S.A.Gogilidze[1], A.M.Khvedelidze[2], V.V.Papoyan[3],
Yu.G.Palii, V.N.Pervushin

DIRAC AND FRIEDMANN OBSERVABLES IN QUANTUM UNIVERSE WITH RADIATION

[1]Permanent address: IHEP, Tbilisi State University, 380086, Tbilisi, Georgia
[2]Permanent address: Tbilisi Mathematical Institute, 380093, Tbilisi, Georgia
[3]Permanent address: Yerevan State University, 375049, Yerevan, Armenia

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1   Statement of the problem

There is a hope of solving fundamental problems of cosmology of the early Universe by help of quantum gravity [1, 2, 3, 4, 5]. The problem of quantization has stimulated the development of the Hamiltonian approach to the theory of gravity and cosmological models of the Universe. A lot of papers and some monographs (see e.g. [6, 7]) have been devoted to the Hamiltonian description of cosmological models of the Universe. The main peculiarity of the Hamiltonian theory of gravity is the presence of nonphysical variables and constraints. They arise due to the diffeomorphism invariance of the theory which is the basis of the difficulties with the solution for the important conceptual problems
- treatment of the observable time in classical cosmology - interpretation of the wave function and its non-normalizability ,
- relations between the observational cosmology (the Hubble law and red shift) and the Dirac observables in the Hamiltonian description of the classical and quantum cosmologies.

One of the possible solution of these problems in the Hamiltonian approach is to reduce the initial constraint system to unconstrained one by separation of pure gauge degrees of freedom from physical ones. In the present paper, we would like to apply recently developed method of the Hamiltonian reduction of singular systems with the full separation of the gauge sector [8, 9] to a simple, but important, cosmological model of the Universe with scalar field to investigated the problems listed above and to compare our reduced quantization with the extended approach [3, 4, 5].

The content of the paper is the following. Section 2 is devoted to observational cosmology. In Section 3, we present the Lagrangian model the equations of which coincide with the ones of the Friedmann Universe filled in by radiation. In Section 4, the gaugeless version of the Dirac Hamiltonian description [8, 9] of the model is expounded, and the phase space reduction is fulfilled by separating of the physical and nonphysical sectors. In Section 5, we establish the relation between the Dirac observables in the Hamiltonian approach and the Friedmann ones in the classical cosmology. Section 6 is devoted to the quantization of the model in the reduced phase space and the description of the cosmological observables in quantum theory. In Section 7, the functional integral is constructed which is adequate to the gaugeless quantization. In Section 8, we show how to modify the Wheeler – DeWitt wave function so that it describes the Friedmann cosmological observables. The conclusion is devoted to the discussion and physical interpretation of the results.

2   Observational cosmology.

2.1   Experimental data.

One of the main facts of the observational cosmology is correlation between the distance of an astronomical object (R_F) to the Earth and the red shift z (in $\hbar = c = 1$ units)

$$z = \frac{\lambda(T_F)}{\lambda(T_F - R_F)} - 1 = \frac{1}{\lambda(T_F)}\frac{d\lambda(T_F)}{dT_F}R_F + \cdots \; ; \quad R_F \ll T_F \qquad (1)$$

where $\lambda(T_F)$ is wave length of photon radiated by an atom on the Earth and $\lambda(T_F - R_F)$ wave length of photon radiated by an atom on an astronomical object at the time $(T_F - R_F)$.

The quantity $\frac{1}{\lambda(T_F)}\frac{d\lambda(T_F)}{dT_F} = H_o$ is known as the "Hubble constant". The present value of this constant [10], [11]

$$H_o \propto (70 \pm 15) \; \frac{\text{km}}{\text{s Mpc}},$$

gives the scale of the observational cosmology.

