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ON HAMILTONIAN ANALYSIS OF BIANCHI IX COSMOLOGY*

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1 Introduction

It is the purpose of this report to state the generalized Hamiltonian dynamics [1] of the spatialy homogeneous Bianchi IX cosmological model without matter sources.

The cosmological models due to the existence of additional rigid spacetime symmetries are incomparably simpler than the underlying theory of gravity. At the same time, they possess the main features of the full theory and thus can be used as a laboratory for testing viability of new ideas and techniques. The traditional method in Hamiltonian analysis of cosmological models that has been extensively used is the Arnowitt-Deser-Misner (ADM) formulation of canonical theory of gravitation [2]. The ADM method is based on certain fixing of coordinate condition (gauge) and solving of constraints. The crucial unsolved problem in this approach is the proof of the independence of observable quantities from any possible choice of the gauge condition. To clarify the problem we shall study the Bianchi cosmological models using the alternative method [3], [4] to construct the observables in the constrained system without supposing any gauge condition. The first step in the application of this gaugeless approach is the abelianization of constraints i.e., the conversion of the initial non-Abelian constraints to the equivalent set of Abelian ones. Below, we shall construct the matrix transforming the constraints to the abelian form for the non-diagonal Bianchi IX cosmology with constraints obeying the $SO(3)$ algebra. The realization of this conversion allows us to find the explicit connection of dynamics of diagonal and non-diagonal Bianchi IX cosmology. The conclusion is that the dynamics differs only for non-physical degrees of freedom and observables for diagonal and non-diagonal cases are one and the same.

2 Spacetime decomposition

Canonical analysis views the Universe in terms of space plus time [5]. Thus, we suppose that the spacetime is a smooth manifold, $\mathcal{M} = \Sigma_t \times R$, endowed with a metric g of signature $(-,+,+,+)$, metric-compatible connection and time function t. The level surfaces of t, Σ_t, are spacelike and form a foliation of a spacetime manifold. This means that they are nonintersecting and fill $\mathcal{M}$. After the foliation of the spacetime manifold it is useful to choose on $\mathcal{M}$ a surface compatible moving coframe $(\mathbf{e}_\perp, e_a)$ [1] with four-dimensional unit-length vector field $\mathbf{e}_\perp$ orthogonal to Σ_t and three dimensional vector fields $e_a = (e_1, e_2, e_3)$ tangent to it. [2] The corresponding dual frame has a time axis orthogonal to the slices Σ_t while the space axes are tangent to them. In this frame, the metric $\mathbf{g}$ reads

$$\mathbf{g} = -\boldsymbol{\theta}^\perp \otimes \boldsymbol{\theta}^\perp + \gamma_{ab}\,\theta^a \otimes \theta^b, \tag{1}$$

with the spatial metric γ induced on Σ_t. To implement the canonical analysis, one can specify a time like vector $\mathbf{e}_0 \equiv \frac{\partial}{\partial t}$; "time flow" vector field on $\mathcal{M}$ which will describe the evolution with the time parameter t. The well-known Dirac-ADM metric [5] follows from (1) after fixing the coordinate coframe $\mathbf{e}_a = (\frac{\partial}{\partial x^a})$ and rewriting the vector field $\mathbf{e}_0$ in terms of the normal vector field $\mathbf{e}_\perp$ and spatial vector field $N^a e_a$ tanget to the hypersurface Σ_t $\mathbf{e}_0 = N\mathbf{e}_\perp + N^a e_a$

$$\mathbf{g} = -(N^2 - N^a N_a)\mathbf{d}t \otimes \mathbf{d}t + 2N_a \mathbf{d}t \otimes dx^a + \gamma_{ab}\, dx^a \otimes dx^b. \tag{2}$$

The spatial metric γ, the lapse function N and shift vector N^a are treated as field configuration variables for the gravitational field. Their classical behaviour

[1] We use boldface to distinguish four-dimensional quantities from three-dimensional ones.

