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INSTANTONS AND GRAVITY

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1 Introduction

Here it is shown the unity of the problems of energy conservation, vacuum structure, radiation processes and quantization in the nonabelian gauge fields geometric theory and gravity. The gravity theory with the high derivatives and General Relativity are regarded as the examples of the nonabelian gauge fields. The single approach to all fundamental interactions is applied. The base of this approach is Lagrange formalism, infinite Lie's groups $G_{\infty r}$, and two Noether's theorems. For the first time this approach was formulated in [1] and after that in [2] and [3].

Unified description of all interactions including the gravity permits to demonstrate the single form of all conservation laws in all nonabelian gauge fields theories. Thanks to this fact the energy conservation problem in GR can be decided in natural analogy with the same problem in the theory of any fundamental interaction. At the same time without exit into unified description of fundamental interactions the gravity energy conservation problem has not decided now. These questions are discussed in [4].

The vacuum structure is the important problem in every gauge field theory. The special configurations of the gauge field equations solutions form the vacuum structure. They are named instantons and are used in nonperturbative methods of quantization. In the frame of unified description of interactions the vacuum structure question is closely connected with correct definition of the gauge field energy-momentum tensor and therefore with Einstein equations. If the same rules produce the energy-momentum tensors of gravity and all other fundamental interactions then the whole class of Einstein equations vacuum solutions becomes the gravity instantons in spite of hyperbolic signature of V_4 metrics. For the single procedure of all nonabelian gauge fields quantization it is necessary to unify the selection of field variables and radiation process equations in all gauge field theory.

The whole range of these questions can be named the prequantization since it concerns the necessary procedures foregoing transition from classical form of some interaction theory to its quantum form. If such transition was done incorrectly then in many cases it is possible to arise the problems which are artifacts. Therefore it seems to me that specific character of classical gravity must be discussed before quantization of this theory. As GR is geometrical theory here other fundamental interactions theories are described in geometrical terms also. In that way the unification of description of all interactions is reached in geometrical form of theories as well as in Lagrange form.

The conception of vacuum and instantons role as nontrivial vacuum solutions of nonabelian gauge fields equations in Riemannian space-time are analyzed. The energy-momentum tensor of nonabelian gauge field and the energy-momentum tensor of the same form of $SO(3,1)$-gravity are constructed. It is demonstrated that all gauge fields instantons correspond to vacuum Einstein equations and describe a vacuum space-time structure. The problems of correct choice of field variables and wave operators in the nonabelian gauge fields and in particular in gravity are discussed.

In the variational and geometric approach which was formulated in [1]-[3] all fundamental interactions are unified by the single principle of constructing the theory of every of them. To produce the lagrangian, equations etc. one can give:

1. the field variables,
2. two kinds of symmetry groups (space-time and internal symmetries),
3. two kinds of the field variables transformations (in correspondence with two kinds of symmetry groups),
4. the order of the field variables derivatives in the lagrangian.

There are no problems in the unification gravity and other interactions in this approach. All gauge fields are different representations of infinite Lie groups $G_{\infty r}$. Any compensation idea is absent.

It is unique gauge theory containing GR as the gauge theory of gravity equally other fundamental interactions. In this scheme GR acts as the lowest approximation and it remains in the theory when the higher $g_{\mu\nu}$ derivatives are included in the Lagrangian. Then the higher symmetry groups arise and GR turnes into the vacuum structure theory for the gauge $SO(3,1)$-gravity and so on.

Thus the problem of including Einstein theory of gravity into the number of S.Lie groups based theories was decided by me in 1960-th. GR can not be included into the number of local gauge theories by Utiyama or Kibble method or by Great Unification without some modifications of itself or method used . Infinite S.Lie groups $G_{\infty r}$ are the only mathematical language which permits us to unify GR with other fundamental interactions consistently. But these groups in contrast to finite-parameters S.Lie groups G_r do not give us the conservation laws and generate only some identical relations (Noether's identities) which become the constraints in the theories with local gauge symmetries. The conservation laws are given by first Noether's theorem, but Noether's identities are given by second Noether's theorem. The construction of energy conservation law in GR is regarded now as one of the big problem of theoretical physics.

Before the quantization of some theory it is necessary to define some important conceptions: what is the energy, what is the vacuum, what is the wave and so on. As the instantons are the solutions of the classical gauge fields equations we shall consider its definition on the classical level.