2.2   Theoretical interpretation.

There are two interpretations of this experimental fact. The recent theoretical cosmology is based on the Friedmann solution of equations of general relativity for the case of homogeneous and isotropic distribution of matter in the Universe [12]. It is important to emphasize that classical cosmology uses the comoving frame of reference with the Friedmann – Robertson – Walker metric

$$(ds_F)^2 = dT_F^2 - a^2(T_F)\gamma_{ij}dx^i dx^j, \qquad (2)$$
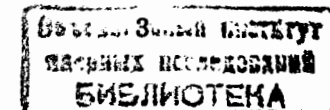
where $a(T_F)$ is cosmic scale factor, $\gamma_{ij}dx^i dx^j$ is the metric of the three-dimensional space of the constant curvature

$$^{(3)}R(\gamma_{ij}) = \frac{-6k}{r_o^2} \; ; \quad k = 0, \pm 1, \tag{3}$$

(r_o is a parameter characterizing a "size" of the Universe). As the consequence of such a choice, one supposes that, in cosmology, physically measured quantities are the ones which evolve in the proper (Friedmann) time T_F. The measured quantity of the metric (2) is the distance $R_F(T_F)$ to cosmic objects:

$$R_F(T_F) = a(T_F)R_c \quad R_c = \int \frac{dr'}{\sqrt{1 - kr'^2/r_o^2}}. \tag{4}$$

and the "Hubble constant"

$$H_o = \frac{1}{\lambda(T_F)} \frac{d\lambda(T_F)}{dT_F} R_F = \frac{1}{a(T_F)} \frac{da(T_F)}{dT_F}. \tag{5}$$

The alternative treatment of Hubble law was developed by Narlikar (see review [13] and the literature cited therein). According to Narlikar the measured quantity is the distance R_c

$$R_c = \int\limits_{0}^{T_c(T_F)} \frac{dT_F}{a(T_c)} = T_c(T_F) \tag{6}$$

in the conformal metric

$$(ds_c)^2 = dT_c^2 - \gamma_{ij}dx^i dx^j. \tag{7}$$

In the conformal (Narlikar) frame the Universe is stationary and the "conformal wave length" of a photon does not change during the time of the photon flight from a "star" to the Earth. However, the " conformal mass"

$$m_c(T_c) = m_F a(T_c) \tag{8}$$

is time dependent and this leads to the red shift. In result the Hubble law has the form

$$z = \frac{m_c(T_c)}{m_c(T_c - R_c)} - 1.$$

3 Model

We begin from the Einstein – Hilbert action with the conformal scalar field

$$W = \int d^4x \sqrt{-g} \left[-\frac{^{(4)}R(g_{\mu\nu})}{16\pi G} \left(1 - \frac{16\pi G}{12}\Phi^2\right) + \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi \right]. \tag{9}$$

The Hamiltonian formulation of gravity is fulfilled in the ADM metric [2]

$$(ds_E)^2 = N^2 dt^2 - g_{ij}\check{d}x^i\check{d}x^j \; ; \quad \check{d}x^i = dx^i + N^i dt. \tag{10}$$

In order to derive a set of equations which is completely equivalent to the Friedmann – Einstein ones we choose the metric

$$(ds)^2 = a^2(t)[N_c^2 dt^2 - \gamma_{ij}dx^i dx^j]. \tag{11}$$

and the ansatz for the scalar field

$$\Phi = \frac{\varphi(t)}{a(t)}. \tag{12}$$

Instead of eq. (9) we get the action in the homogeneous approximation

$$W^F = \int\limits_{t_1}^{t_2} dt \left[-\beta \left(\frac{\dot{a}^2}{2N_c} - \frac{ka^2}{2r_o^2}N_c\right) + V_{(3)}\left(\frac{\dot{\varphi}^2}{2N_c} - \frac{k\varphi^2}{2r_o^2}N_c\right) + \right.$$
$$\left. + \frac{\beta}{2}\frac{d}{dt}\left(\frac{\dot{a}a}{N_c}\right) \right]. \tag{13}$$

We retained here one of the total derivatives arising from the gravitational part of the action (9)); $V_{(3)}$ is the volume of the three-dimensional space with the constant curvature and β is a constant coefficient

$$\beta = V_{(3)}\frac{6}{2\pi G} \; ; \quad V_{(3)}|_{k=+1} = 2\pi^2 r_o^3. \tag{14}$$

One can easily be convinced that the set of equations of the model (13) is equivalent to the Friedmann Universe filled in by the matter with the equation of state of the radiation [15].