[2] According to this decomposition, the Lie derivative $\mathcal{L}_{\mathbf{e}_\perp}$, derivative with respect the proper time along the normal to Σ_t, will describe the evolution having physical meaning.

is determined by varying the Hilbert-Einstein action

$$A[N, N^a, \gamma] = \int_{\mathcal{M}} \boldsymbol{\eta}\mathbf{R}, \tag{3}$$

where $\mathbf{R}$ is the spacetime curvature scalar and $\boldsymbol{\eta} = \sqrt{-\boldsymbol{g}}\,\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3$ is the four-dimensional volume element.

To proceed further from this general canonical formulation of the gravitation, let us consider the spacetime that contain some rigid symmetry and use this symmetry to restrict the gravitational configuration space in the Hilbert-Einstein action (3). In the case of large symmetry the gravitational degrees of freedom are reduced to finite number and this circumstance essentially relieves the analysis of the theory. Below we shall investigate the restriction of gravitational configuration space by the requirement of spatial homogeneity that leads to so-called Bianchi cosmological models. [3]

3 Model description

By definition, in spatial homogeneous spacetime a three-dimensional Lie group G_3 acts on spacetime as a group of isometries, such that each orbit on which G_3 acts simply transitively is a spacelike hypersurface. The advantage of considering simply transitive action is that we can put the element of G_3 into one-to-one correspondence with the points of Σ_t. After this identification the spacetime is considered topologically as the product space $G_3 \times R$. After mentioning this observation it is clear that instead of usual coordinate coframes we need to choose a new space coframe e_a adapted to the Lie group structure of the three-dimensional hypersurface Σ_t. The algebra of infinitesimal generators of isometries, i.e., Killing fields $\boldsymbol{\xi}_a$, $a = 1, 2, 3$

$$[\boldsymbol{\xi}_a, \boldsymbol{\xi}_b] = C^c{}_{ab}\boldsymbol{\xi}_c \tag{4}$$

[3] For details we refer to one of many comprehensive reviews [6].

dictates this choice. The vector fields e_0 and ξ_a provide a basis of a coframe invariant under the isometries $\mathcal{L}_{\xi_a} e_0 = 0$, $\mathcal{L}_{\xi_a} e_a = 0$. In this case, one finds the form of a space metric for the Bianchi models $\gamma = \gamma_{ab}\,\omega^a \otimes \omega^b$ with a group invariant frame ω^a whose structure coefficients $\frac{1}{2}C^a_{\cdot bc} = d\omega^a(e_b, e_c)$ are structure constants of the homogeneity group G_3. The preferable role of this choice for a coframe is clear: from the Killing equation $\mathcal{L}_{\xi_a}\mathrm{g} = 0$ it immediately follows that the functions N, N^a and γ_{ab} depend only on the time parameter t. Due to this simplification the initial variational problem for Bianchi A models [4] is restricted to a variational problem of the "mechanical" system

$$L\,(N, N_a, \gamma_{ab}, \dot\gamma_{ab}) = \int_{t_1}^{t_2} dt \sqrt{\gamma} N \left[{}^3R - K_a{}^a K_b{}^b + K_{ab}K^{ab} \right], \tag{5}$$

where ${}^3R$ is the curvature scalar formed from the spatial metric γ

$${}^3R = -\frac{1}{2}\gamma^{ab}C^c{}_{da}C^d{}_{cb} - \frac{1}{4}\gamma^{ab}\gamma^{cd}\gamma_{ij}C^i{}_{ac}C^j{}_{bd}, \tag{6}$$

and

$$K_{ab} = -\frac{1}{2N}\left((\gamma_{ad}C^d{}_{bc} + \gamma_{bd}C^d{}_{ac})N^c + \dot\gamma_{ab} \right) \tag{7}$$

is the extrinsic curvature of the slice Σ_t defined by the relation $K_{ab} = -\frac{1}{2}\mathcal{L}_{e_\perp}\gamma_{ab}$. The Lagrangian (5) belongs to the class of so-called degenerate ones. Thus, to deal with the Hamiltonian description we need the Dirac generalization of Hamiltonian dynamics [1].