In the present paper the universal approach is proposed which is suitable to all gauge fields. In consequence with such approach the construction of the conservation laws and the quantization procedure must be same for all nonabelian gauge fields including gravity. The role of instantons in creating of vacuum structure must be same also in every nonabelian gauge field theory. Because any vacuum is a long-distance structure in the space-time it must be closely connected with the gravity vacuum structure. This fact will be demonstrated below in the frame of the unified geometric gauge field theory ([5]).

2 The gravity field variables choice

This problem is generated by two reasons:

1. the general covariant coordinates transformations $G_{\infty 4}$

$$x^{\mu\prime} = f^{\mu}(x^{\nu}) \tag{1}$$

where $f^{\mu}(x^{\nu})$ - arbitrary functions of x^{ν}, and

2. the equivalence principle.

In the unified variational and geometric scheme for all fundamental interactions which is under discussion here the gravity (and GR also) is the gauge field. When $g_{\mu\nu}$ and the gauge group $G_{\infty 4}$ are chosen as the field variables and the invariance group of theory we obtain GR as the gauge theory of the gravity field. Analogously when $g_{\mu\nu}$ and Ricci connection coefficients $\Delta_{\mu}(ik)$ are chosen as the field variables and the local gauge groups $G_{\infty 4}$ and $G_{\infty 6}$ are chosen as the theory invariance groups we obtain the gauge $SO(3,1)$-gravity. Other cases are characterized by proper choice of the field variables and the theory invariance groups.

Which choice is the best to describe the real physical situation in the gravity and proper experiments?

The best choice corresponds to the situation when the devices properties and the measurements methods are taken into account in the theory structure.

Can $g_{\mu\nu}$ reflect the properties of the real gravitational field?

In one point the metrics $g_{\mu\nu}$ may always be transformed into the Euclidean form by $G_{\infty 4}$-transformations. Consequently in one point Riemannian space may always be considered Euclidean space, and noneuclidean form of $g_{\mu\nu}$ in one point does not reflect the existence of real gravity.

Can the connection coefficients $\Gamma^{\lambda}_{\mu\nu}$ reflect the properties of the real gravitational field?

In one point in correspondence with the equivalence principle the connection coefficients $\Gamma^{\lambda}_{\mu\nu}$ may always be eliminated by $G_{\infty 4}$-transformations and hence can not reflect the properties of the real gravity.

Can the Riemannian curvature tensor $R^{\lambda}_{\mu\nu\tau}$ reflect the properties of the real gravity?

The components of the curvature tensor can not be eliminated by $G_{\infty 4}$- transformations even in one point. The curvature tensor reflects the properties of the whole region of Riemannian space in neighbourhood of fixed point. Therefore the curvature tensor $R^{\lambda}_{\mu\nu\tau}$ can reflect the properties of the real gravity field. Just its components are experimentally measurable.

3 The gravity waves equations choice

1. *The metrics waves equations*

   As the equations for the metrics gravity waves the linearized Einstein equations are usually being used ([6]). They are considered the weak gravity waves equations over the given flat external metrics $\tilde{g}_{\mu\nu}$ (Euclidean background). Then many questions arise about the physical sense of such equations solutions. If this approach is generalized into nonflat background the same questions remain.

2. *The connection coefficients waves equations*

   In consequence of the equivalence principle and nontensorial nature of the connection coefficients the waves equations for $\Gamma^{\lambda}_{\mu\nu}$ or $\Delta_{\mu}(ik)$ are beyond any discussion.

3. *The gravity waves equations as the equations for the components of Riemanian curvature tensor.*

   Such approach is supported by K.S.Thorne and others ([7]) who use the geodesic deviation equations as the gravity waves that. These equations describe the real physical situation when the experimental data are receiving from two points measurements. Then the gravity waves existence is identified with the periodical variations of the distance between two points. Such variations is proportional to the curvature tensor $R^{\lambda}_{\mu\nu\tau}$.

4 The gravity waves equations in the unified geometrical gauge fields theory

In the unified geometrical gauge fields theory the gravity waves equations arise as the special case of the gauge fields equations when the field variables are $g_{\mu\nu}$ and Ricci connection coefficients $\Delta_{\mu}(ik)$. The local gauge groups of the theory invariance are $G_{\infty 4}$ and the local Lorentz group $G_{\infty 6}$.