The variation of the action (13) with respect to the matter field leads to equation of motion φ

$$\frac{\delta W}{\delta \varphi} = 0 \quad \Rightarrow \quad -\frac{d}{N_c dt}\left(\frac{d\varphi}{N_c dt}\right) - \frac{k\varphi}{r_o^2} = 0. \tag{15}$$

The consequence of this equation is the integral of motion

$$E_c(\varphi) = V_{(3)}\left[\frac{1}{2}\left(\frac{d\varphi}{N_c dt}\right)^2 + \frac{k\varphi^2}{2r_o^2}\right] ; \quad \frac{d}{dt}E_c(\varphi) = 0, \tag{16}$$

which plays a role of the conformal energy E_c for the massless scalar field.

The equation on the variable N_c coincides with the known Einstein balance of energy of expanding space and matter

$$\frac{\delta W}{\delta N_c} = 0 \quad \Rightarrow \quad \beta\left[\left(\frac{da}{N_c dt}\right)^2 + \frac{ka^2}{2r_o^2}\right] = E_c(\varphi). \tag{17}$$

The Friedmann evolution results from equations (15), (16) and (17) when the convention about the definition of the proper time of observing (5)

$$dT_F = aN_c dt = adT_c \tag{18}$$

is added to these equations. Substituting eq.(18) into (17) and solving this equation under T_F, we get the Hubble law of the radiation dominant Universe in the parametric form

$$a(T_c) = \sqrt{\frac{2E_c(\varphi)r_o^2}{\beta}} S_k\left(\frac{T_c}{r}\right) ; \quad T_F(T_c) = \int_o^{T_c} dT_c a(T_c) \tag{19}$$

where

$$S_{k=1}(\eta) = \sin \eta ; \quad S_{k=-1}(\eta) = \sinh \eta ; \quad S_{k=0}(\eta) = \eta . \tag{20}$$

Our problem is to find out the connection between the cosmological observables and the Dirac observables of the Hamiltonian approach to the model (13) and establish a bridge between the classical evolution and the wave function of the Universe determined by the WDW equation [3, 4]

$$[-\frac{1}{2a\beta}\frac{d^2}{da^2} + \beta\frac{ka}{2r_o^2} - \frac{E_c(\varphi)}{a}]\Psi_{WDW}(a,\varphi) = 0 \tag{21}$$

which is the quantum analogy of the energy balance equation (17).

4 Gaugeless Hamiltonian reduction

According to the Dirac classification [16] the action (13) is a singular. Following to the generalized Hamiltonian approach to singular theories this action can be rewritten in the form

$$W^F[p_\varphi, \varphi; p_a, a] = \int dt \left\{ p_\varphi\dot{\varphi} - \left[p_a\dot{a} - \frac{1}{2}\frac{d}{dt}(p_a a)\right] - N_c\mathcal{H}_{Ec} \right\}, \tag{22}$$

where

$$\mathcal{H}_{Ec} = -\left(\frac{p_a^2}{2\beta} + \frac{ka^2}{2r_o^2}\beta\right) + \mathcal{H}_\varphi \tag{23}$$

is the conformal version of the Einstein energy and

$$\mathcal{H}_\varphi = \left(\frac{p_\varphi^2}{2V_{(3)}} + \frac{k\varphi^2}{2r_o^2}V_{(3)}\right)$$

is the part describing a homogeneous scalar field (matter).

The considered model (22) faces principal difficulties of the theory of gravity. The main of these difficulties is the presence of nonphysical (ignored) variables. In the phase space $p_\varphi, \varphi; p_a, a$, one of the momenta depends on the others due to the constraint

$$\mathcal{H}_{Ec} = 0.$$

Let us discuss the Hamiltonian reduction in the case when an independent variable is chosen as a matter momentum. For the complete separation of the physical sector from the nonphysical one, we apply the method developed in papers [8, 9]. In accordance with this method, such a separation can be fulfilled using the canonical transformation to new variables

$$(p_a, a) \quad \to \quad (\Pi_a, \eta_a), \tag{24}$$

so that the gravitation part of the constraint for these variables becomes a new momentum

$$\frac{p_a^2}{2\beta} + \frac{ka^2}{2r_o^2}\beta = \Pi_a. \tag{25}$$

There are two possible canonical transformations

$$p_{a(\pm)} = \pm\sqrt{2\beta\Pi_a}C_k(\eta_a) \; ; \; a_{(\pm)} = \pm\sqrt{\frac{2\Pi_a r_o^2}{\beta}}S_k(\eta_a) \qquad (26)$$

where

$$(C_{+1}(\eta_a) = \cos\eta_a \; ; \; C_{-1}(\eta_a) = \cosh\eta_a \; ; \; C_0(\eta_a) = 1). \qquad (27)$$