4 Hamiltonian formulation

Implementing the Legendre transformation on variables N, N_a and γ_{ab} we get the canonical Hamiltonian $H_C = N\mathcal{H} + N^a\mathcal{H}_a$, the primary $P^a = 0$, $P^0 = 0$

[4]Writing the structure constants of the isometry Lie group in the general form, $C^d_{ab} = \epsilon_{lab}S^{ld} + A_{[a}\delta^d_{b]}$, class A models are those for which $C^d_{ad} = A_a = 0$.

and secondary constraints

$$\mathcal{H} = \frac{1}{\sqrt{\gamma}}\left[\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^a{}_a\pi^b{}_b \right] - \sqrt{\gamma}\,{}^3R, \tag{8}$$

$$\mathcal{H}_a = 2\,C^d{}_{ab}\pi^{bc}\gamma_{cd}. \tag{9}$$

Due to the reparametrization symmetry of (5) inherited from the diffeomorphism invariance of the initial Hilbert-Einstein action, the evolution of the system is unambiguous and it is governed by the total Hamiltonian

$$H_T = N\mathcal{H} + N^a\mathcal{H}_a + u_0P^0 + u_aP^a, \tag{10}$$

with four arbitrary functions $u_a(t)$ and $u_0(t)$. One can verify that the secondary constraints are first class and obey the algebra

$$\{\mathcal{H}, \mathcal{H}_b\} = 0, \quad \{\mathcal{H}_a, \mathcal{H}_b\} = -C^d{}_{ab}\mathcal{H}_d. \tag{11}$$

To provide the explicit construction of true dynamical degrees of freedom without gauge fixing according to the general scheme [3], [4], it is necessary to implement two operations: to convert the constraints (8) to the new equivalent set of "commuting" constraints

$$\Phi_a = \mathcal{C}_{ad}\mathcal{H}_d \tag{12}$$

i.e., to abelianize the first class constraints (11) and to perform the canonical transformation to new coordinates so that a part of the new canonical variables coincides with the new Abelian constraints.

5 Hamiltonian reduction: canonical transformation and abelianization

Let us concentrate our attention on the Bianchi type IX model. [5] For this model it is very useful at first to realize certain canonical transformation that essentially

[5]For the Bianchi IX model the symmetric matrix S is the unit matrix.

simplifies the procedure of abelianization. This canonical transformation means a passing from coordinates (γ_{ij}, π^{ij}) to the well-known Misner representation [5]. In Misner's representation the spatial metric is given by

$$\gamma_{ij} = R_0^2 e^{-2\Omega} e^{2\beta}{}_{ij}, \tag{13}$$

where β_{ij} is a 3×3 symmetric traceless matrix and Ω is a scalar, both being functions of time parameter only, R_0 is a constant. Whenever Ω is a monotonic function of time, one can choose Ω as a scale factor for cosmology related to the volume through $R_0^6 e^{-6\Omega} = det\gamma$. To realize the abelianization it is convenient [6] to use for nondegenerate symmetric matrix β the following decomposition: $\beta = \mathcal{R}^{-1}(\phi, \theta, \psi)\mathcal{D}\mathcal{R}(\phi, \theta, \psi)$ with the $SO(3)$ matrix

$$\mathcal{R}(\phi, \theta, \psi) = e^{\phi k_3} e^{\theta k_1} e^{\psi k_3} \tag{14}$$

parametrized with the Euler angles and diagonal traceless matrix

$$\mathcal{D} = diag(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+). \tag{15}$$

In terms of new canonical coordinates $\theta, p_\theta; \phi, p_\phi; \psi, p_\psi; , \beta_-, p_-; \beta_+, p_+$ the momentum constraints can be rewritten as

$$\mathcal{H}_a = \left(\mathcal{R}^{-1}\mathcal{T}\Phi\right)_a, \tag{16}$$

where the $\Phi_a = (p_\theta, p_\psi, p_\phi)$ and $\mathcal{T}$ is the following matrix

$$\mathcal{T} = \begin{pmatrix} 0 & 1 & 0 \\ \sin\psi & \cot\theta\cos\psi & \frac{\cos\psi}{\sin\theta} \\ \cos\psi & \cot\theta\sin\psi & \frac{\sin\psi}{\sin\theta} \end{pmatrix}. \tag{17}$$

