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0. On a $*$-algebra $\mathfrak{A}$ of bounded operators on a Hilbert space one defines different topologies, which are all important for investigation of the structure of the algebra. Three of such topologies are:
the uniform topology, defined by the norm

$$\tau_{\mathcal{H}}: \qquad \|A\| = \sup_{\|\phi\|\le 1} \|A\phi\|,$$

the strong topology, defined by the seminorms

$$\sigma^{\mathcal{H}}: \qquad p_{\phi}(A) = \|A\|_{\phi} = \|A\phi\| \quad \text{for all } \phi \in \mathcal{H},$$

and the weak topology, defined by the seminorms

$$\sigma_{\mathcal{H}}: \qquad p_{\phi,\psi}(A) = |\langle \phi, A\psi\rangle| \quad \text{for all } \phi,\psi \in \mathcal{H}.$$

If the $*$-algebra $\mathfrak{A}$ of bounded operators is infinite dimensional, then all three topologies are different.

In this paper we remark the interesting fact, that for a $*$-algebra of unbounded operators the uniform topology $\tau_{\mathfrak{D}}$, generalizing the norm, can coincide with the strong topology $\sigma^{\mathfrak{D}}$. This is of some importance for continuity considerations in observable algebras with unbounded operators, since the strong topology is directly defined by the states ϕ and the square root of the expectation values

$$\|A\|_{\phi} = \|A\phi\| = \langle \phi, A^{+}A\phi\rangle^{1/2}$$

without any limiting procedure.

1. Let us repeat some definitions and properties /1/. Let $\mathfrak{D}$ be a unitary space with scalar product $\langle .,. \rangle$ and let $\|\phi\| = \langle \phi,\phi\rangle^{1/2}$ be the norm in $\mathfrak{D}$. $\mathcal{H}$ denotes the Hilbert space which is the completion of $\mathfrak{D}$.

By $\mathcal{L}^+(\mathcal{D})$ we denote the $*$-algebra of all operators $\mathcal{A}$ with $A\mathcal{D}\subset\mathcal{D}$, $\mathcal{D}(A^*)\supset\mathcal{D}$, $A^*\mathcal{D}\subset\mathcal{D}$. The involution is given by $A\to A^+ = A^*|_{\mathcal{D}}$. An Op^*-algebra $\mathcal{A}(\mathcal{D})$ is a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$ with identity 1. An Op^*-algebra can be equipped with different topologies (cf. /1, 2/), first with the uniform topology $\tau_{\mathcal{D}}$, defined by all seminorms

$$\sigma_{\mathcal{D}}: \qquad \|A\|_{\mathfrak{M}} = \sup_{\phi,\psi\in\mathfrak{M}} |\langle\phi, A\psi\rangle| \qquad \mathfrak{M}\subset\mathcal{D}$$

where $\mathfrak{M}$ runs over all subsets $\mathfrak{M}\subset\mathcal{D}$, for which $\|A\|_{\mathfrak{M}}<\infty$ for all $A\in\mathcal{A}$. Second we have also the strong topology $\sigma^{\mathcal{D}}$, defined quite analogously as in the bounded case by all seminorms

$$\sigma^{\mathcal{D}}: \qquad \|A\|_{\phi} = \|A\phi\| \qquad \phi\in\mathcal{D}$$

and also the weak topology $\sigma_{\mathcal{D}}$, which we do not regard in what follows.

In this paper we consider only a special case of Op^*-algebras. Let $\mathcal{P}(x) = \{p(x) = \sum_{n\geq 0} a_n x^n, a_n\in\mathbf{C}\}$ be the algebra of all polynomials in one variable x. A realization of $\mathcal{P}(x)$ as an Op^*-algebra $\mathcal{A}(\mathcal{D}) = \mathcal{P}(T)$ is given by the mapping $x\to T = T^+\in\mathcal{L}^+(\mathcal{D})$:

$$\mathcal{P}(T) = \{p(T) = \sum_{n\geq 0} a_n T^n, \qquad a_n\in\mathbf{C}\}$$

Now on the polynomial algebra $\mathcal{P}(T)$ the uniform topology $\tau_{\mathcal{D}}$ coincides with the strongest locally convex topology $\tau_\sim$ on $\mathcal{P}(T)$ (cf. /1/, Theorem 5.1). The strongest locally convex topology $\tau_\sim$ is given by the set of all seminorms $\{\|p(T)\|_{(\gamma_n)}, (\gamma_n)$ an arbitrary sequence of non-negative numbers $\}$:

$$\|p(T)\|_{(\gamma_n)} = \sum_{n\geq 0}\gamma_n|a_n|, \quad \text{where } p(T) = \sum_{n\geq 0} a_n T^n. \tag{1}$$

It is clear that we can restrict ourselves to sequences (γ_n) with $1\leq\gamma_0\leq\gamma_1\leq\dots$.