It is interesting to note, that the surface term of the gravitational part of the Einstein – Hilbert action (15) is completely absorbed by the new canonical structure [17]

$$-(p_a\dot{a} - \frac{d}{dt}(p_a a)) = \mp\Pi_a\dot{\eta}_a r_o. \qquad (28)$$

In terms of the new variables (26) the action (22) reads

$$W_{(\pm)}^F(\Pi_a,\eta_a,p_\varphi,\varphi,N_c) = \int dt\,[p_\varphi\dot{\varphi} \mp \Pi_a\dot{\eta}_a r_o - N_c(-\Pi_a + \mathcal{H}_\varphi)]. \qquad (29)$$

Expression (29) leads to the Hamiltonian equation describing the non-physical sector of the variables (Π_a, η_a)

$$\frac{\delta W_{(\pm)}^F}{\delta\eta_a} = 0 \quad\Rightarrow\quad \pm\dot{\Pi}_a = 0 \qquad (30)$$

$$\frac{\delta W_{(\pm)}^F}{\delta\Pi_a} = 0 \quad\Rightarrow\quad r_o d\eta_a = \pm N_c dt \qquad (31)$$

and the physical (by the Dirac definition [16]) one

$$\frac{\delta W_{(\pm)}^F}{\delta p_\varphi} = 0 \quad\Rightarrow\quad \frac{d\varphi}{N_c dt} = \pm\{\mathcal{H}_\varphi, \varphi\}. \qquad (32)$$

$$\frac{\delta W_{(\pm)}^F}{\delta\varphi} = 0 \quad\Rightarrow\quad \frac{dp_\varphi}{N_c dt} = \pm\{\mathcal{H}_\varphi, p_\varphi\}. \qquad (33)$$

From equation (31) we can see that after transformation (24) the new ignored variable η_a turns into the parameter of time of the evolution of the

Dirac physical variables in the reduced phase space p_φ, φ. This parameter is invariant under the time-reparametrization group transformations of the initial time t which is not observable. We can call the parameter η_a *the Dirac observable time*. The role of the Dirac Hamiltonian in the reduced space is played by the matter part of the Einstein Hamiltonian E_c (23) which coincides with the conventional definition of the matter Hamiltonian in the flat space. For the description of the Dirac physical sector, we can restrict ourselves to the action obtained from (29) by the substitution of the constraint

$$\Pi_a = \mathcal{H}_\varphi. \qquad (34)$$

As a result, we get the reduced action

$$W_\pm^F|_{\Pi_a=\mathcal{H}_\varphi} = W_\pm^{Red} = \int_{t_1}^{t_2} (p_\varphi d\varphi \mp \mathcal{H}_\varphi r_o d\eta_a) \qquad (35)$$

which describes excitations of the scalar field in a cavity of the conformal space with the constant metric γ_{ij}.

Thus, instead of the extended phase space $N, P_N; a, p_{a;\varphi}, p_\varphi$ and the initial action invariant under reparametrizations of the coordinate time $(t \longmapsto t' = t'(t))$, we have got the reduced phase space which contains only the fields of matter and the reduced action with the conformal Hamiltonian $\mathcal{H}_\varphi$ describing the evolution of these fields in the stationary conformal space (7) with respect to the conformal time. All these quantities are invariant under the coordinate time reparametrizations and can be called the Dirac observables [16], including the conformal time.

Our main conclusion is the following: the gaugeless Hamiltonian reduction, satisfying the correspondence principle, leads to the Narlikar conformal frame of reference [13] where the observable space seems stationary and the observable time $(T_c = r_o\eta_a)$ is monotonic for all types of the space [18, 19].