It is clear that exept when $\sin\theta = 0$ the matrix $\mathcal{T}^{-1}\mathcal{R}$ is just the matrix of abelianization C in (12). So, after implementing the Dirac transformation to the initial constraints $\mathcal{H}_a$, the equivalent set of Abelian constraints is

$$\Phi_1 = p_\psi = 0, \qquad \Phi_2 = p_\theta = 0, \qquad \Phi_3 = p_\phi = 0 \tag{18}$$

To complete the abelianization stage of reduction let us rewrite energy constraint in new coordinates

$$\mathcal{H} = \frac{1}{6}(p_+^2 + p_-^2 - p_\Omega^2) + W(\phi, \theta, \psi, p_\phi, p_\theta, p_\psi) - {}^3R(\beta_+, \beta_-)\ R_0^4 e^{-4\Omega}, \tag{19}$$

where

$$W = \frac{1}{2}\left(\frac{p_\phi^2}{\sinh^2(2\sqrt{3}\beta_-)} + \frac{(\sin\phi\sin\theta\ p_\theta + \cos\phi\cos\theta\ p_\phi - \cos\phi\ p_\psi)^2}{\sinh^2(3\beta_+ + \sqrt{3}\beta_-)\sin^2\theta} + \right.$$
$$\left. + \frac{(\cos\phi\sin\theta\ p_\theta - \sin\phi\cos\theta\ p_\phi + \sin\phi\ p_\psi)^2}{\sinh^2(3\beta_+ - \sqrt{3}\beta_-)\sin^2\theta}\right), \tag{20}$$

The three-dimensional scalar curvature ${}^3R$ for the Bianchi IX model is

$${}^3R = -\frac{1}{2}\left(e^{4(\beta_+ + \sqrt{3}\beta_-)} + e^{4(\beta_+ - \sqrt{3}\beta_-)} + e^{-8\beta_+} - \right.$$
$$\left. -2e^{-2(\beta_+ + \sqrt{3}\beta_-)} - 2e^{-2(\beta_+ - \sqrt{3}\beta_-)} - 2e^{4\beta_+}\right). \tag{21}$$

The price for the passing to new constraints (18) is that the energy constraint (19) does not commute with them

$$\{\mathcal{H}, \Phi_a\} = f_a^b \Phi_b,$$
$$\{\Phi_a, \Phi_b\} = 0, \tag{22}$$

where f_a^b are a certain functions on phase space. However, one can again apply the Dirac equivalent transformation of constraints. Keeping the momentum constraints Φ_a unchanged and shifting the energy constraint

$$\overline{\mathcal{H}} = \mathcal{H} + p_\theta C_\theta + p_\psi C_\psi + p_\phi C_\phi \tag{23}$$

we shall get the set of Abelian momentum constraints (18) and new energy constraint

$$\overline{\mathcal{H}} = \frac{1}{6}(p_+^2 + p_-^2 - p_\Omega^2) + {}^3R(\beta_+, \beta_-)\ R_0^4 e^{-4\Omega}, \tag{24}$$

The arbitrariness of the functions u in the total Hamiltonian (10) reflects the presence in the theory of variables whose dynamics is governed in an arbitrary way. The conversion to the equivalent Abelian set of constraints allows us to separate these ignorable variables from physical one whose classical behavior is uniquely determined. It is clear that in the case of the Bianchi IX model ψ, ϕ, θ are just these ignorable coordinates. The fourth ignorable coordinate is connected with the remaining energy constraint (24) It is worth to note that this constraint coincides with the corresponding Hamiltonian ones for so-called diagonal Bianchi IX cosmological model. This means that in terms of the Misner variables after abelianization of constraints the dynamics of diagonal and non-diagonal Bianchi IX cosmology differs only for the non-physical degrees of freedom ψ, ϕ, θ while the dynamics for the physical variables is one and the same.