The next theorem gives a quite general condition on the operator T and on the domain $\mathcal{D}$ which is sufficient for $\sigma^{\mathcal{D}} = \tau_{\mathcal{D}} = \tau_\sim$.

2. We formulate

Theorem

Let T be a symmetric unbounded operator on the domain $\mathcal{D}$ with $T\mathcal{D}\subset\mathcal{D}$ and $\mathcal{P}(T)$ the Op^*-algebra, generated by T. If for any sequence (δ_n), $\delta_n\geq 0$ there exists a $\phi\in\mathcal{D}$ such that

$$\|\phi\|^2 \geq \delta_0 \tag{2}$$

$$\langle T^{2n}\phi,\phi\rangle \geq \delta_n\{\max\{|\langle T^i\phi,\phi\rangle|, 0\leq i\leq 2n-1\}\}^{n+1} \tag{3}$$

then the topologies $\sigma^{\mathcal{D}}$ and $\tau_\sim$ coincide, $\sigma^{\mathcal{D}} = \tau_\sim$.

Proof

1. $\sigma^{\mathcal{D}}\subset\tau_\sim$ is trivial, because $\tau_\sim$ is the strongest locally convex topology.

2. To show $\tau_\sim\subset\sigma^{\mathcal{D}}$, we consider the system of seminorms

$$\|p(T)\|_{(\gamma_n)} = (\sum\gamma_n^2|a_n|^2)^{1/2}$$

which gives the same topology $\tau_\sim$ as (1).

We show now that for any seminorm $\|\;\|_{(\gamma_n)}$ where exists a vector $\phi\in\mathcal{D}$ such that

$$\|p(T)\|_{(\gamma_n)} \leq C\|p(T)\|_{\phi} \qquad \text{for all } p(T)\in\mathcal{P}(T). \tag{4}$$

The inequality (4) can be written as

$$\sum_{n\geq 0}\gamma_n^2|a_n|^2 \leq \langle\sum_{n\geq 0} a_n T^n\phi, \sum_{m\geq 0} a_m T^m\phi\rangle = \sum_{n,m}\bar a_n a_m\langle T^{n+m}\phi,\phi\rangle,$$

i.e.,

$$\sum_{n,m}(\langle T^{n+m}\phi,\phi\rangle - \gamma_n^2\delta_{nm})\bar a_n a_m \geq 0$$

or

$$\sum_{n,m}(t_{n+m} - \gamma_n^2\delta_{nm})\bar a_n a_m \geq 0, \tag{5}$$

where

$$t_{n+m} = \langle T^{n+m}\phi,\phi\rangle. \tag{6}$$

The quadratic form (5) is positive, if the matrix

$$D = \begin{pmatrix} t_0 - \gamma_0^2 & t_1 & t_2 & \cdot & \cdot & \cdot \\ t_1 & t_2 - \gamma_1^2 & t_3 & \cdot & \cdot & \cdot \\ t_2 & t_3 & t_4 - \gamma_2^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

is positive definite.

A sufficient condition for D to be positive definite if

$$\Lambda_n = \begin{vmatrix} t_0 - \gamma_0^2 & t_1 & \cdot & \cdot & \cdot & t_n \\ t_1 & t_2 - \gamma_1^2 & \cdot & \cdot & \cdot & t_{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ t_n & t_{n+1} & \cdot & \cdot & \cdot & t_{2n} - \gamma_n^2 \end{vmatrix}$$

is positive for all n.

Let $\delta_n = 3(n+1)! \prod_{i=0}^{n} \gamma_i^2$ and choose $\phi \in \mathcal{D}$ such that (2) and (3) hold for this sequence (δ_n). By induction we show that $\Lambda_n \geq 1$ for all n, which proves our Theorem.

$\Lambda_0 = t_0 - \gamma_0^2 \geq 1$

is trivial.

Let $\Lambda_{n-1} \geq 1$, then for Λ_n we have:

$$\Lambda_n = (t_{2n} - \gamma_n^2)\Lambda_{n-1} + r_n$$

$$\geq (t_{2n} - \gamma_n^2) - (n+1)! \prod_{i=0}^{n-1} \gamma_i^2 [\max\{1, \langle T^i\phi,\phi\rangle, 0 \leq i \leq 2n-1\}]^{n+1}$$

$$\geq t_{2n} + 1 - 3(n+1)! \prod_{i=0}^{n} \gamma_i^2 [\max\{1, \langle T^i\phi,\phi\rangle, 0 \leq i \leq 2n-1\}]^{n+1}$$

$$\geq t_{2n} + 1 - \delta_n [\max\{1, \langle T^i\phi,\phi\rangle, 0 \leq i \leq 2n-1\}]^{n+1} \geq 1$$

Q.E.D.

