formulas, i.e. the necessity of use of $W'^{\,r(d)}_{JN(d),M}$ with $M > 1$.

We have seen that in the induction process from Theorem I to Theorem 2 we come to the question of skew-Hermitian Schur-realizations of the compact Lie algebra $u(p,q)$ in $W_{2N}^r \simeq W_{2N,1}^r$. It can be proved that no such a nontrivial realization of $u(p,q)$ exists in $W_{2N,1}^r$ and so some extension of $W_{2N,1}^r$ is necessary. We could look for such an extension, e.g., in the quotient division ring of the Weyl algebra /8/ wherein W_{2N}^r is also contained. As we, however, do not know any example of nontrivial skew-Hermitian Schur-realization of a compact Lie algebra in this structure (with the usual involution) we have taken the other, more simple, extension of W_{2N}^r, the algebra $W_{2N,M}^r$, $M \geq 1$.

If one is interested in applications of the canonical realizations for representation theory there is also a further reason to take $W_{2N,M}^r$, $M \geq 1$. The iterative process to construct realizations of $u(p,q)$ starts from the well-known classification of skew-Hermitian representations of the Lie algebra of the unitary group $U(p,q)$.

If we accept $W_{2N,M}^r$ as the best structure for our purposes we may ask for further skew-Hermitian Schur-realizations of $u(p,q)$ here, not contained in $Mat_M \simeq W_{0,M}^c \subset W_{2N,M}^r$. It is an interesting property of $W_{2N,M}^r$ that no such realization of a compact Lie algebra exists in $W_{2N,M}^r$. This assertion generalizes a result known for $W_{2N} \simeq W_{2N,1}^c$ (see /7/) and $W_{2N,M} \simeq W_{2N,M}^c$ (see /3/) to the localization $W_{2N,M}^r$, and it will be proved together with more detailed study of the presented realizations of $u(p,q)$ elsewhere.

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