The structure of such equations is nonlinear generalization of the Maxwell equations for the electromagnetic waves. In the gravity the role of Riemannian curvature tensor is similar to that of the field tensor $F_{\mu\nu}$ in electrodynamics. The connection coefficients $\Delta_\mu(ik)$ are the analog of the electromagnetic vector-potential A_μ. To find the wave solutions of the gravitational equations

$$R^{\lambda}_{\mu\nu\tau\ ;\lambda} = 0 \tag{2}$$

it is necessary to fix the gauge of $\Delta_\mu(ik)$. What is the gauge needed? To answer this question it is necessary to consider the geometrical and topological properties of equations.

In the Maxwell electrodynamics Lorentz gauge $A_\mu^{\ ;\mu} = 0$ transforms the equations of the field tensor $F_{\mu\nu}$:

$$F_{\mu\nu}^{\ \ ;\nu} = 0 \tag{3}$$

into Laplacian operator for A_μ:

$$\Delta_4 A_\mu = 0 \qquad (i.e. \Box A_\mu = 0) \tag{4}$$

Laplacian operator on the 4D manifold V_4 with metrics $g_{\mu\nu}$ of hyperbolic signature (+++-) is Dalambertian operator which has the wave solutions.

However in Euclidean space Laplacian and Dalambertian operators are linear. What is the wave as the solution of the nonlinear equation? This is the solution possessing other properties. To quantize some field it is necessary to construct Green's function of its equation. The operator for which Green's function exists in the Riemannian space is the topological Laplacian ([8])

$$\Delta = d\delta + \delta d \tag{5}$$

where d, δ - invariant operators of external differentiation and codifferentiation. These operators act on the external forms.

The metrics $g_{\mu\nu}$ is symmetrical second rank tensor but not the external form. Therefore the invariant wave operator (the topological Laplacian) in the Riemannian space doesn't exist for it. But for the curvature tensor such operator exists ([5]):

$$\Box R^{\lambda}_{\mu\nu\tau} = 0 \tag{6}$$

or

$$\Delta R^{\lambda}_{\mu\nu\tau} = 0 \tag{7}$$

The topological Laplacian for $R^{\lambda}_{\mu\nu\tau}$ can be constructed by differentiation of the equations system

$$R^{\lambda}_{\mu[\nu\tau;\sigma]} = 0 \qquad R^{\lambda}_{\mu\nu\tau;\lambda} = 0 \longmapsto \Box R^{\lambda}_{\mu\nu\tau} = 0 \tag{8}$$

in full analogy with the procedure in electrodynamics which leads to the wave equations for the real electric (**E**) and magnetic (**H**) components of the field tensor $F_{\mu\nu}$:

$$F_{[\mu\nu;\sigma]} = 0 \qquad F^{\mu}_{\nu\ ;\mu} = 0 \longmapsto \Box F_{\mu\nu} = 0 \tag{9}$$

5 GR and gauge gravity vacuum structure

In the Wheeler's unified theory of electrodynamics and gravity (the geometrodynamics [9]) the main equations system is

$$F_{\mu\nu}^{\ \ ;\nu} = 0 \tag{10}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa (F_{\mu\tau} F^{\tau}_{\ \nu} - \frac{1}{4} g_{\mu\nu} F_{\lambda\tau} F^{\lambda\tau}) \tag{11}$$

This equations system consists of Maxwell equations in Riemannian V_4 and Einstein equations with energy-momentum tensor of electromagnetic field

$$T^{(em)}_{\mu\nu} = (F_{\mu\tau} F^{\tau}_{\ \nu} - \frac{1}{4} g_{\mu\nu} F_{\lambda\tau} F^{\lambda\tau}) \tag{12}$$

as the source of the gravity field.

In the unified geometric theory of the gauge fields the main equations system is analog of that in the Wheeler's geometrodynamics ([5]):

$$F^{a\ \ ;\nu}_{\mu\nu} = 0 \tag{13}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa (F^{a}_{\mu\tau} F^{\tau}_{a\nu} - \frac{1}{4} g_{\mu\nu} F^{a}_{\lambda\tau} F^{\lambda\tau}_{a}) \tag{14}$$

This equations system consists of the gauge field equations in Riemannian V_4 and Einstein equations with energy-momentum tensor of the gauge field in the right side as the source of the gravity field:

$$T^{(gf)}_{\mu\nu} = (F^{a}_{\mu\tau} F^{\tau}_{a\nu} - \frac{1}{4} g_{\mu\nu} F^{a}_{\lambda\tau} F^{\lambda\tau}_{a}) = (F^{a}_{\mu\tau} - i^{*} F^{a}_{\mu\tau})(F_{a\ \nu}^{\ \tau} + i^{*} F_{a\ \nu}^{\ \tau}) \tag{15}$$