3. The following examples show how rich is the class of domains $\mathcal{D}$ and operators T such that the assertion of the Theorem is valid.

Example 1: (cf. Schmüdgen /3/, Theorem 10, chap. 4).

Let $T = T^+ \in \mathcal{L}^+(\mathcal{D})$ unbounded and such that

$$\mathcal{D} = \bigcap_{n \geq 0} \mathcal{D}(\bar{T}^n)$$

(i.e., $\mathcal{D}[t_{\mathcal{P}(T)}]$ is complete, where $t_{\mathcal{P}(T)}$ is the topology given by the seminorms $\|\phi\|_A = \|A\phi\|$, $A \in \mathcal{P}(T)$) then for $\mathcal{P}(T)$:

$$\sigma^{\mathcal{D}} = \tau_\infty .$$

Example 2:

Let N be the number operator defined on the Schwartz space s (or on $\mathcal{S}$) and (ψ_n) the orthonormal basis of ℓ^2 (or L^2) consisting of the eigenvectors of N:

$$N\psi_n = n\psi_n$$

It is clear that for $\mathcal{P}(N)$ $\tau_\infty = \sigma^{\mathcal{D}}$ holds, because $s = \bigcap_{n \geq 0} \mathcal{D}(N^n)$ is complete in the topology given by the seminorms $\| \ \|_A$, $A \in \mathcal{P}(N)$ (cf. example 1). The choice of ϕ is even constructive, one may choose

$$\phi = \sum_{\ell \geq 0} \phi_\ell ,$$

where $\phi_\ell = a_{n_\ell} \psi_{n_\ell}$ for appropriate n_ℓ depending on the given sequence $\{\delta_n\}$. For a_{n_ℓ} one can take for example:

$$a_{n_\ell} = (n_\ell)^{-\ell+1/2}$$

The next example shows that not only in the case of $t_{\mathcal{P}(T)}$-complete domains $\mathcal{D}$ the assumptions (2) and (3) of the Theorem are fulfilled but also in other cases.

Example 3:

Regard the position operator x on the Schwartz space $\mathcal{S}$ of rapidly decreasing functions, i.e., $\mathcal{D} = \mathcal{D}(x)$, $\mathcal{D} = \mathcal{S}$, $\mathcal{S} \subset L^2(R^1)$, $(xf)(x) = x\, f(x)$ for all $f \in \mathcal{S}$. Remark, that $\mathcal{S}[x\varphi]$ is not complete, because for example the characteristic function of an arbitrary finite interval $[a, b]$ is contained in the closure of $\mathcal{S}[x\varphi]$ but not in $\mathcal{S}[x\varphi]$. Now we want to show: For a given sequence $\{\delta_n\}$ there exists a $\phi \in \mathcal{S}$ such that the assumptions (2) and (3) of the Theorem are fulfilled, i.e.,

$$\|\phi\|^2 \geq \delta_0 \tag{2}$$

$$\langle x^{2n}\phi,\phi\rangle \geq \delta_n[\max\{1, \langle x^i\phi,\phi\rangle,\ 0 \leq i \leq 2n-1,\ n \geq 1\}]^{n+1}. \tag{3}$$

For this, let

$$f(x) \in C_0^\infty, \qquad 0 \leq f(x) \leq 1$$

$$f(x) = \begin{cases} 1 & \text{for } x \in [1,2] \\ 0 & \text{for } x \notin [0,3]. \end{cases}$$

We put

$$a_n = \sup_{x \in [0,3]} |f^{(n)}(x)|.$$

Now we choose a sequence $\{\ell_n\}$ of naturals with the condition

$$\ell_0 = 0, \quad \ell_n + 3 < \ell_{n+1} \qquad n \geq 0 \tag{7}$$

and form

$$\phi_0(x) = \lambda f(x),$$

$\lambda \geq 1$ and such that (2) holds

$$\phi_n(x) = (a_0 + a_1 + \dots + a_n + 1)^{-1} \ell_n^{-n+1/4} f(x - \ell_n) \qquad n \geq 1$$

We put

$$\phi = \Sigma\, \phi_n .$$

Because of (7)

$$\langle x^i \phi_n, x^j \phi_m \rangle = 0 \qquad \text{for all } i, j, n \neq m .$$

If we choose the $\{\ell_n\}$ in an appropriate way, then ϕ satisfies the assumptions (2) and (3) too. For this we have to show:

$$\sum_m \langle x^{2n}\phi_m,\phi_m\rangle \geq \delta_n[\max\{1, \sum_m \langle x^i\phi_m,\phi_m\rangle,\ 1 \leq i \leq 2n-1\}]^{n+1} \tag{8}$$

for all $n \geq 1$.