5 Construction of the Friedmann observables in the Hamiltonian scheme

The Friedmann evolution of the Universe is based on the Einstein convention about the observable (proper) time (2)

$$dT_F = a(t)N_c dt \equiv a(\eta_a)r_o d\eta_a, \qquad (36)$$

proper distance

$$R_F = a(\eta_a)R_c, \qquad (37)$$

and proper energy

$$E_F = \frac{E_c}{a(\eta_a)}. \qquad (38)$$

Such an evolution is described by the quantity $a(\eta_a)$ defined through the canonical transformation (26) on the constraint surface (34)

$$a_{(\pm)} = \pm\sqrt{\frac{2E_c r_o^2}{\beta}} S_k(\eta_a), \qquad (39)$$

where E_c is a value of the energy.

6 Quantization in the reduced phase space

As in the case of a relativistic particle, two solutions of the energy constraint corresponding to two reduced actions W_+^{Red}, W_-^{Red} mean that the total wave function of the Universe represents the superposition of two wave functions constructed from these actions

$$\Psi_{Red}(\eta_a, \varphi) = A^+ \Psi_{Red}^{(+)}(\eta_a, \varphi) + A^- \Psi_{Red}^{(-)}(\eta_a, \varphi). \qquad (40)$$

The functions $\Psi^{(\pm)}$ satisfy the Schrödinger equations

$$\pm\frac{1}{ir_o d\eta_a}\Psi_{Red}^{(\pm)} = \hat{\mathcal{H}}_\varphi \Psi_{Red}^{(\pm)}(\eta_a, \varphi), \qquad (41)$$

and the coefficients A^+, A^- can be treated as creation operators of the Universe and anti-Universe [20].

The wave function $\Psi^{(\pm)}(\eta_a, \varphi)$ can be represented in the form of the spectral representation over the complete set of eigenfunctions $< \varphi|n >$ of the reduced Hamiltonian

$$\Psi_{Red}^{(+)}(\eta_a, \varphi) = \sum_{\varepsilon(n)} e^{-i\varepsilon(n)\eta_a r_o} < \varphi|n > \qquad (42)$$

$$\Psi_{Red}^{(-)}(\eta_a, \varphi) = \sum_{\varepsilon(n)} e^{+i\varepsilon(n)\eta_a r_o} < \varphi|n > \qquad (43)$$

where $< \varphi|n >$ satisfies the equations

$$\hat{\mathcal{H}}_\varphi < \varphi|\eta_a >= \varepsilon(n) < \varphi|\eta_a >, \qquad (44)$$

$$\int d\varphi < n_1|\varphi >< \varphi|n_2 >^* = \delta_{n_1 n_2}, \qquad (45)$$

n being set of conserved quantum numbers. There are two Universes the evolution of which is governed by the invariant parameter of the conformal time η_a. In the following, we shall consider the case of a closed space $k = +1$, where $< \varphi|n >$ are the Hermite polynomials, and

$$\varepsilon(n) = \frac{1}{r_o}(n + \frac{1}{2}); \quad n = 0, 1, 2, 3, \ldots . \qquad (46)$$

The spectral decompositions (42), (43) represent wave functions of quantum excitations of the massless scalar field, in a closed cavity of the conformal space. The wave lengths of the excitations (and the region of validity of quantum theory) coincide with the size of the space occupied by the Universe.

Let us show that the obtained wave function (42), in contrast with the WDW one (21), has a direct relation to the Friedmann classical evolution with respect to the Friedmann observable time (36)

$$dT_F^\pm = a_\pm(\eta_a)r_o d\eta_a, \qquad (47)$$

where the scale factor a is expressed through the parameter η_a by the formula of the canonical transformation (26)

$$a_{(\pm)}(\eta_a) = \pm\sqrt{\frac{2\varepsilon(n)r_o^2}{\beta}} \sin(\eta_a), \qquad (48)$$

for each term of the spectral decomposition (42). Taking into account the connection of the Friedmann time (47) with the conformal one $T_c = \eta_a r_o$, we can verify that the result of the variation of the wave function (42) with respect to the Friedmann time determines the Friedmann observable energy of the red shift $E_F = \pm \frac{\varepsilon(n)}{a_\pm}$ (7) for each term of the spectral decomposition (42)

$$
-\frac{d}{idT_F^\pm(a)}\Psi_{Red}^{(\pm)}(\eta_a, \varphi) = \left(\frac{dT_F(a)}{d\eta_a(a)}\right)^{-1}\frac{d}{id\eta_a}\Psi_{Red}^{(\pm)}(\eta_a, \varphi) =
$$
$$
= \sum_{\varepsilon(n)} \frac{\varepsilon(n)}{a_\pm} e^{-i\varepsilon(n)\eta_a r_o} <\varphi|n> . \quad (49)
$$