To find the fourth ignorable coordinate it is necessary to analyze the energy constraint $\overline{\mathcal{H}}$. For this purpose it is useful to implement the set of canonical transformations

$$\beta_1 = \beta_+ + \sqrt{3}\beta_- - \Omega \qquad p_1 = \frac{1}{6}\,p_+ + \frac{1}{2\sqrt{3}}\,p_- - \frac{1}{3}\,p_\Omega \tag{25}$$

$$\beta_2 = \beta_+ - \sqrt{3}\beta_- - \Omega \qquad p_2 = \frac{1}{6}\,p_+ - \frac{1}{2\sqrt{3}}\,p_- - \frac{1}{3}\,p_\Omega \tag{26}$$

$$\beta_3 = -2\beta_+ - \Omega \qquad p_3 = -\frac{1}{3}\,p_+ - \frac{1}{3}\,p_\Omega \tag{27}$$

and

$$b_i = \exp(\beta_i) \qquad \mathcal{P}_i = p_i b_i, \qquad i = 1,2,3. \tag{28}$$

In this new canonical basis the energy constraint reads

$$\overline{\mathcal{H}} = \frac{1}{2}\sum_{i=1}^{3} b_i^2\mathcal{P}_i^2 - \sum_{i<j}^{3} b_i\mathcal{P}_i\, b_j\mathcal{P}_j + \frac{1}{2}\sum_{i=1}^{3} b_i^4 - \sum_{i<j}^{3} b_i^2\, b_j^2 \tag{29}$$

Starting with this representation one can reduce the problem of the Hamiltonian description of our constrained system to the analysis of motion of "free particle"

with zero "energy" on the three-dimensional hyperbolic manifold with a certain metric. To achieve this let us again perform the canonical transformation that absorbs the "potential term" in (24)

$$\begin{aligned} b_i &= \sqrt{2\Pi_i}\sin\eta_i, \\ \mathcal{P}_i &= \sqrt{2\Pi_i}\cos\eta_i. \end{aligned} \tag{30}$$

As a result the energy constraint becomes

$$\overline{\mathcal{H}} = \Pi_a\, \mathcal{G}^{ab}\Pi_b,$$

where the "metric"

$$\mathcal{G}^{ab} = 2\sin\eta_a\ \sin\eta_b\Big(\delta_{ab} + (\delta_{ab} - 1)\ \cos(\eta_a - \eta_b)\Big) \tag{31}$$

has been introduced. Now it is clear that one can easily determine the corresponding ignorable coordinates if this metric possesses the Killing vector. For example, there is a simple case when this metric admits the symmetry. If one suppose, that $\eta \equiv \eta_1 = \eta_2 = \eta_3$ then the metric $\mathcal{G}$ can be transformed to the diagonal form

$$\overline{\mathcal{G}} = \left(4sin^2\eta\right)\,\mathrm{diag}\,(-1/2, 1, 1)$$

with the help of the constant orthogonal transformation $\mathcal{G} = \ \mathcal{O}^T\,\overline{\mathcal{G}}\,\mathcal{O}$. After the implementing the canonical transformation

$$\begin{aligned} \Pi^*_a &= \mathcal{O}_{ac}\Pi^c, \\ \eta^*_a &= \mathcal{O}_{ac}\eta_c. \end{aligned} \tag{32}$$

the energy constraint reduces to the simple diagonal form

$$\overline{\mathcal{H}}_0 = -1/2\Pi_1^{*2} + \Pi_2^{*2} + \Pi_3^{*2} = 0 \tag{33}$$

and this means that our reduced system is equivalent to the motion of a free " massless relativistic particle " in the three-dimensional flat Minkowski space-time. It is interesting that the same reduced system has been obtained in [7] for the so-called diagonal, intrinsically multiply transitive models (DIMT).

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