Regard the left hand side:

$$\sum_m \langle x^{2n}\phi_m,\phi_m\rangle \geq \langle x^{2n}\phi_n,\phi_n\rangle$$

$$= \int_{\ell_n}^{\ell_n+3} x^{2n} \frac{f^2(x-\ell_n)}{(a_0+\dots+a_n+1)^2 \ell_n^{2n-1/2}}\, dx$$

$$\geq \frac{1}{(a_0+\dots+a_n+1)^2} \frac{(\ell_n+1)^{2n}}{\ell_n^{2n-1/2}}$$

$$\geq \frac{\ell_n^{1/2}}{(a_0+\dots+a_n+1)^2} .$$

This can be obtained by more use of the properties of $f(x)$. On the other hand we get for the right hand side:

$$\delta_n[\max\{1, \sum_m \langle x^i \phi_m,\phi_m\rangle,\quad 0 \leq i \leq 2n-1\}]^{n+1}$$

$$= \delta_n[\max\{\sum_m \int_{\ell_m}^{\ell_m+3} x^i \frac{f^2(x-\ell_m)}{(a_0+a_1+\dots+a_m+1)\ \ell_m^{2m-1/2}}\, dx\}]^{n+1}$$

$$\le \delta_n 3^{n+1}\left[\sum_{m=0}^{n-1}\frac{(\ell_m+3)^{2n-1}}{\ell_m^{2m-\frac12}} + \sum_{m=n}^{\infty}\frac{(\ell_m+3)^{2n-1}}{\ell_m^{2m-\frac12}}\right]^{n+1}.$$

The first sum in the brackets is a function $A_{n-1}(\ell_1,\dots,\ell_{n-1})$ of the ℓ_i for $i \le n-1$ and the second sum is smaller than 2^{2n}, since $\ell_m + 3 \le 2\ell_m$. Therefore

$$\delta_n\left[\max\{1,\ \sum_m \langle x^i\phi_m,\phi_m\rangle,\quad 0\le i\le 2n-1\}\right]^{n+1}$$

$$\le \delta_n 3^{n+1}\left[A_{n-1}(\ell_1,\dots,\ell_{n-1}) + 2^{2n}\right]^{n+1}.$$

From (8) by the above estimations it follows that

$$\langle x^{2n}\phi,\phi\rangle \ge \sum_m \langle x^{2n}\phi_m,\phi_m\rangle$$

$$\ge (a_0+\dots+a_n+1)^{-2}\,\ell_n^{\frac12}$$

$$\ge \delta_n 3^{n+1}\left[A_{n-1}(\ell_1,\dots,\ell_{n-1}) + 2^{2n}\right]^{n+1}$$

$$\ge \delta_n\left[\max\{1,\ \langle x^i\phi,\phi\rangle,\quad 0\le i\le 2n-1\}\right]^{n+1}$$

can be fulfilled for large ℓ_n.

Now we have still to show that $\phi = \sum\phi_n$ is contained in the Schwartz space $\mathcal{S}$. It is clear that $\phi\in C^\infty(R^1)$. We show:

$$\sup_{x\in R^1}|x^r\phi^{(s)}(x)| < \infty \qquad \text{for all } r,s\in N$$

We write

$$\sup|x^r\phi^{(s)}(x)| = \sum_n \sup|x^r\phi_n^{(s)}(x)|$$

$$\le \sum_{n\le s}\sup|x^r\phi_n^{(s)}(x)| + \sum_{n\le s}\frac{(\ell_n+3)^r}{\ell_n^{n-\frac14}}\ \frac{a_s}{(a_0+\dots+a_n+1)}$$

$$< \sum_{n\le s}\sup|x^r\phi_n^{(s)}(x)| + C < \infty$$

because

$$\frac{a_s}{(a_0+\dots+a_n+1)} \le 1$$

and

$$\sum \frac{(\ell_n+3)^r}{\ell_n^{n-\frac14}}$$

converges.

So we have constructed a $\phi\in\mathcal{S}$ which satisfies the assumptions (2) and (3) of our Theorem.

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