We can see that the wave functions (42), (43) constructed by the gaugeless reduction have the correct correspondences with the classical evolutions, in both the frames of reference: the Einstein (2) and Narlikar (7). In the conformal frame (7), the Dirac observables coincide with the cosmological ones. In the Einstein frame (2), the cosmological observables connected with the Dirac ones by the conformal transformations with the cosmic scale factor a.

7 Functional integral

To connect the reduced wave function with the WDW one (21) it is useful to write the spectal decomposition of the Green function

$$
G(\eta_1, \eta_2|\varphi_1, \varphi_2) = \sum_{\varepsilon(n)} \Big[e^{i\varepsilon(n)(\eta_1-\eta_2)} <\varphi_1|n><n|\varphi_2>^* +
$$
$$
+ e^{-i\varepsilon(n)(\eta_1-\eta_2)} <\varphi_1|n>^*<n|\varphi_2>\Big] \quad (50)
$$

in the form of the functional integral over the variables of the Dirac physical sector:

$$
G(\eta_1, \eta_2|\varphi_1, \varphi_2) = \int_{\varphi_{\eta_1}=\varphi_1}^{\varphi_{\eta_2}=\varphi_2} D\varphi Dp_\varphi \Big[e^{+iW_+^{Red}(p_\varphi, \varphi)} + e^{iW_-^{Red}(p_\varphi, \varphi)}\Big], \quad (51)
$$

where $W_\pm^{Red}$ are given by formula (35)

$$
W_\pm^{Red} = \int_{\eta_1}^{\eta_2} d\eta_a \left(p_\varphi \frac{d\varphi}{d\eta_a} \mp \mathcal{H}_\varphi r_o\right), \quad (52)
$$

and the role of time is played by the parameter η_a. Formally, we can return to the initial time coordinate t so that $\varphi_1 = \varphi(t_1)$, $\varphi_2 = \varphi(t_2)$ and the action (52) has the form

$$
W_\pm^{Red} = \int_{\eta_a(t_1)=\eta_1}^{\eta_a(t_2)=\eta_2} dt \left(p_\varphi \frac{d\varphi}{dt} \mp \mathcal{H}_\varphi r_o \frac{d\eta_a}{dt}\right) \quad (53)
$$

In the following, we should take into account that η_a is not the variable but the parameter. With the help of the functional δ-function

$$
\int D\Pi_a \delta(-\Pi_a + \mathcal{H}_\varphi) = \int DN_c D\Pi_a \exp\left(i\int_{t_1}^{t_2} dt N_c(-\Pi_a + \mathcal{H}_\varphi)\right) \quad (54)
$$

one can also introduce additional integration and restore all variables of the extended phase space except for the ignored "variable" η_a

$$
G(\eta_1, \eta_2|\varphi_1, \varphi_2) = \int_{\eta_1, \varphi_1}^{\eta_2, \varphi_2} D\varphi Dp_\varphi D\Pi_a DN_c \left[e^{+iW_+^F} + e^{iW_-^F}\right], \quad (55)
$$

where the actions $W_\pm^F$ are defined by eq. (29). Integration over the ignored "variable" η_a results in the infinite gauge factor

$$
\int_{t_1<t<t_2} D\eta_a(t) = \triangle. \quad (56)
$$

This factor can be removed by introducing a δ-function

$$
\int_{t_1<t<t_2} D\eta_a(t)\delta(\eta_a(t)). \quad (57)
$$

We call this additional constraint the "canonical" gauge. Let us insert (57) into (55) and make the transformation to initial variables (p_a, a)

$$\Pi_a(p_a, a) = \left(\frac{p_a^2}{2\beta} + \frac{ka^2}{2r_o^2}\beta \right) \; ; \quad \eta_a(p_a, a)|_{k=1} = \arctan\left(\frac{\beta a}{r_o^2 p_a} \right). \quad (58)$$