If the gauge field is the gravity field and its local gauge group is $SO(3,1)$ (in addition to $G_{\infty 4}$) and consequently its Lagrangian has Yang-Mills type form plus Einstein part: $L = R + \lambda^2 R^{\tau\lambda}_{\mu\nu} R^{\mu\nu}_{\tau\lambda}$ then above-mentioned equations system takes the form ([5]):

$$R^{\tau\lambda\ \ ;\nu}_{\mu\nu} = 0 \tag{16}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa (R^{\tau\lambda}_{\mu\sigma} R^{\sigma}_{\nu\tau\lambda} - \frac{1}{4} g_{\mu\nu} R^{\alpha\beta}_{\lambda\tau} R^{\lambda\tau}_{\alpha\beta}) \tag{17}$$

This equations system describes the theory of gravity with higher derivatives. Such a theory takes into account the extent of real objects and describes the real gravity forces acting on them, i.e. tidal forces. As the energy-momentum tensor of such gravity forces it is necessary to admit the following expression:

$$T^{(g)}_{\mu\nu} = (R^{\tau\lambda}_{\mu\sigma} R^{\sigma}_{\nu\tau\lambda} - \frac{1}{4} g_{\mu\nu} R^{\alpha\beta}_{\lambda\tau} R^{\lambda\tau}_{\alpha\beta}) \tag{18}$$

If the energy-momentum tensor vanishes then it is vacuum state by definition. There are nontrivial solutions of equations $T_{\mu\nu} = 0$ besides trivial that. These solutions obey the duality equations which in electrodynamics have the following form

$$F_{\mu\nu} = \pm i^* F_{\mu\nu} \tag{19}$$

Nontrivial solutions of duality equations are named the instantons. They minimize the action integral $S = \int F_{\mu\nu} F^{\mu\nu} dV$ and transform it into the topological constant.

In the case of the gauge field nontrivial solutions of the equations $T^{(gf)}_{\mu\nu} = 0$ and duality equations

$$F_{\mu\nu} = \pm i^* F_{\mu\nu} \tag{20}$$

are instantons which minimize the action integral $S = \int F^a_{\mu\nu} F^{\mu\nu}_a dV$ and transform it into the topological constant.

In the case of the gauge gravity the equation $T^{(g)}_{\mu\nu} = 0$ implies arising of the vacuum state of the real gravity and the transition to GR. All solutions of Einstein equations are the solutions of the gauge gravity equations. But instead of duality equations

$$R^{\tau\lambda}_{\mu\nu} = \pm i^* R^{\tau\lambda}_{\mu\nu} \tag{21}$$

they are the solutions of twice dual equations

$$R_{\tau\lambda\mu\nu} = \pm^* R^*_{\tau\lambda\mu\nu} \tag{22}$$

and therefore

$$R_{\tau\nu} = \pm^* R^*_{\tau\nu} \tag{23}$$

The duality equations (21) which are analog of electromagnetic conditions of duality have only trivial solutions in the case of gravity (Euclidean V_4).

Taking into account that equations (17) followed by $R = 0$ we can transform them to the form ([5])

$$R^\mu_\nu = -\kappa (R^{\mu\sigma\tau\lambda} -^* R^{*\mu\sigma\tau\lambda})(R_{\nu\sigma\tau\lambda} +^* R^*_{\nu\sigma\tau\lambda}) \tag{24}$$

Therefore $T^{(g)}_{\mu\nu} = 0$ if either

$$R^{\tau\lambda}_{\mu\nu} = +^* R^{*\tau\lambda}_{\mu\nu} \quad \text{and} \quad R_{\tau\nu} = +^* R^*_{\tau\nu} = R_{\tau\nu} - \frac{1}{2} g_{\tau\nu} R \longmapsto R = 0 \tag{25}$$

and we have not any new solution, or

$$R^{\tau\lambda}_{\mu\nu} = -^* R^{*\tau\lambda}_{\mu\nu} \quad \text{and} \quad R_{\tau\nu} = -^* R^*_{\tau\nu} \longmapsto R_{\tau\nu} = 0 \tag{26}$$

that is Einstein gravity.

Hence we have vacuum Einstein equations which solutions are gravity instantons by definition in the frame of $SO(3,1)$-gauge gravity theory. Therefore all

solutions of GR-equations describe the vacuum structure of the gauge gravity theory and Schwarzschild solution is one of them. The hyperbolical signature is not an obstacle to being instanton.

Thus it is shown that the gravity has to be consider the gauge field in the single scheme with other interactions and the gravity waves quantization procedure has to be analogous to that of any nonabelian gauge field. It is necessary to note that under condition $T_{\mu\nu} = 0$ in the right side of Einstein equations we obtain always the Einstein gravity vacuum equation independently of the gauge field type. Thus all gauge field instantons can take part in creation of space-time vacuum structure.

6 Conclusions

Thus we can resume following:

1. Using the local gauge groups $G_{\infty 4}$ and $G_{\infty 6}$ we obtain the gravity Lagrangian in the form of sum of Einstein's and Weyl's parts:

$$L = R + \lambda^2 R_{\mu\nu\tau\lambda} R^{\mu\nu\tau\lambda}$$

2. The Euler-Lagrange equations for this Lagrangian are the extensions of Wheeler's geometrodynamics equations.

3. The energy-momentum tensor $T_{\mu\nu}$ of each gauge field equally the gravity has the same form for all gauge fields.

4. The instantons are the solutions of self-duality equations for nongravitational fields and twice self-duality equations for Riemannian curvature tensor in the gravity case.

5. Such instantons determine the vacuum structure of each gauge field equally the gravity because self-duality and twice self-duality equations are followed by $T_{\mu\nu} = 0$. Moreover they turn the integrals of form

$$S = \int F^a_{\mu\nu} {}^* F^{\mu\nu}_a dV$$

and

$$S = \int R_{\mu\nu\tau\lambda} {}^* R^{*\mu\nu\tau\lambda} dV$$

into the surface integrals (that is the topological constants).

6. The vacuum Einstein's equations can be regarded as the contracted twice self-duality equations of Riemannian curvature tensor. Consequently they

can be used with Weyl's Lagrangian under the conditions of twice self-duality of solutions (without Einstein's part of Lagrangian!). In this case the second coupling constant does not arise.

7. The quantization procedure of gravity must be the same that for all non-abelian gauge fields equally the gravity and must be performed over Einstein vacuum background.

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Литература

[1] N.P.Konopleva, in: Gravitation and the Theory of Relativity, Nos.4-5, Kazan State University, (1968) 67 in Russian

[2] N.P.Konopleva, V.N.Popov, Gauge fields, Atomizdat, Moscow, (1972) in Russian

[3] N.P.Konopleva, V.N.Popov, Gauge fields, Harwood academic publishers, Chur-London-New York, (1981)

[4] N.P.Konopleva, The conservation laws in GR, in: The gravity energy and gravity waves, Proc. of VI sem., Dubna, 1994, 60, in Russian

[5] N.P.Konopleva, On geometrodynamics and compensating fields, in: Problems of the Theory of Gravitation and Elementary particles, iss.3, Atomizdat, Moscow, 1970, 103, in Russian

[6] L.D.Landau, E.M.Lifshitz, Theory of field, Nauka, Moscow, 1973, in Russian

[7] C.W.Misner, K.S.Thorne, J.A.Wheeler, Gravitation, W.H.Freeman and Company, San Francisco, 1973

[8] G. de Rham, Variétés Différentiables, Hermann, Paris, 1955

[9] J.A.Wheeler, Geometrodynamics, Academic Press, New York, 1962

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Коноплева Н.П. E2-96-459
Инстантоны и гравитация

Обсуждаются проблемы, связанные с применением непертурбативных методов квантования калибровочных полей и гравитации. Объединение взаимодействий рассматривается в рамках геометрической теории калибровочных полей. Обсуждается понятие вакуума в единой теории взаимодействий и роль инстантонов в структуре вакуума. Демонстрируется роль вакуумных решений уравнений Эйнштейна в определении вакуума калибровочных полей.

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Konopleva N.P. E2-96-459
Instantons and Gravity

The problems of application of nonperturbative quantization methods in the theories of the gauge fields and gravity are discussed. Unification of interactions is considered in the frame of the geometrical gauge fields theory. Vacuum conception in the unified theory of interactions and instantons role in the vacuum structure are analyzed. The role of vacuum solutions of Einstein equations in definition of the gauge field vacuum is demonstrated.

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