Two items in (55) can be joined keeping in mind that

$$\delta\left(\frac{p_a^2}{2\beta} - (\Pi_a - \frac{a^2}{2r_o^2}\beta) \right) = $$
$$\sqrt{2\beta} \left[\delta\left(p_a - \sqrt{\Pi_a - \frac{a^2}{2r_o^2}\beta} \right) + \delta\left(p_a + \sqrt{\Pi_a - \frac{a^2}{2r_o^2}\beta} \right) \right].$$

As a result, we got a functional integral in the Faddeev – Popov form [5] in the canonical gauge (57)

$$G(\eta_1, \eta_2|\varphi_1, \varphi_2) = $$
$$\int\limits_{\eta_1,\varphi_1}^{\eta_2,\varphi_2} D\varphi Dp_\varphi Dp_a Da DN_c \delta\left[\arctan\left(\frac{\beta a}{r_o^2 p_a} \right) \right] e^{iW^F(p_\varphi,\varphi;p_a,a;N_c)} (59)$$

We remind that the naive functional integral over the whole phase space has the form

$$G(a_1, a_2|\varphi_1, \varphi_2) = \int\limits_{\varphi_1,a_1}^{\varphi_2,a_2} Dp_\varphi D\varphi DN_c Dp_a Da e^{iW_{ADM}^F} \quad (60)$$

and differs from the canonical result (59) as follows:
   i) by the infinite gauge factor (56),
   ii) by the surface term

$$W_{ADM}^F = W^F - \int dt \left[\frac{1}{2}\frac{d}{dt}(p_a a) \right]. \quad (61)$$

Just the same naive functional integral including the infinite gauge factor has been discussed by Hartle and Hawking [21] and until now is used by many others (see, for example, [22]).

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8   Wheeler – DeWitt equation

In this section we shall discuss the connection between the obtained wave function (42), (43) and the solution for the Wheeler – DeWitt equation (21).

A conformal version of this equation has the following form:

$$\left[-\left(\frac{p_a^2}{2\beta} + \frac{ka^2\beta}{2r_o^2} \right) + \hat{\mathcal{H}}_\varphi \right] \Psi_{WDW}(\varphi, a) = 0. \quad (62)$$

The conserved quantum number (ε_n) corresponds to classical integral of motion $(\mathcal{H}_\varphi)$. Then, factorization of the wave function takes place

$$\Psi_{WDW}(\varphi, a) = \sum_{\varepsilon(n)} \Psi(a, \varepsilon(n)) < n|\varphi >, \quad (63)$$

and $\Psi(a, \varepsilon_n)$ satisfies the equation

$$\left[-\left(\frac{p_a^2}{2\beta} + \frac{ka^2\beta}{2r_o^2} \right) + \varepsilon(n) \right] \Psi(a, \varepsilon(n)) = 0. \quad (64)$$

A solution of this equations will coincide with the wave functions of the Hamiltonian reduction (42), (43) if we use the ordering of the operators in (64), so that the momentum $(\hat{p})$ acts later than variable (a), and add a phase multiplier resulting from the surface term (61) contained in the initial Einstein – Hilbert action.

The prescribed ordering rule leads to the solution

$$\Psi_{WDW} = A^+ \Psi_{WDW}^+ + A^- \Psi_{WDW}^- \quad (65)$$

$$\Psi_{WDW}^\pm(a, \varepsilon(n)) = \exp\left[\pm i\sqrt{2\beta} \int\limits_o^a da' \sqrt{\varepsilon(n) - \frac{a'^2\beta}{2r_o^2}} \right] \quad (66)$$

$$\hat{p}_a \Psi_{WDW}^\pm = \pm\sqrt{2\beta} \sqrt{\varepsilon(n) - \frac{a^2\beta}{2r_o^2}} \Psi_{WDW}^\pm. \quad (67)$$

Taking into account the phase multiplier with the phase formed by the surface term in eq. (61)

$$S(a) = \frac{\sqrt{2\beta}}{2} a \sqrt{\varepsilon(n) - \frac{a^2\beta}{2r_o^2}} \quad (68)$$

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leads to the relation between the wave functions: the reduced one (42) and the WDW function (21)

$$\Psi^{\pm}_{Red}(a,\varphi) = e^{\pm i\varepsilon(n)r_0\eta_a(a)} <\varphi|n> = e^{\mp iS(a)}\Psi^{\pm}_{WDW}(a,\varepsilon(n)) <\varphi|n> . \tag{69}$$

As the variable a turns into a parameter, we can demand normalizability of this function only for the variables of the reduced phase space, for the variable φ in this case.

Thus, we show that a solution of the WDW equation can coincide with the wave function of the Dirac gaugeless quantization in the reduced phase space where the wave function is normalizable and describes the geometric evolution of the Universe for the Einstein comoving frame of reference (49):

$$\frac{d}{idT_F(a)}\left[\Psi^{\pm}_{WDW}(a,\varepsilon(n))e^{\mp iS(a)}\right] = \frac{\varepsilon(n)}{a(T_F)}\left[\Psi^{\pm}_{WDW}(a,\varepsilon(n))e^{\mp iS(a)}\right]. \tag{70}$$

We see that variation of such a WDW function with respect to the Friedmann time T_F gives the "observable" Friedmann energy (49).

9　Interpretation and conclusion

The aim of the present paper is to investigate relations between the Friedmann cosmological observables and the Dirac physical ones in the Hamiltonian approach to quantization of the Universe using a simple but important example of the homogeneous Universe filled in by the scalar field excitations.

An essential difference of the research presented here from the analogous papers on the Hamiltonian dynamics of cosmological models is complete separation of the sector of physical invariant variables from the pure gauge sector by the application of the gaugeless reduction [8, 9]. The main point is that in the process of the reduction one of variables converts in to the observable invariant time. We have shown that this conversion of the variable to the time parameter leads to the normalizability of the Wheeler – DeWitt (WDW) wave function and removes an infinite factor in the Hartle – Hawking functional integral. The gaugeless reduction gives us the definite mathematical and physical treatment of

the WDW wave function and clears up its relation to the observational cosmology.

The obtained wave function of the Friedmann Universe filled in by the homogeneous scalar field is nothing but the one of the scalar field excitations in the finite cavity of the conformal space occupied by the Universe. The time evolution is governed by the reduced Hamiltonian that coincides with the conventional Hamiltonian for the massless excitation in field theory.

The considered gaugeless reduction distinguishes the conformal ([13]) frame of reference. It was shown that, in both the classical and quantum theories, the Friedmann observables are connected with Dirac ones by the conformal transformations with the cosmic scale factor.

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Гогилидзе С.А. и др. E2-96-475
Дираковские и фридмановские наблюдаемые в квантовой вселенной с радиацией

При помощи бескалибровочной гамильтоновой редукции устанавливается отношение между фридмановскими наблюдаемыми в расширяющейся вселенной (закон Хаббла и красное смещение) и дираковскими наблюдаемыми в обобщенном гамильтоновом подходе для фридмановской космологической модели вселенной, заполненной полевыми возбуждениями, имитирующими радиацию.

Выполняется полное отделение физического сектора от калибровочного при помощи бескалибровочной гамильтоновой редукции, в которой гравитационная часть гамильтоновой связи преобразуется в новый импульс.

Показано, что эта редукция устраняет бесконечный калибровочный фактор из функционального интеграла Хартли — Хокинга, обеспечивает нормируемость волновой функции Уиллера — Де Витта относительно инвариантных переменных физического сектора, делает ясным их отношение к наблюдательной космологии и выделяет конформную систему отсчета Нарликара.

Работа выполнена в Лаборатории теоретической физики им.Н.Н.Боголюбова ОИЯИ.

Gogilidze S.A. et al. E2-96-475
Dirac and Friedmann Observables in Quantum Universe with Radiation

Relations between the Friedmann observables of the expanding Universe and the Dirac observables in the generalized Hamiltonian approach are established for the Friedmann cosmological model of the Universe with the field excitations imitating radfiation.

A full separation of the physical sector from the gauge one is fulfilled by the method of the gaugeless reduction in which the gravitational part of the energy constraint is considered as a new momentum. We show that this reduction removes an infinite factor from the Hartle — Hawking functional integral, provides the normalizability of the Wheeler — DeWitt wave function, clarifies its relation to the observational cosmology, and picks out a conformal frame of Narlikar.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.